Composing Polynomials with Planar Transformations

Let's compose the function $f(x) = x^3 - 2x + 9$ with the function $g(x) = 4 - x$.

$$
\begin{aligned}
f \circ g(x) &= f(g(x)) \\
&= g(x)^3 - 2g(x) + 9 \\
&= (4 - x)^3 - 2(4 - x) + 9
\end{aligned}
$$

To get the same answer in perhaps a slightly different way, first we write the formula for $f(x)$.

$$x^3 - 2x + 9$$

Second, we think of g as the function that replaces x with $4 - x$.

$$x \longmapsto 4 - x$$

Third, the formula for $f \circ g(x)$ can be obtained be rewriting the formula for $f(x)$, except that we'll replace every x with $4 - x$.

$$(4-x)^3 - 2(4-x) + 9$$

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Suppose $p : \mathbb{R}^2 \to \mathbb{R}$ is the polynomial $p(x, y) = 3x^2 - xy + 2$. Recall that $A_{(-3,4)} : \mathbb{R}^2 \to \mathbb{R}^2$ is the planar transformation given by the formula $A_{(-3,4)}(x, y) = (x - 3, y + 4)$.

The composition $p \circ A_{(-3,4)} : \mathbb{R}^2 \to \mathbb{R}$ will again be a polynomial in two variables. On this page, we'll find the formula for $p \circ A_{(-3,4)}(x, y)$.

First, write down the formula for $p(x, y)$.

$$3x^2 - xy + 2$$

Second, write down what $A_{(-3,4)}$ replaces each of the coordinates of the vector (x, y) with.

$$A_{(-3,4)}(x, y) = (x - 3, y + 4)$$

$$x \longmapsto x - 3$$

$$y \longmapsto y + 4$$

Third, the formula for $p \circ A_{(-3,4)}(x, y)$ is found by rewriting the formula for p, except that we'll replace each x with $x - 3$, and replace each y with $y + 4$.

$$3(x-3)^2 - (x-3)(y+4) + 2$$

In conclusion,

$$p \circ A_{(-3,4)}(x, y) = 3(x - 3)^2 - (x - 3)(y + 4) + 2$$

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Let's compose $q(x, y) = x + y - 1$ with the matrix

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

That is, we'll find $q \circ M(x, y)$.

First, write the formula for q.

$$x + y - 1$$

Second, write what M replaces each of the coordinates of the vector (x, y) with.

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+y \end{pmatrix}$$

$$x \longmapsto x+2y$$

$$y \longmapsto 3x+y$$

Third, the formula for $q \circ M(x, y)$ is found by rewriting the formula for q, except that we'll replace each x with $x + 2y$ and each y with $3x + y$.

$$(x+2y) + (3x+y) - 1$$

In conclusion

$$q \circ M(x, y) = (x + 2y) + (3x + y) - 1$$

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As we saw in the previous chapter, if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a planar transformation, and if S is the set of solutions of $p(x, y) = q(x, y)$, then $T(S)$ is the set of solutions of $p \circ T^{-1}(x, y) = q \circ T^{-1}(x, y)$. We called this (POTS), and we can describe it more economically as follows:

The equation for S composed with T^{-1}
is an equation for $T(S)$.

To compose an equation with T^{-1}, there are three steps to be followed.

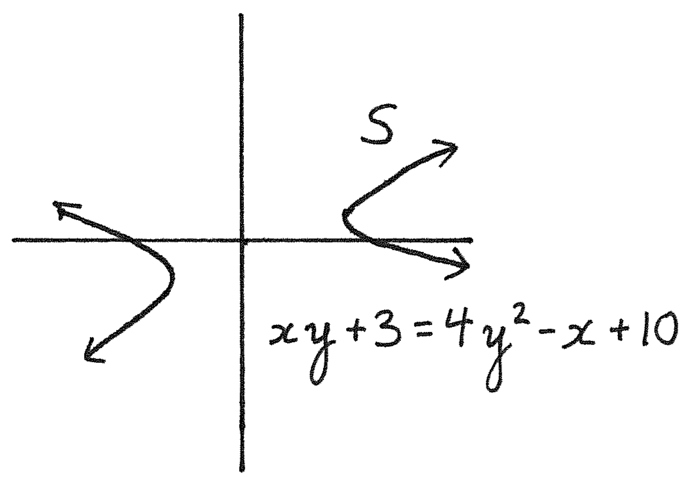
Step 1: Write the original equation.

Step 2: Find T^{-1} and write what T^{-1} replaces each of the coordinates of the vector (x, y) with.

Step 3: Rewrite the equation from Step 1, except replace every x and every y with the formulas identified in Step 2.

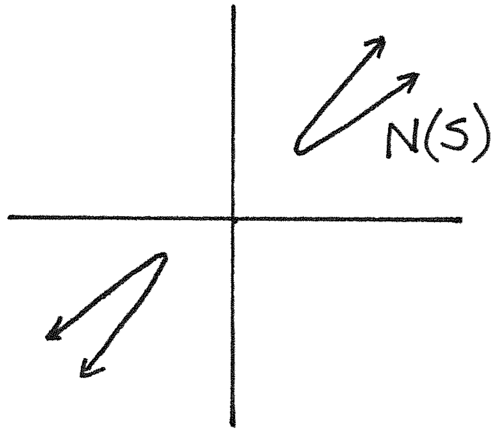
We'll practice (POTS) using these three steps in the next problem.

Problem: Suppose that S is the subset of the plane that is the set of solutions of the equation $xy + 3 = 4y^2 - x + 10$.



Let N be the invertible matrix $N = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Give an equation that

$N(S)$ is the set of solutions of.



Solution: To find an equation for $N(S)$, we have to compose the equation for S with N^{-1}.

First, write down the equation for S.

$$xy + 3 = 4y^2 - x + 10$$

Second, find N^{-1} and write what N^{-1} replaces each of the coordinates of the vector (x, y) with.

$$N^{-1} = \frac{1}{2(1)-1(1)}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ -x+2y \end{pmatrix}$$

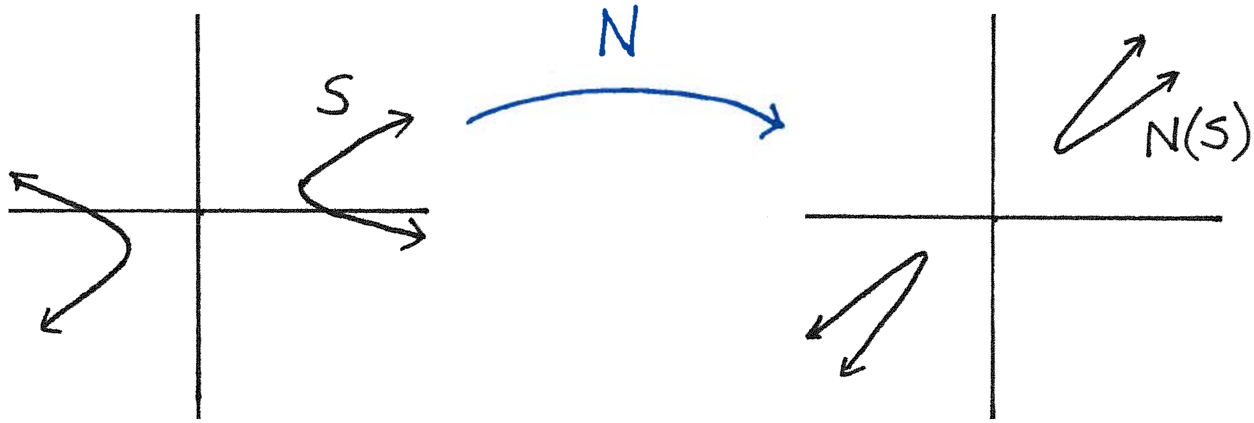
$$x \longmapsto x - y$$

$$y \longmapsto -x + 2y$$

Third, rewrite the equation for S, except replace each x with $x - y$ and each y with $-x + 2y$.

$$(x-y)(-x+2y)+3=4(-x+2y)^2-(x-y)+10$$

We can summarize what we've written above with the following diagram.



$$xy+3 = 4y^2-x+10 \xrightarrow[\substack{x \mapsto x-y \\ y \mapsto -x+2y}]{N^{-1}} (x-y)(-x+2y)+3 = 4(-x+2y)^2-(x-y)+10$$

S is the set of solutions of $xy + 3 = 4y^2 - x + 10$, so an equation for the set $N(S)$ is found by composing $xy + 3 = 4y^2 - x + 10$ with N^{-1}. That is, $N(S)$ is the set of solutions of $(x - y)(-x + 2y) + 3 = 4(-x + 2y)^2 - (x - y) + 10$.

Exercises

For #1-5, match the numbered planar transformation on the left with its lettered inverse on the right.

1.) $A_{(-6,1)}$

2.) $\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

3.) $A_{(4,-5)}$

4.) $\begin{pmatrix} \frac{1}{8} & 0 \\ 0 & 7 \end{pmatrix}$

5.) $\begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}$

A.) $A_{(-4,5)}$

B.) $\begin{pmatrix} 1 & -9 \\ 0 & 1 \end{pmatrix}$

C.) $\begin{pmatrix} 8 & 0 \\ 0 & \frac{1}{7} \end{pmatrix}$

D.) $A_{(6,-1)}$

E.) $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix}$

For #6-10, use your answers from #1-5 to find the given vectors.

6.) $A^{-1}_{(-6,1)}(x, y)$

7.) $\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$

8.) $A^{-1}_{(4,-5)}(x, y)$

9.) $\begin{pmatrix} \frac{1}{8} & 0 \\ 0 & 7 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$

10.) $\begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$

For #11-15, S is the subset of the plane that is the set of solutions of the equation

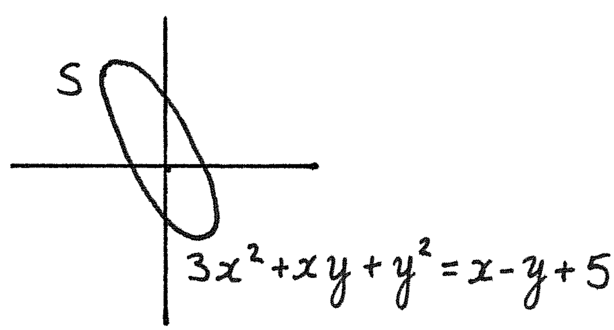
$$3xy = x^2 - y + 2$$

Match the numbered subset of the plane on the left with its lettered equation on the right. You'll be using your answers from #6-10 to answer these questions. You'll also be using (POTS), which says that an equation for $T(S)$ can be found by composing the equation for S with T^{-1}.

11.) $A_{(-6,1)}(S)$	A.) $3(8x)(\frac{y}{7}) = (8x)^2 - (\frac{y}{7}) + 2$
12.) $\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}(S)$	B.) $3(\frac{x}{3})(2y) = (\frac{x}{3})^2 - (2y) + 2$
13.) $A_{(4,-5)}(S)$	C.) $3(x-4)(y+5) = (x-4)^2 - (y+5) + 2$
14.) $\begin{pmatrix} \frac{1}{8} & 0 \\ 0 & 7 \end{pmatrix}(S)$	D.) $3(x-9y)y = (x-9y)^2 - y + 2$
15.) $\begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}(S)$	E.) $3(x+6)(y-1) = (x+6)^2 - (y-1) + 2$

For #16-19, suppose that S is the set of solutions of the equation

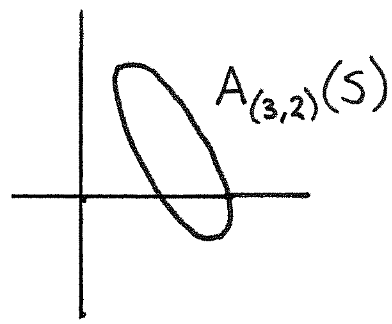
$$3x^2 + xy + y^2 = x - y + 5$$



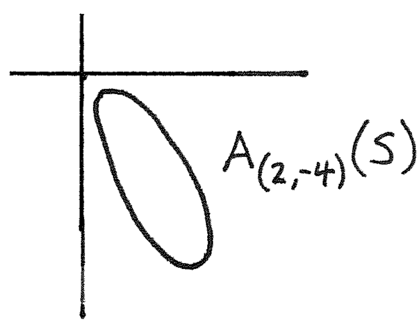
Use that an equation for $T(S)$ can be found by composing the equation for S with T^{-1} to find answers for #16-19. Write all polynomials in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

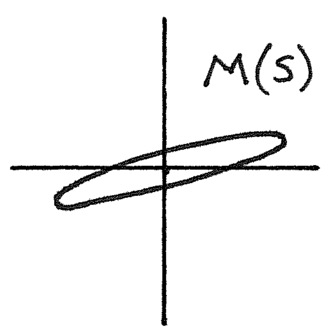
16.) What is $A_{(3,2)}^{-1}$? What is the equation for $A_{(3,2)}(S)$?



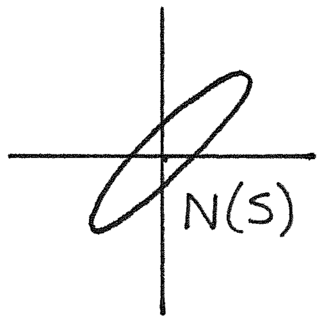
17.) What is $A_{(2,-4)}^{-1}$? What is the equation for $A_{(2,-4)}(S)$?



18.) Suppose $M = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$. What is M^{-1}? What is the equation for $M(S)$?



19.) Suppose $N = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$. What is N^{-1}? What is the equation for $N(S)$?



For #20-28, let

$$f(x) = \begin{cases} x-1 & \text{if } x \in (-\infty, 0); \\ x^2 & \text{if } x \in [0,4]; \quad \text{and} \\ 57 & \text{if } x \in (4, \infty). \end{cases}$$

Find the following values.

20.) $f(-2)$

21.) $f(-1)$

22.) $f(0)$

23.) $f(1)$

24.) $f(2)$

25.) $f(3)$

26.) $f(4)$

27.) $f(5)$

28.) $f(6)$

Multiply the following matrices.

29.) $\begin{pmatrix} 3 & 4 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 1 & 1 \end{pmatrix}$

30.) $\begin{pmatrix} 2 & -2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}$

Find the solutions of the following equations in one variable.

31.) $\log_e(x)^2 - 5\log_e(x) + 6 = 0$

32.) $(x^3 - 3x^2 + 2x - 3)^2 = -1$