Classification of Conics

A conic is the set of solutions of a quadratic equation in two variables, an equation of the form

$$a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0} = 0$$

where $a_{2,0}, a_{1,1}, a_{0,2}, a_{1,0}, a_{0,1}, a_{0,0} \in \mathbb{R}$, and $a_{2,0}$, $a_{1,1}$, $a_{0,2}$ don't all equal 0. (If $a_{2,0}$, $a_{1,1}$, $a_{0,2}$ all equal 0, then the equation can be written more simply as $a_{1,0}x + a_{0,1}y + a_{0,0} = 0$, and this is a linear or constant equation, not a quadratic equation.)

For example, the sets of solutions of the equations $x^2 - xy = 0$, or $2x^2 + 4xy + 7y^2 + 3x + 2y - 8 = 0$, or $x^2 + y^2 - 1 = 0$, or $x^2 + x - 3 = 0$ are conics.

We have seen several examples of conics, and we'll see in this chapter that we've essentially seen every possible type of conic. Precisely, we'll see that if you have a conic, you can rotate it, flip it over, shift it around in the plane, and scale the x- and y-axes, to transform the conic into one of 11 different examples of conics that we've seen before. Of those 11 basic examples of conics, there are only 8 different shapes:

- an ellipse
- a hyperbola
- a parabola
- two lines that intersect
- two lines that don't intersect
- a single line
- a single point
- no points at all

The three shapes on the left—ellipses, hyperbolas, and parabolas—are called *nondegenerate conics*. The remaining shapes are called *degenerate conics*.

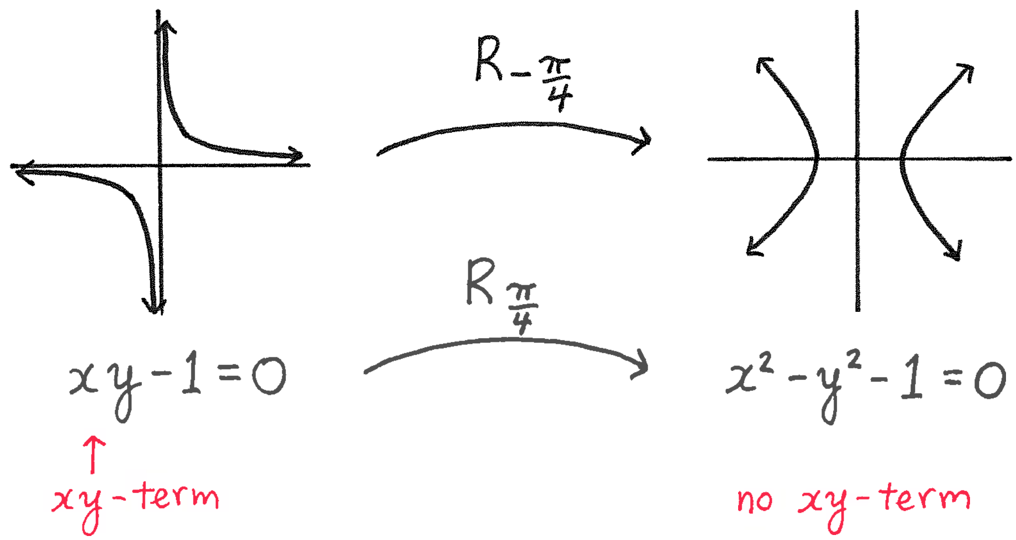
In the following pages, we'll outline nine steps (Steps A-I below) for using planar transformations to transform any conic into one of the 8 basic shapes listed above. We'll start with a mystery conic, the set of solutions of

$$a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0} = 0$$

where $a_{2,0}$, $a_{1,1}$, $a_{0,2}$ don't all equal 0, and at each step of the process we'll apply a planar transformation to obtain a conic whose equation is simpler.

A. Rotate to remove xy-term

We saw at the end of the Rotation Matrix chapter that if we rotate the hyperbola given by the equation $xy-\frac{1}{2}=0$ (or equivalently $xy=\frac{1}{2}$) clockwise by an angle of $\frac{\pi}{4}$ then the rotated hyperbola has the equation $x^2-y^2-1=0$ (equivalently, $x^2-y^2=1$).



Notice that there is no xy-term in the equation of the rotated conic, the equation $x^2-y^2-1=0$. There is only an x^2-term, a y^2-term, and a constant term. This is a specific example of a more general principle. Whenever we have a conic, we can rotate the conic so that the equation for the rotated conic does not have an xy-term.

Precisely, suppose we have a conic $S \subseteq \mathbb{R}^2$ that is the set of solutions of

$$a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0} = 0$$

where $a_{2,0}$, $a_{1,1}$, $a_{0,2}$ don't all equal 0. If $a_{2,0}=a_{0,2}$, then rotate S clockwise by an angle of $\frac{\pi}{4}$, and the rotated conic $R_{-\frac{\pi}{4}}(S)$ will have an equation that does not have an xy-term.

If $a_{2,0} \neq a_{0,2}$, then we let

$$\theta = -\frac{1}{2}\arctan\left(\frac{a_{1,1}}{a_{2,0}-a_{0,2}}\right)$$

The conic S rotated by an angle of θ, the conic $R_\theta(S)$, has an equation that has no xy-term. We know all of the techniques needed to check that this is true, and the interested reader is encouraged to check it, but we'll omit writing it here because it's a little long. (Too check it you'd need to use the double angle formulas.)

Form of the resulting equation after Step A: In summary, we can always rotate a conic to obtain a conic whose equation does not have an xy-term. After performing this rotation, we will be left with an equation of the form

- $a_{2,0}x^2 + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0} = 0$

B. Flip over to have an x^2-term

The equation that we have at the end of Step A must either have an x^2-term or a y^2-term, because it is a quadratic. If it has an x^2-term, for example if our equation is of the form

$$4x^2 + 5y^2 + 7x + 3y + 2 = 0,$$

then we're happy, and there's nothing to do in this step.

If our equation does not have an x^2-term, then it must have a y^2-term. For example, our conic S might have the equation

$$3y^2 + 2x + y + 5 = 0$$

In this case, we'll transform our conic S by flipping it over the $y = x$ line. The equation for the flipped conic is obtained by precomposing the equation for S with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is the matrix that interchanges the x- and y-coordinates. Thus, the equation for $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(S)$ is

$$3x^2 + 2y + x + 5 = 0$$

This technique applies to any equation of the form described at the end of Step A.

Form of the resulting equation after Step B: After perhaps flipping our conic from the end of Step A, we will be left with an equation that has an x^2-term. That is, we'll be left with an equation of the form

- $a_{2,0}x^2 + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0} = 0 \quad \text{where} \quad a_{2,0} \neq 0$

C. Shift horizontally to remove x-term

If we have a conic $S \subseteq \mathbb{R}^2$ that is the solution of an equation of the form given at the end of Step B, then we can shift S horizontally to remove the x-term. Precisely, if S is a conic whose equation has the form

$$a_{2,0}x^2 + a_{0,2}y^2 + a_{1,0}x + a_{0,1}y + a_{0,0} = 0 \quad \text{where} \quad a_{2,0} \neq 0$$

then the conic

$$A_{\left(\frac{a_{1,0}}{2a_{2,0}},0\right)}(S)$$

has no x-term.

For example, if S is the set of solutions of $2x^2 + y^2 + 8x + 2y + 7 = 0$, then $A_{(2,0)}(S)$ has an equation obtained by precomposing $2x^2+y^2+8x+2y+7 = 0$ with $A_{(2,0)}^{-1} = A_{(-2,0)}$. The result is the equation

$$2(x-2)^2 + y^2 + 8(x-2) + 2y + 7 = 0$$

which expands to

$$(2x^2 - 8x + 8) + y^2 + (8x - 16) + 2y + 7 = 0$$

and then simplifies to

$$2x^2 + y^2 + 2y - 1 = 0$$

which is an equation that does not have an x-term.

Form of the resulting equation after Step C: After shifting our mystery conic horizontally, we'll have a conic whose equation has no x-term. The equation will be of the form

- $a_{2,0}x^2 + a_{0,2}y^2 + a_{0,1}y + a_{0,0} = 0 \quad \text{where} \quad a_{2,0} \neq 0$

D. If there's a y^2-term, shift vertically to remove y-term

If after Step C our equation has a y^2-term, then we can shift our conic vertically using the planar transformation $A_{\left(0,\frac{a_{0,1}}{2a_{0,2}}\right)}$ to remove the y-term. This process is exactly as in Step C, except that it affects the y-coordinate instead of the x-coordinate.

Form of the resulting equation after Step D: Our equation from Step C might not have a y^2-term. If it doesn't, leave it alone. If it does, we can shift our conic vertically so that the resulting equation won't have a y-term. There are two possible forms that our equation might take at this point:

- $a_{2,0}x^2 + a_{0,1}y + a_{0,0} = 0$ where $a_{2,0} \neq 0$
- $a_{2,0}x^2 + a_{0,2}y^2 + a_{0,0} = 0$ where $a_{2,0} \neq 0$

E. Equivalent equation has constant term 0 or 1

Whichever of the two forms of equation that we have at the end of Step D, it might be the case that the constant term, $a_{0,0}$, equals 0. If it does, that's fine. If it does not equal 0, then we can divide the entire equation by $a_{0,0}$, to get an equivalent equation whose constant term equals 1. For example, if we have an equation of the form

$$8x^2 + 6y^2 + 2 = 0$$

then we can divide by 2 to obtain the equation

$$4x^2 + 3y^2 + 1 = 0$$

whose constant term is 1.

Form of the resulting equation after Step E: After this step, we have an equation of the form described at the end of Step D, except that now $a_{0,0}$ is either 0 or 1. There are four possible forms for an equation that fits this description:

- $a_{2,0}x^2 + a_{0,1}y = 0$ where $a_{2,0} \neq 0$
- $a_{2,0}x^2 + a_{0,1}y + 1 = 0$ where $a_{2,0} \neq 0$
- $a_{2,0}x^2 + a_{0,2}y^2 = 0$ where $a_{2,0} \neq 0$
- $a_{2,0}x^2 + a_{0,2}y^2 + 1 = 0$ where $a_{2,0} \neq 0$

F. Scale x-axis so that the x^2 coefficient is 1 or -1

If S is the conic for one of the four types of equations at the end of Step E, then we can scale the x-axis by $\sqrt{|a_{2,0}|}$ using the diagonal matrix $\begin{pmatrix} \sqrt{|a_{2,0}|} & 0 \\ 0 & 1 \end{pmatrix}$. The resulting conic, $\begin{pmatrix} \sqrt{|a_{2,0}|} & 0 \\ 0 & 1 \end{pmatrix}(S)$ will have a coefficient that is either 1 or -1. For example, if S is the conic for the equation

$-9x^2 + 7y + 1 = 0$, then $\sqrt{|-9|} = \sqrt{9} = 3$ and the equation for $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}(S)$ is obtained by precomposing the equation with $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$, which is the transformation that replaces x with $\frac{x}{3}$. Thus, the equation for $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}(S)$ is

$$-9\left(\frac{x}{3}\right)^2 + 7y + 1 = 0$$

which simplifies to

$$-x^2 + 7y + 1 = 0$$

This is an equation that has an x^2-coefficient of -1.

Form of the resulting equation after Step F: After scaling the x-axis, we have an equation of one of the four forms described at the end of Step E, except that we know that the x^2-coefficient is either 1 or -1. Rewriting the four options from Step E with this added information leaves us with the following possible options for our equation:

- $\pm x^2 + a_{0,1}y = 0$
- $\pm x^2 + a_{0,1}y + 1 = 0$
- $\pm x^2 + a_{0,2}y^2 = 0$
- $\pm x^2 + a_{0,2}y^2 + 1 = 0$

G. Scale y-axis so y^2 coefficient is 0, 1, or -1

In any of the four options of equations that are listed at the conclusion of the previous step, there might not be a y^2-term, and that's fine. If there is a y^2-term, then we can scale the y-axis by applying the transformation $\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{|a_{0,2}|} \end{pmatrix}$. Similar to the previous step, the result will be to make the y^2-coefficient equal to 1 or -1.

Form of the resulting equation after Step G: After scaling the y-axis, the y^2-coefficient in each of the four forms of equation from Step F will equal

0, 1, or -1. This change only affects the third and fourth form from Step F. The new list of possible equations is:

- $\pm x^2 + a_{0,1}y = 0$
- $\pm x^2 + a_{0,1}y + 1 = 0$
- $\pm x^2 = 0$
- $\pm x^2 \pm y^2 = 0$
- $\pm x^2 + 1 = 0$
- $\pm x^2 \pm y^2 + 1 = 0$

H. Scale y-axis so y coefficient is 0 or 1

The first two possibilities from the previous step might not have a y-term. That is, it might be that $a_{0,1} = 0$. If $a_{0,1} = 0$, that's fine. If on the other hand $a_{0,1} \neq 0$, then apply the matrix $\begin{pmatrix} 1 & 0 \\ 0 & a_{0,1} \end{pmatrix}$ to our conic. The resulting equation would have the variable y replaced with $\frac{y}{a_{0,1}}$, and this would leave us with an equation whose y-coefficient is 1.

Form of the resulting equation after Step H: After scaling the y-axis, we can make it so that any y-coefficient equals 1 if it doesn't already equal 0. Of the six forms of equation listed at the conclusion of Step G, this step only affects the first two. The new collection of possible equations for our conic is

- $\pm x^2 = 0$
- $\pm x^2 + y = 0$
- $\pm x^2 + 1 = 0$
- $\pm x^2 + y + 1 = 0$
- $\pm x^2 \pm y^2 = 0$
- $\pm x^2 \pm y^2 + 1 = 0$

I. If there's a y-term, shift vertically to remove constant

The fourth form of equation from the previous list, the form $\pm x^2+y+1=0$, isn't needed. If we had an equation of this form, we could shift our conic up by 1 using the addition function $A_{(0,1)}$. The affect on the equation of the conic would be to to replace y with $y-1$, and that would give us an equation of the form

$$\pm x^2+(y-1)+1=0$$

which simplifies to

$$\pm x^2+y=0$$

a form of equation that is already included in the list at the end of the previous step. It's the second form on the list.

Form of the resulting equation after Step I: After shifting vertically by 1, we can remove the fourth equation from the list of six possible equations at the end of Step H. We are left with the following five possible forms:

- $\pm x^2=0$
- $\pm x^2+y=0$
- $\pm x^2+1=0$
- $\pm x^2\pm y^2=0$
- $\pm x^2\pm y^2+1=0$

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List of possible conics resulting from Steps A-I

The first column of the chart on the next page is a list of the possible equations for a conic that can appear as a result of having applied the planar transformations described in Steps A-I. It's a list of the 14 possible equations that fit the 5 forms of equation described at the end of Step I.

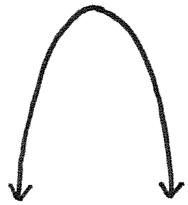
The second column of the chart gives an equivalent version of each equation, a version that we've seen before.

Three of the equations in the second column are listed twice. The third column lists the same equations as the second column, except that the three repeated equations are removed from the list. In the end, we have 11 possible equations for a conic after the conic has been rotated, scaled, shifted, and flipped as described in Steps A-I. We've seen each of these 11 equations and each of their conics before.

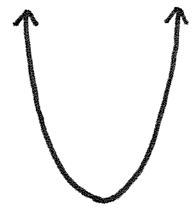
The page following the chart, page 332, shows the 11 equations along with pictures of their conics.

possible equations	equivalent equations	without repetition
$x^2 = 0$	$x^2 = 0$	$x^2 = 0$
$-x^2 = 0$	$x^2 = 0$	
$x^2 + y = 0$	$y = -x^2$	$y = -x^2$
$-x^2 + y = 0$	$y = x^2$	$y = x^2$
$x^2 + 1 = 0$	$x^2 = -1$	$x^2 = -1$
$-x^2 + 1 = 0$	$x^2 = 1$	$x^2 = 1$
$x^2 + y^2 = 0$	$x^2 + y^2 = 0$	$x^2 + y^2 = 0$
$x^2 - y^2 = 0$	$x^2 - y^2 = 0$	$x^2 - y^2 = 0$
$-x^2 + y^2 = 0$	$x^2 - y^2 = 0$	
$-x^2 - y^2 = 0$	$x^2 + y^2 = 0$	
$x^2 + y^2 + 1 = 0$	$x^2 + y^2 = -1$	$x^2 + y^2 = -1$
$x^2 - y^2 + 1 = 0$	$y^2 - x^2 = 1$	$y^2 - x^2 = 1$
$-x^2 + y^2 + 1 = 0$	$x^2 - y^2 = 1$	$x^2 - y^2 = 1$
$-x^2 - y^2 + 1 = 0$	$x^2 + y^2 = 1$	$x^2 + y^2 = 1$

$x^2 = 0$

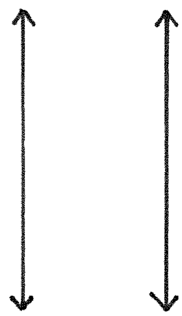


$y = -x^2$



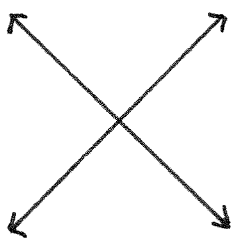
$y = x^2$

$x^2 = -1$



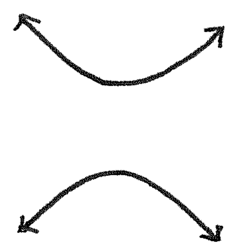
$x^2 = 1$

$x^2 + y^2 = 0$

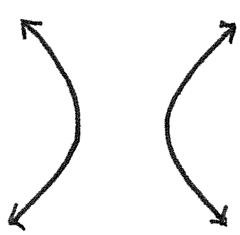


$x^2 - y^2 = 0$

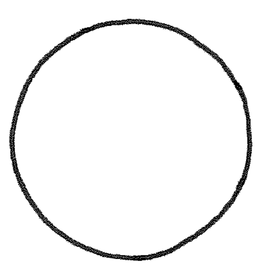
$x^2 + y^2 = -1$



$y^2 - x^2 = 1$



$x^2 - y^2 = 1$



$x^2 + y^2 = 1$

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Chapter Summary

A conic is the set of solutions in the plane of a quadratic polynomial in 2 variables. In this chapter we've seen that any conic, after being rotated, flipped, shifted, and scaled, can be transformed into on of the 11 basic examples of conics on the previous page. Of those 11 examples, 2 are parabolas, 2 are hyperbolas, and two are the emptyset. Thus, there are only 8 distinct shapes that appear as conics. They are:

- an ellipse
- a hyperbola
- a parabola
- two lines that intersect
- two lines that don't intersect
- a single line
- a single point
- no points at all

Of the eight possible shapes of conics, only the three listed in the left column are shapes that have some curviness to them. (Lines don't curve, they're straight, and a single point definitely doesn't curve.) These three shapes that curve—ellipses, hyperbolas, and parabolas—are called *nondegenerate conics*.

The remaining five conics from the right column above—two lines that do or don't intersect, a single line, a single point, or no points—are called *degenerate conics*.

Exercises

The exercises for this chapter are examples of the steps A-I used in this chapter to show that any conic can be rotated, flipped, shifted around, and scaled to be one of eight different shapes.

A.) Let S be the conic given by the equation $4x^2+2xy+4y^2+\sqrt{2}\,x+5\sqrt{2}\,y+4=0$. Find the equation for $R_{-\frac{\pi}{4}}(S)$. (The equation won't have an xy-term.)

B.) Let S be the conic given by the equation $2y^2+3x+4y+1=0$. Find the equation for the conic obtained by flipping S over the $y=x$ line, the conic $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(S)$. (The equation will have an x^2-term.)

C.) Let S be the conic given by the equation $x^2+3y^2+10x+4y+2=0$. Find the equation for $A_{(5,0)}(S)$. (The equation will not have an x-term.)

D.) Let S be the conic given by the equation $3x^2+2y^2+4y+5=0$. Find the equation for $A_{(0,1)}(S)$. (The equation will not have a y-term.)

E.) Let S be the conic given by the equation $4x^2+8y^2+2=0$. What's the equivalent equation obtained by dividing both sides of the equation by 2? (The equation will have 1 as its constant term.)

F.) Let S be the conic given by the equation $25x^2+7y^2+1=0$. Find the equation for $\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}(S)$. (The x^2-coefficient of the equation will equal 1.)

G.) Let S be the conic given by the equation $x^2-4y^2+1=0$. Find the equation for $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}(S)$. (The y^2-coefficient of the equation will equal -1.)

H.) Let S be the conic given by the equation $x^2-4y+1=0$. Find the equation for $\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}(S)$. (The y-coefficient of the equation will equal 1.)

I.) Let S be the conic given by the equation $x^2+y+1=0$. Find the equation for $A_{(0,1)}(S)$. (The equation will not have a constant term.)