Equations in One Variable II

In this chapter, we'll learn two more techniques for determining solutions of equations in one variable.

No solutions due to the range of a function

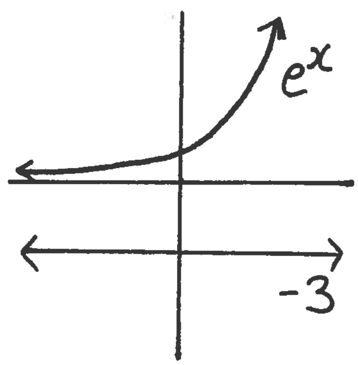
If $f(x)$ is a function, and if $c \in \mathbb{R}$ is a number that is not in the range of $f(x)$, then the equation

$$f(x) = c$$

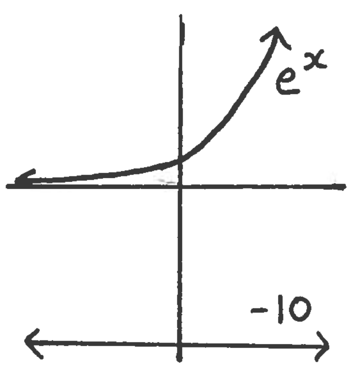
has no solution. That is, the set of solutions of the equation $f(x) = c$ is $\emptyset$.

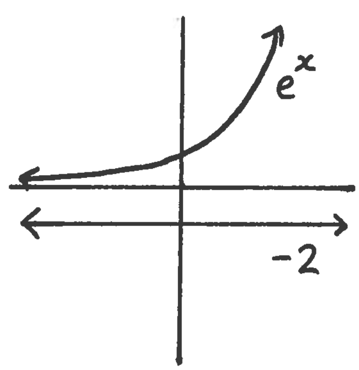
Examples.

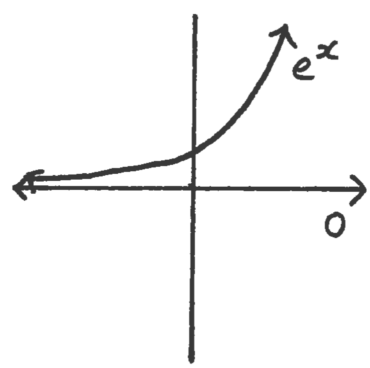
• The range of the exponential function e^x is the set of positive numbers. Whatever we choose as an input for the exponential function, the output is always positive. That means that there is no value of x for which e^x equals, say, -3. That is, the equation $e^x = -3$ has no solution.



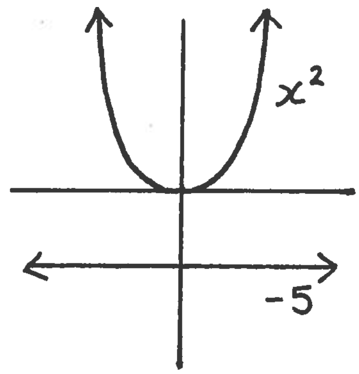
• The equations $e^x = -10$, $e^x = -2$, and $e^x = 0$ each have no solution. That's because -10, -2, and 0 are not in the range of the function e^x. The exponential function only has positive numbers as outputs.



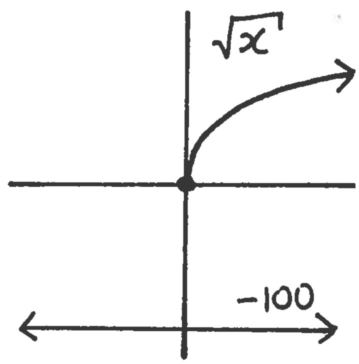




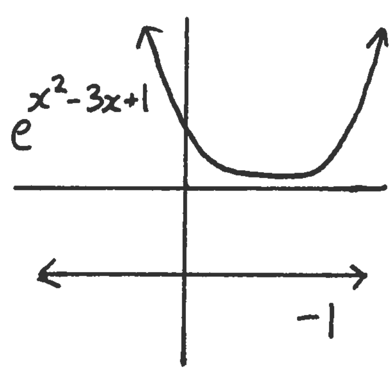
• The equation $x^2 = -5$ has no solution. That's because you can't square a number to get a negative value. The range of the function x^2 is $[0, \infty)$, and $-5 \notin [0, \infty)$.



• The range of $\sqrt{x}$ is $[0, \infty)$. Because $-100 \notin [0, \infty)$, the equation $\sqrt{x} = -100$ has no solution.



• No matter what we use as the value for x, e^x is always positive. Even if the value we use for x is another function. For example, e^{x^2-3x+1} is always positive. Therefore, the equation $e^{x^2-3x+1} = -1$ has no solution.



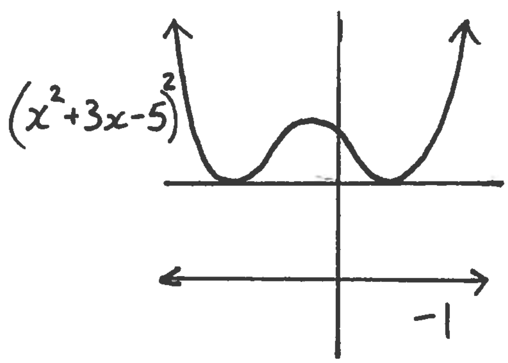
- Expanding on the last example:

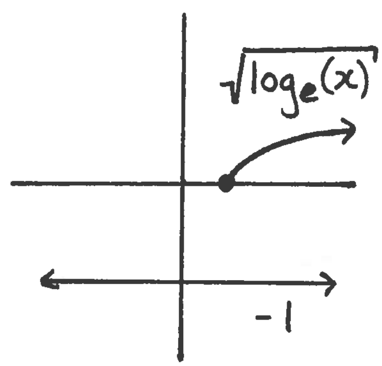
An equation of the form $e^{f(x)} = c$ has no solution if c is a number with $c \leq 0$.

- We have a similar rule for squaring and taking square roots:

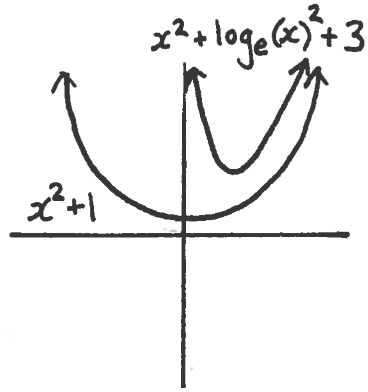
Equations of the form $f(x)^2 = c$ or $\sqrt{f(x)} = c$ have no solution if c is a number with $c < 0$.

As particular examples of the rule above, $(x^2 + 3x - 5)^2 = -1$ and $\sqrt{\log_e(x)} = -1$ each have no solution.





- The equation $x^2+\log_e(x)^2+3 = x^2+1$ is equivalent by addition to the equation $\log_e(x)^2 = -2$. Because -2 is negative, and because the square of a number can never be negative, the equation $\log_e(x)^2 = -2$ has no solution. Because equivalent equations have the same solutions, our original equation $x^2 + \log_e(x)^2 + 3 = x^2 + 1$ also has no solution.



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Using the quadratic formula as an intermediate step

- The equation $3x^2 + 4x - 5 = 0$ is called a *quadratic equation in* x, meaning that it's a quadratic equation, and x is the variable.

- $3y^2+4y-5 = 0$ is a quadratic equation in y, because y is the variable.

- $3z^2 + 4z - 5 = 0$ is a quadratic equation in z.

- $3w^2 + 4w - 5 = 0$ is a quadratic equation in w.

- $3\clubsuit^2 + 4\clubsuit - 5 = 0$ is a quadratic equation in $\clubsuit$.

- $3(x-3)^2 + 4(x-3) - 5 = 0$ is a quadratic equation in $(x-3)$.

- $3\log_e(x)^2 + 4\log_e(x) - 5 = 0$ is a quadratic equation in $\log_e(x)$.

- $3f(x)^2 + 4f(x) - 5 = 0$ is a quadratic equation in $f(x)$.

If we have a quadratic equation in x, a quadratic equation of the form $ax^2+bx+c = 0$ (where $a \neq 0$), then the quadratic formula tells us that there are no solutions of the equation if $b^2 - 4ac < 0$, and that if $b^2 - 4ac \geq 0$ then the solutions of the equation are

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula is the same regardless of the variable used in a quadratic equation. That is, if we have a quadratic equation in $f(x)$, an equation of the form $a\,f(x)^2 + b\,f(x) + c = 0$, then the quadatic formula tells us what $f(x)$ must be. The solutions are

$$f(x) = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad f(x) = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

It doesn't matter what the function $f(x)$ is.

Examples.

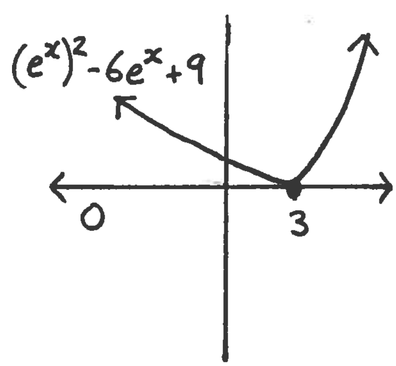
• Let's determine the solutions of the equation $(e^x)^2 - 6e^x + 9 = 0$. This is a quadratic equation in e^x, and the quadratic formula tells us that we must have that

$$e^x = \frac{-(-6) - \sqrt{(-6)^2 - 4(1)(9)}}{2(1)} = \frac{6 - \sqrt{0}}{2} = 3$$

This is only an intermediate step. Our ultimate goal is not to find values of e^x that are solutions to an equation, but rather to find values of x that are solutions to an equation. We now know that $e^x = 3$, and we must continue.

The equation $e^x = 3$ is equivalent by invertible function to the equation $x = \log_e(3)$. Thus, our original equation $(e^x)^2 - 6e^x + 9 = 0$ has $\log_e(3)$ as its one and only solution.

$$(e^x)^2 - 6e^x + 9 = 0 \longrightarrow e^x = 3 \longrightarrow x = \log_e(3)$$



• Let's determine the solutions of the equation $\log_e(x)^2 = \log_e(x) + 2$.

We begin by noting that the domain of the given equation is $(0, \infty)$, because we can only take a logarithm of a positive number.

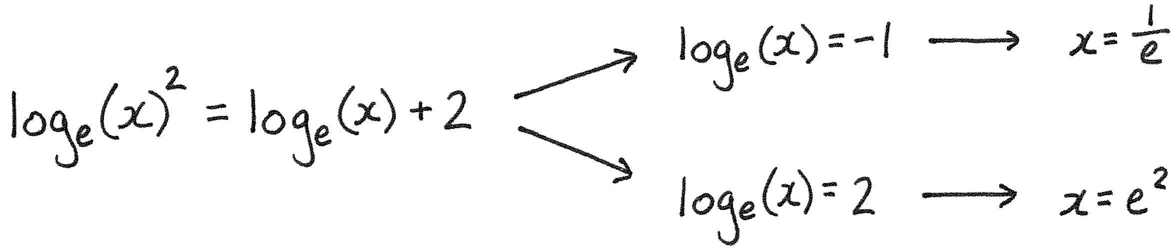
The given equation is equivalent by addition to $\log_e(x)^2 - \log_e(x) - 2 = 0$. This is a quadratic equation in $\log_e(x)$. The quadratic formula tells us either

$$\log_e(x) = \frac{-(-1) - \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{1 - \sqrt{1+8}}{2(1)} = \frac{1-3}{2} = -1$$

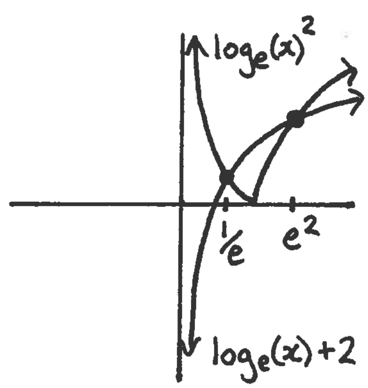
or

$$\log_e(x) = \frac{-(-1) + \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{1 + \sqrt{1+8}}{2(1)} = \frac{1+3}{2} = 2$$

To find all of the solutions to our original equation, we'll now have to find the solutions to two different equations: $\log_e(x) = -1$ and $\log_e(x) = 2$.

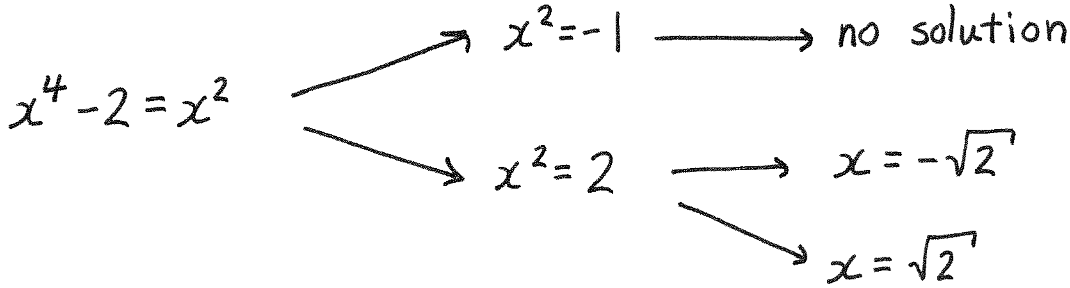


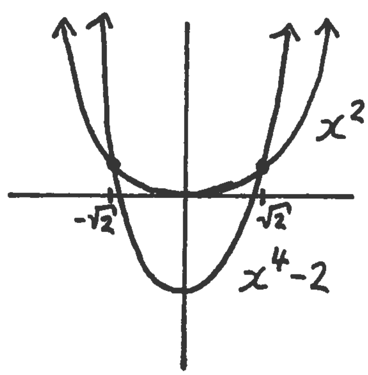
The only solution to the equation $\log_e(x) = -1$ is $x = e^{-1} = 1/e$. The only solution to the equation $\log_e(x) = 2$ is $x = e^2$. Both $1/e$ and e^2 are in the domain of our original equation, $(0, \infty)$, so they are both solutions of our original equation. That is, the set of solutions of the equation $\log_e(x)^2 = \log_e(x) + 2$ is $\{ 1/e \, , e^2 \}$.



• To determine the solutions of the equation $x^4 - 2 = x^2$, first write the equation in its equivalent form as $x^4 - x^2 - 2 = 0$. Second, notice that $x^4 = (x^2)^2$, so that we can write the equation as $(x^2)^2 - (x^2) - 2 = 0$. This is a quadratic equation in x^2, and the quadratic formula tells us that $x^2 = -1$ or $x^2 = 2$.

Now we have to find solutions for both $x^2 = -1$ and $x^2 = 2$. The former has no solutions, since squares can not be negative. The latter has solutions $x = -\sqrt{2}$ and $x = \sqrt{2}$. Thus, our original equation $x^4 - 2 = x^2$ has exactly two solutions, $-\sqrt{2}$ and $\sqrt{2}$.





Exercises

Find the solutions of the equations given in #1-8.

1.) $e^{x^2} = -1$

2.) $\log_e(x)^2 = -10$

3.) $e^{x^2-2} = 5$ where $x < 0$

4.) $\sqrt{\log_e(x)} = -4$

5.) $\log_e(4-3x) = -1$ where $x > 0$

6.) $e^{-3x+2} = 4$

7.) $x + \sqrt{x} + 2 = x$

8.) $(5x^3 - 4x + 1)^2 = -3$

For #9-15, find the set of solutions for the given equations. You can use the quadratic formula to help you with each of these problems.

9.) $x^2 = 4 \quad$ with $x < 0$

10.) $(e^x)^2 - 4 = 0$

11.) $\log_e(x) + 1 = -\log_e(x)^2$

12.) $x^4 - 2x^3 + x^2 = 2(x^2 - x) \quad$ (Hint: What's $(x^2-x)^2$?)

13.) $9 + (3^x)^2 = 6(3^x)$

14.) $x^4 + 3x^2 - 4 = 0$

15.) $\log_2(x)^2 + 4 = 5\log_2(x) \quad$ where $x > 10$

Let's look at the following piecewise defined function

$$f(x) = \begin{cases} x - 4 & \text{if } x \neq 2; \quad \text{and} \\ 279 & \text{if } x = 2. \end{cases}$$

The function $f(x)$ is comprised of two pieces. The first is $f(x) = x - 4$ as long as $x \neq 2$. The second is $f(x) = 279$ when $x = 2$.

For #16-21 match the number, x, on the left to piece of f on the right that you would use to determine $f(x)$.

16.) 3	A.) $f(x) = x - 4$ as long as $x \neq 2$
17.) -1	B.) $f(x) = 279$ when $x = 2$
18.) 0	
19.) 4	
20.) 2	
21.) 5	

Use the piecewise defined function $f(x)$ given above, and your answers from #16-21, to find the following values.

22.) $f(3)$

23.) $f(-1)$

24.) $f(0)$

25.) $f(4)$

26.) $f(2)$

27.) $f(5)$