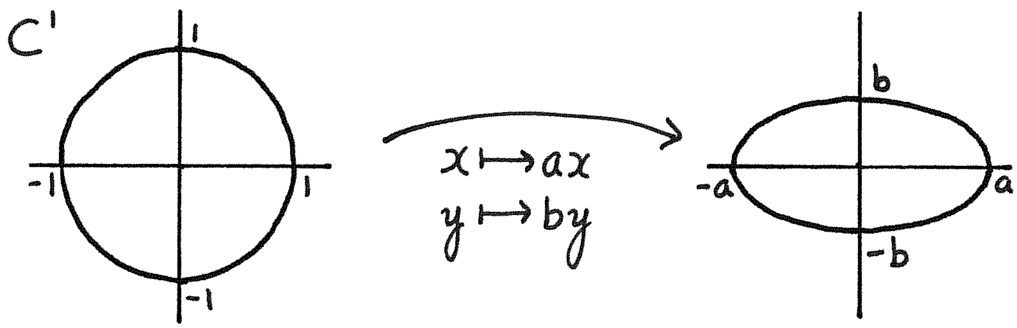
Ellipses and Hyperbolas

In this chapter we'll see three more examples of conics.

Ellipses

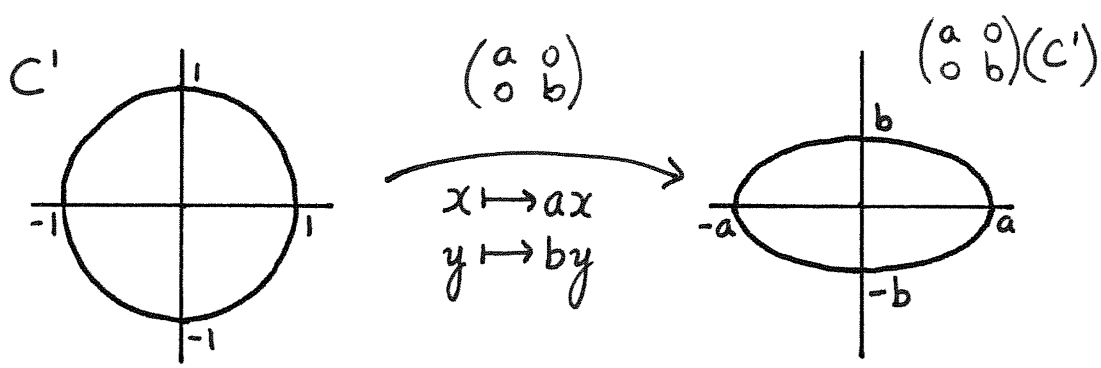
If you begin with the unit circle, C^1, and you scale x-coordinates by some nonzero number a, and you scale y-coordinates by some nonzero number b, the resulting shape in the plane is called an *ellipse*.



Let's begin again with the unit circle C^1, the circle of radius 1 centered at $(0,0)$. We learned earlier that C^1 is the set of solutions of the equation

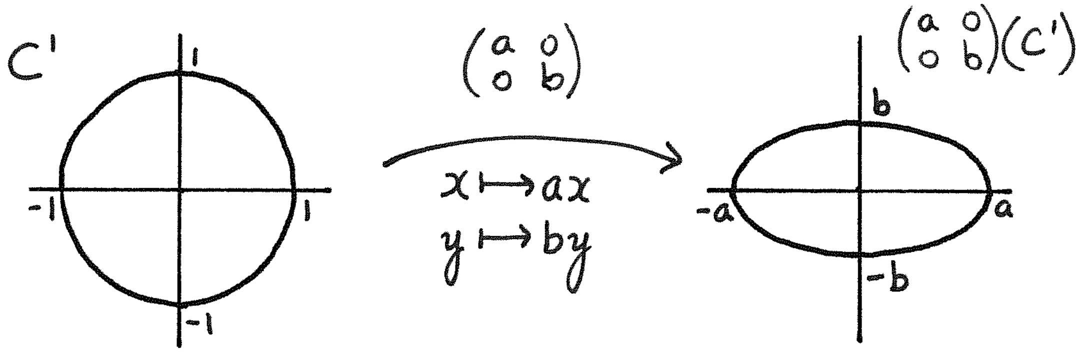
$$x^2 + y^2 = 1$$

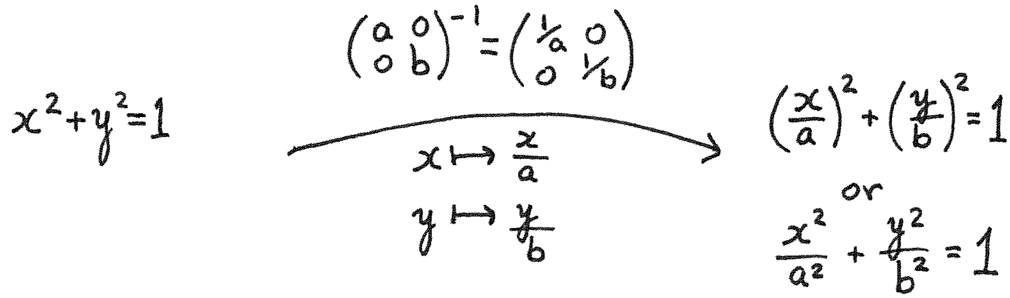
The matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ scales the x-coordinates in the plane by a, and it scales the y-coordinates by b. What's drawn below on the right is $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}(C^1)$, which is the shape resulting from scaling C^1 horizontally by a and vertically by b. It's an example of an ellipse.



Using POTS, the equation for this distorted circle, the ellipse $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}(C^1)$, is obtained by precomposing the equation for C^1, the equation $x^2 + y^2 = 1$, by the matrix

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}$$





Because $\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}$ replaces x with $\frac{x}{a}$ and y with $\frac{y}{b}$, the equation for the ellipse $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}(C^1)$ is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

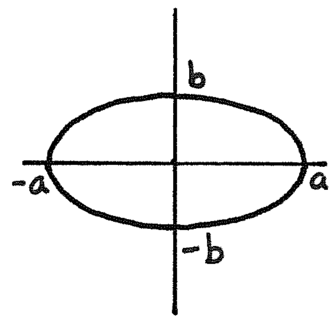
which is equivalent to the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The equation for an ellipse described above is important. It's repeated at the top of the next page.

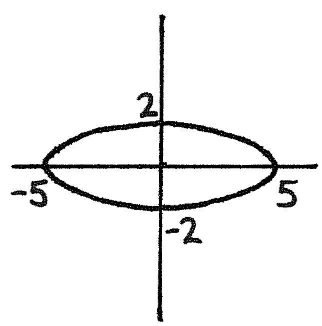
Suppose $a, b \in \mathbb{R} - \{0\}$. The equation for the ellipse obtained by scaling the unit circle by a in the x-coordinate and by b in the y-coordinate is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Example.

• The equation for the ellipse shown below is $\frac{x^2}{25} + \frac{y^2}{4} = 1$. In this example, we are using the formula from the top of the page with $a = 5$ and $b = 2$.

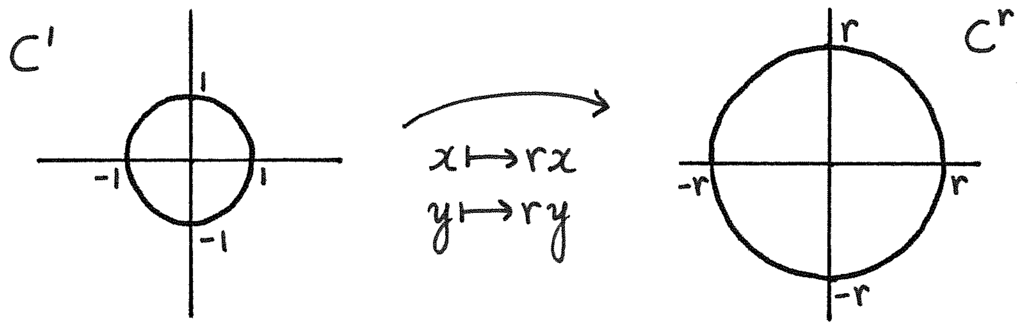


* * * * * * * * * * * * *

Circles are ellipses

A circle is a perfectly round ellipse.

If we scale the unit circle by the same number $r > 0$ in both coordinates, then we'll stretch the unit circle evenly in all directions.



The result will be another circle, one whose equation is

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

We can multiply both sides of this equation by r^2 to obtain the equivalent equation

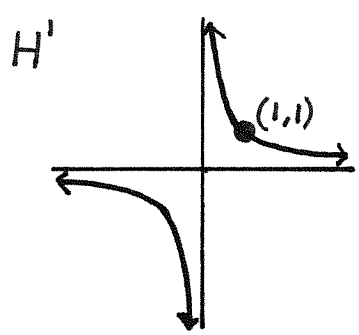
$$x^2 + y^2 = r^2$$

which we had seen earlier as the equation for C^r, the circle of radius r centered at $(0, 0)$.

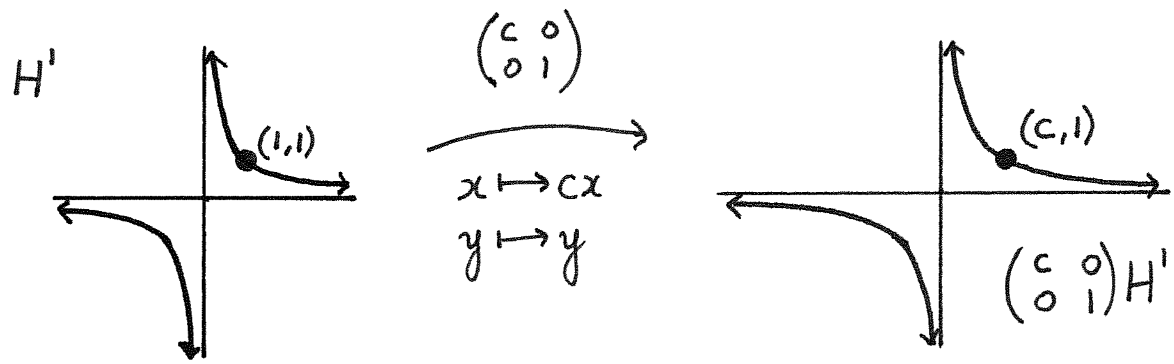
* * * * * * * * * * * * *

Hyperbolas from scaling

We've seen one example of a hyperbola, namely the set of solutions of the equation $xy = 1$. We'll call this hyperbola H^1, the *unit hyperbola*.



Suppose $c > 0$. We can horizontally stretch the unit hyperbola H^1 using the diagonal matrix $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$. This is the matrix that scales the x-coordinate by c and does not alter the y-coordinate. The resulting shape in the plane, $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}(H^1)$, is also called a hyperbola.



$$xy = 1 \xrightarrow[\substack{x \mapsto \frac{x}{c} \\ y \mapsto y}]{\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/c & 0 \\ 0 & 1 \end{pmatrix}} \left(\frac{x}{c}\right) y = 1 \quad \text{or} \quad xy = c$$

The equation for the hyperbola $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}(H^1)$ is obtained by precomposing the equation for H^1, the equation $xy = 1$, with $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix}$. This is the matrix that replaces x with $\frac{x}{c}$ and does not alter y. Thus, the equation for the hyperbola $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}(H^1)$ is

$$\left(\frac{x}{c}\right) y = 1$$

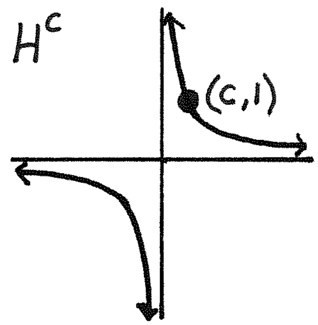
which is equivalent to

$$xy = c$$

From now on, we'll call this hyperbola H^c.

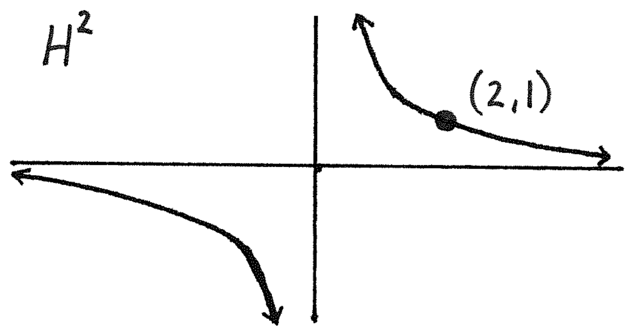
To summarize:

Let $c > 0$. Then H^c is the hyperbola
that is the set of solutions of the equation $xy = c$.



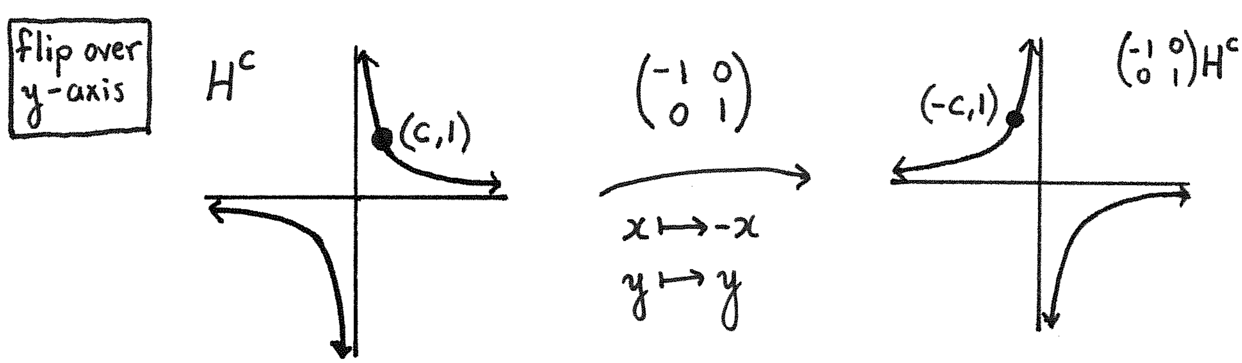
Example.

• The equation for the hyperbola H^2, obtained by scaling the unit hyperbola by 2 in the x-coordinate is $xy = 2$.



Hyperbolas from flipping

We can flip the hyperbola H^c over the y-axis using the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, the matrix that replaces x with $-x$ and does not alter y.



The shape resulting from flipping the hyperbola H^c over the y-axis is also called a hyperbola. Its equation is obtained by precomposing the equation for H^c, the equation $xy = c$, with the inverse of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ itself, the matrix that replaces x with $-x$.

$$xy=c \xrightarrow[\substack{x \mapsto -x \\ y \mapsto y}]{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{matrix} (-x)y=c \\ \text{or} \\ xy=-c \end{matrix}$$

So the equation for H^c after being flipped over the y-axis is

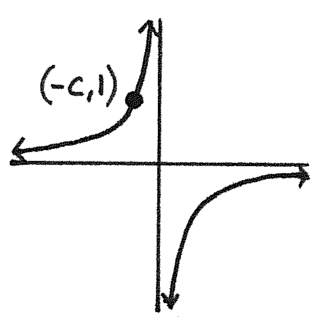
$$(-x)y = c$$

which is equivalent to

$$xy = -c$$

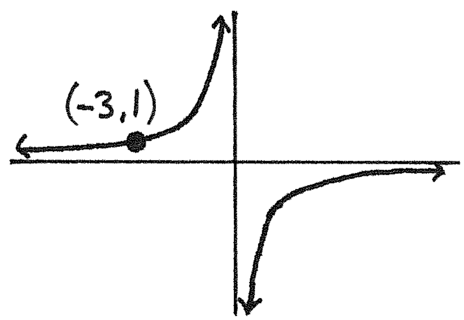
To summarize:

Let $c > 0$. The equation for H^c flipped over the y-axis is $xy = -c$.



Example.

- The equation for H^3 after being flipped over the y-axis is $xy = -3$.



* * * * * * * * * * * * *

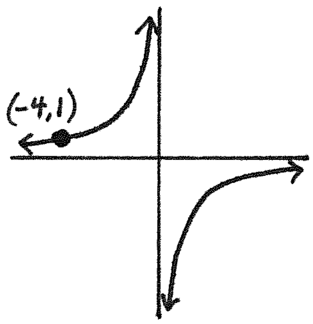
Ellipses and hyperbolas are conics

The equations that we've seen for ellipses and hyperbolas in this chapter are quadratic equations in two variables: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $xy = c$, and $xy = -c$. Thus, the solutions of these equations, the ellipses and hyperbolas above, are examples of conics.

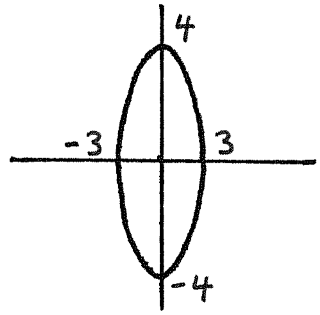
Exercises

For #1-6, match the numbered pictures with the lettered equations.

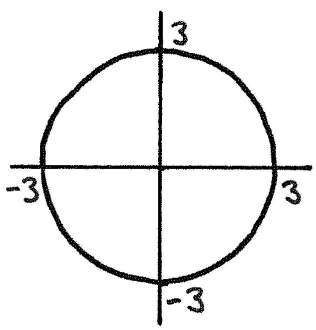
1.)



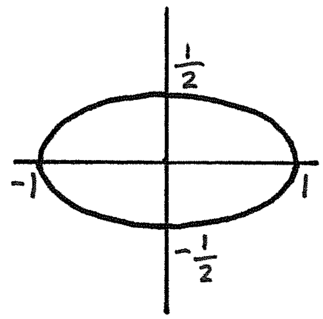
4.)



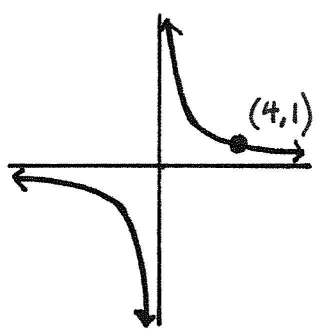
2.)



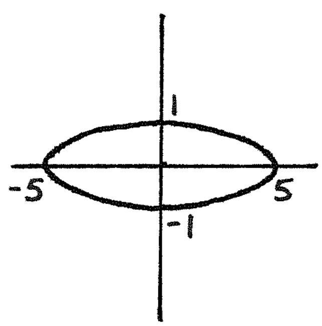
5.)



3.)



6.)

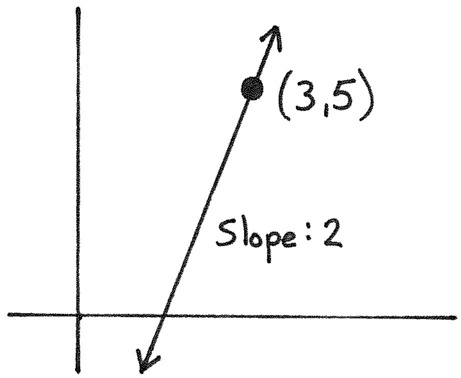


A.) $xy = 4$ B.) $x^2 + 4y^2 = 1$ C.) $x^2 + y^2 = 9$

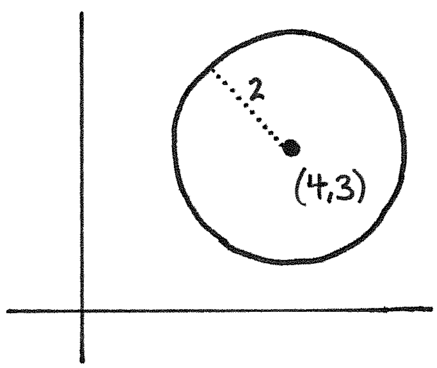
D.) $xy = -4$ E.) $\frac{x^2}{9} + \frac{y^2}{16} = 1$ F.) $\frac{x^2}{25} + y^2 = 1$

For #7-10, write an equation for each of the given shapes in the plane. Your answers should have the form $(y-q)=a(x-p)$, $(x-p)^2+(y-q)^2=r^2$, $(x-p)(y-q)=c$, or $\frac{(x-p)^2}{a^2}+\frac{(y-q)^2}{b^2}=1$. (As with the x- and y-axes, the dotted lines are not part of the shapes. They're just drawn to provide a point of reference.)

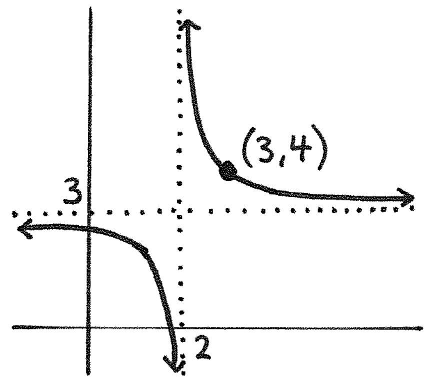
7.)



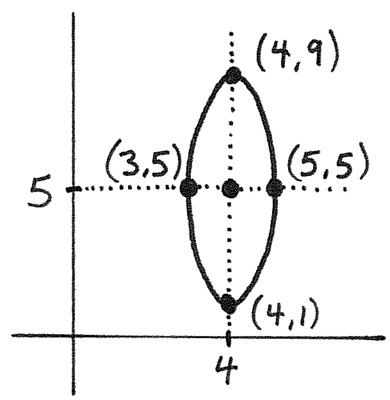
8.)



9.)



10.)



For #11-14, multiply the matrices.

11.) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 2 & -3 \end{pmatrix}$

12.) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$

13.) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 8 & 9 \end{pmatrix}$

14.) $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$