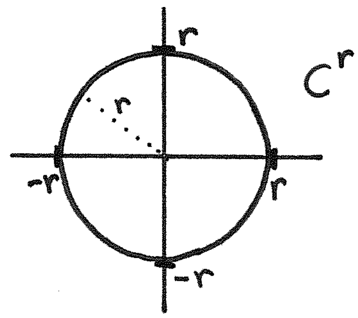
Circles and π

Let $r > 0$. We'll write C^r to denote the set of all vectors in the plane whose norm equals r. That is, C^r is the circle of radius r centered at the point $(0, 0)$.



Proposition (3). The circle C^r is a conic. It's the set of solutions of the quadratic equation

$$x^2 + y^2 = r^2$$

Proof: Recall that the definition of the norm of a vector is $||(x, y)|| = \sqrt{x^2 + y^2}$, which is the distance between (x, y) and $(0, 0)$.

Also notice that $x^2 + y^2$ is never negative, and that r is also not negative. Because the square-root function and the squaring function are inverse functions for nonnegative numbers, the equation $\sqrt{x^2 + y^2} = r$ is equivalent by invertible function to the equation $x^2 + y^2 = r^2$ (just square both sides of the former equation to obtain the latter equation). Therefore, $\sqrt{x^2 + y^2} = r$ and $x^2 + y^2 = r^2$ have the same set of solutions.

Putting everything together, we have

$$\begin{aligned} C^r &= \{\, (x, y) \in \mathbb{R}^2 \mid ||(x, y)|| = r \,\} \\ &= \{\, (x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} = r \,\} \\ &= \{\, (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2 \,\} \end{aligned}$$

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Example.

- The circle of radius 3 centered at $(0, 0)$ is the set of solutions of the quadratic equation $x^2 + y^2 = 9$.

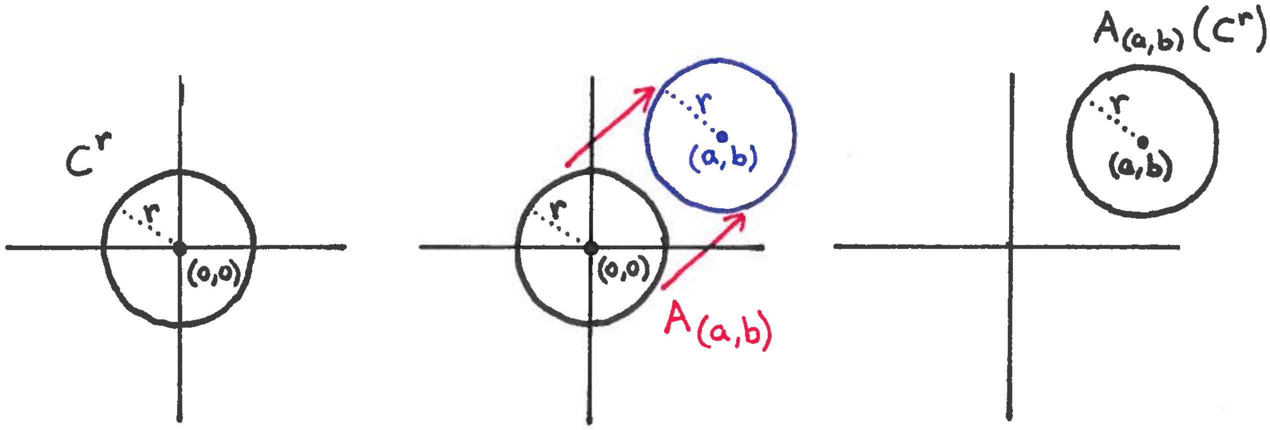
Circles centered at any point

We can use Proposition 3 to describe the equation of any circle in the plane.

Corollary (4). The circle of radius r centered at the point $(a, b) \in \mathbb{R}^2$ is a conic. It's the set of solutions of the equation

$$(x - a)^2 + (y - b)^2 = r^2$$

Proof: C^r is the circle of radius r centered at $(0, 0)$. The planar transformation $A_{(a,b)} : \mathbb{R}^2 \to \mathbb{R}^2$ move points horizontally by a and vertically by b. Thus, $A_{(a,b)}(C^r)$ is the circle of radius r centered at the point (a, b).



Using POTS, the equation for $A_{(a,b)}(C^r)$ is given by precomposing the equation for C^r, $x^2 + y^2 = r^2$, with the function $A_{(a,b)}^{-1} = A_{(-a,-b)}$ – the function that replaces x with $x - a$ and y with $y - b$. The resulting equation is

$$(x - a)^2 + (y - b)^2 = r^2$$

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Example.

- The circle of radius 5 centered at $(-2, 4)$ is the set of solutions of the quadratic equation $(x + 2)^2 + (y - 4)^2 = 25$.

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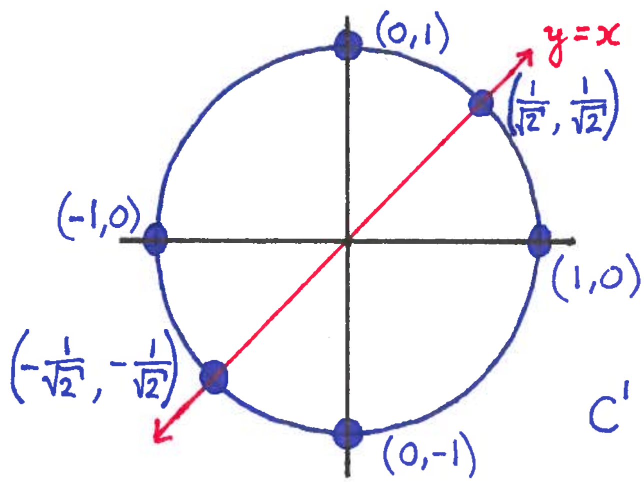
The Unit circle

C^1 is called the *unit circle*. It's the circle of radius 1 centered at the point $(0, 0)$. Half of the unit circle is above the x-axis, and half is below the x-axis.

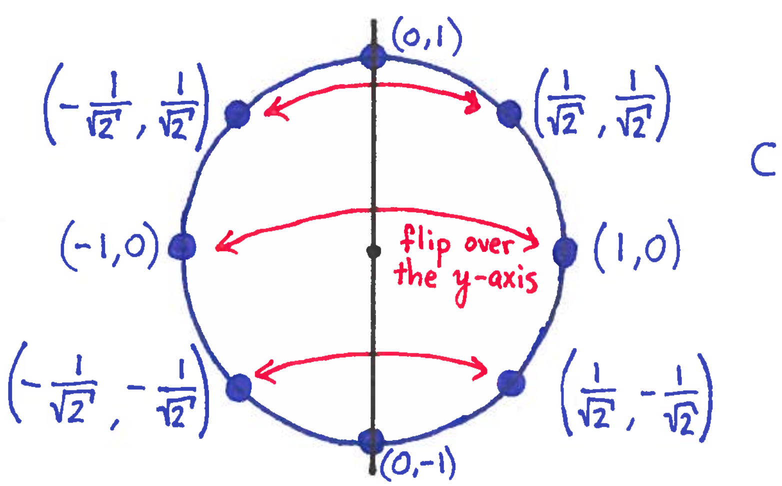
Some of the points in the unit circle include $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$. These are vectors of norm 1. The point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is also a vector of norm 1

$$\left|\left|\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right|\right| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$$

so $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is also on the unit circle C^1. It's one of two points that are on the unit circle and on the line $y = x$. The other is $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.



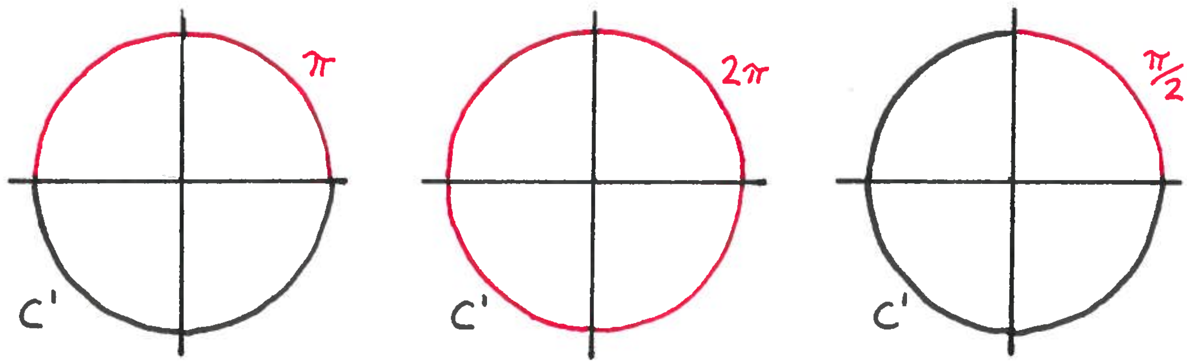
We can flip the circle over the y-axis using the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. This is the matrix that assigns the vector (x,y) to the vector $(-x,y)$, which shows us two more points on the circle: $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.



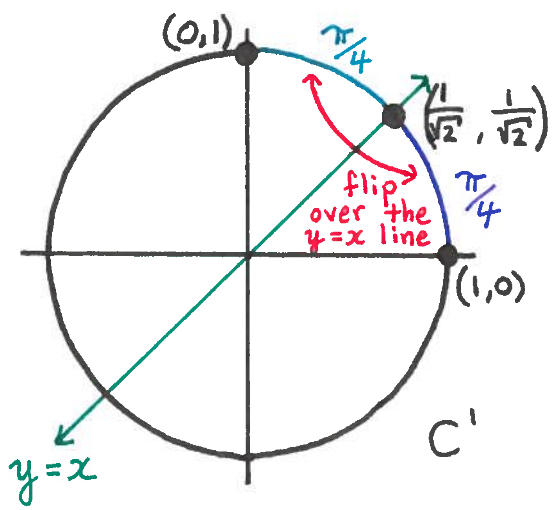
The Definition of π

π is the length of the top half of the unit circle.

Because π is the length of half of the circle, the length around the entire circle – called the *circumference* of the circle – equals 2π. The length of the portion of the circle between the points $(1, 0)$ and $(0, 1)$ is one quarter of the circumference of the circle, so it has length $\frac{1}{4}(2\pi) = \frac{\pi}{2}$.

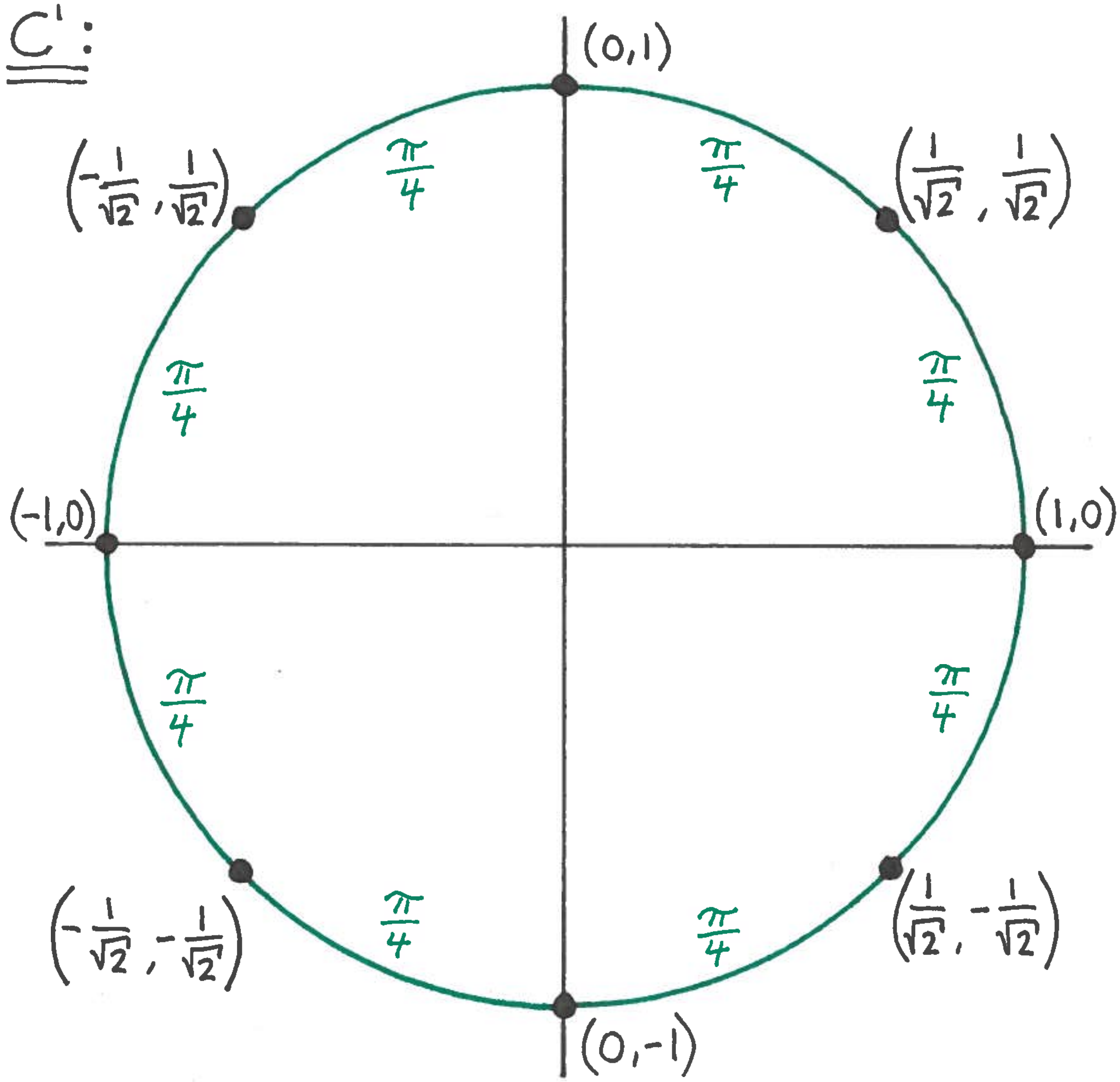


The flip over the $y = x$ line interchanges the portions of the circle between $(1, 0)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and between $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $(0, 1)$. Thus, these two portions have the same length, namely half of the total length between $(1, 0)$ and $(0, 1)$, or $\frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$.



The Circle divided into eight equal segments

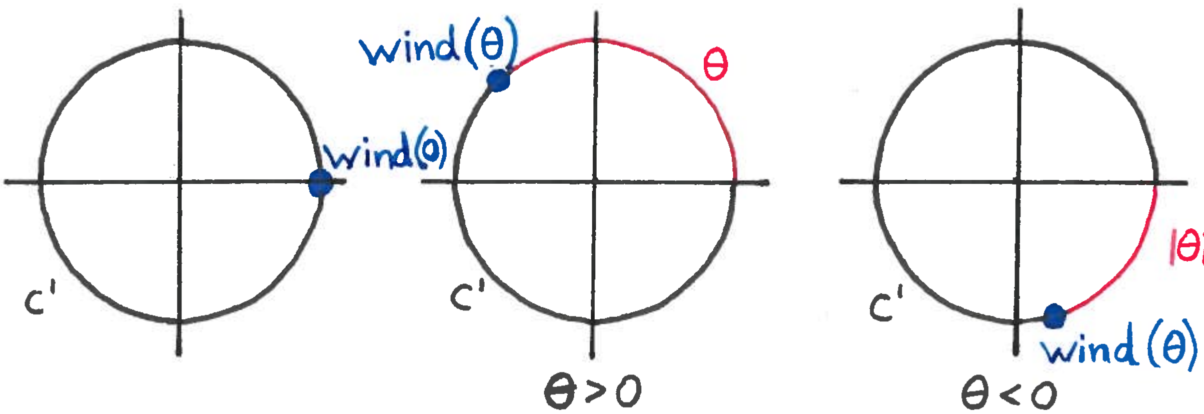
The picture below shows the unit circle divided into eight segments, each of length $\frac{\pi}{4}$.



It will probably help to refer back to this page when answering #11-18 in the exercises for this chapter.

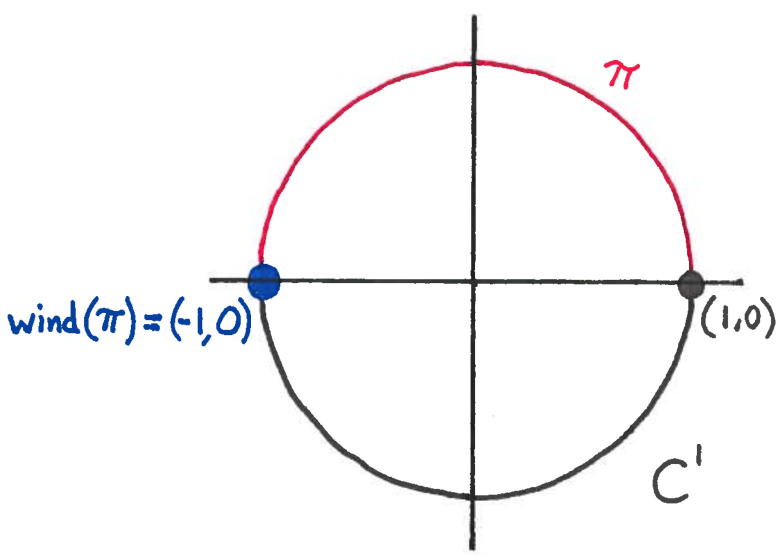
The Winding function

We define the function wind : $\mathbb{R} \to C^1$ as follows: We let wind$(0) = (1, 0)$. If $\theta > 0$, then wind(θ) is the point on C^1 obtained by beginning at the point $(1, 0)$ and traveling counterclockwise around the unit circle a length of θ. If $\theta < 0$, then wind(θ) is the point obtained by beginning at $(1, 0)$ and traveling clockwise around C^1 a length of $|\theta|$.

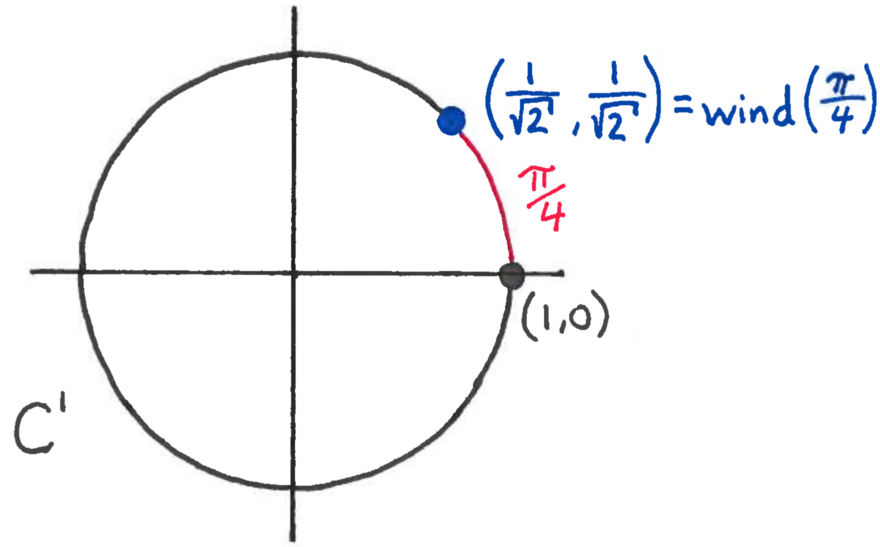


Examples.

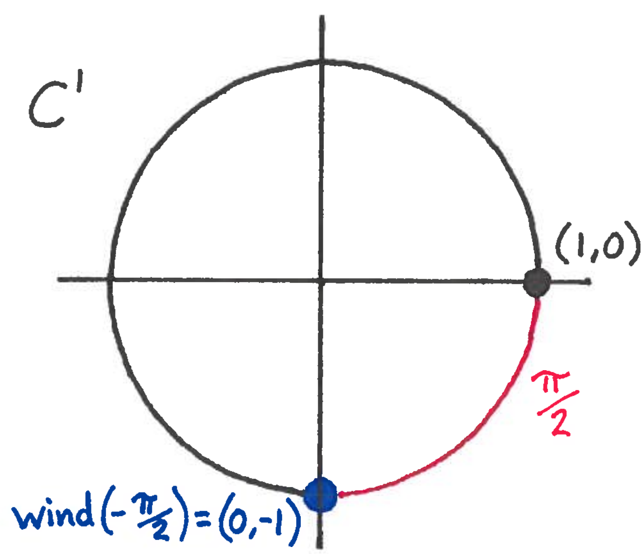
• wind(π) is the point on C^1 obtained by beginning at the point $(1, 0)$ and traveling counterclockwise around the unit circle a length of π. That would take us exactly half-way around the circle to the point $(-1, 0)$. Therefore, wind$(\pi) = (-1, 0)$.



• wind$\left(\frac{\pi}{4}\right)$ is the point on the unit circle obtained by traveling a length of $\frac{\pi}{4}$ from the point $(1,0)$ counterclockwise around the unit circle. As discussed previously, that point is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. That is, wind$\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.



• Because $-\frac{\pi}{2}$ is negative, wind$\left(-\frac{\pi}{2}\right)$ is the point arrived at by traveling a length of $\frac{\pi}{2} = |-\frac{\pi}{2}|$ *clockwise* around the circle from the point $(1,0)$. Thus, wind$\left(-\frac{\pi}{2}\right) = \left(0,-1\right)$.



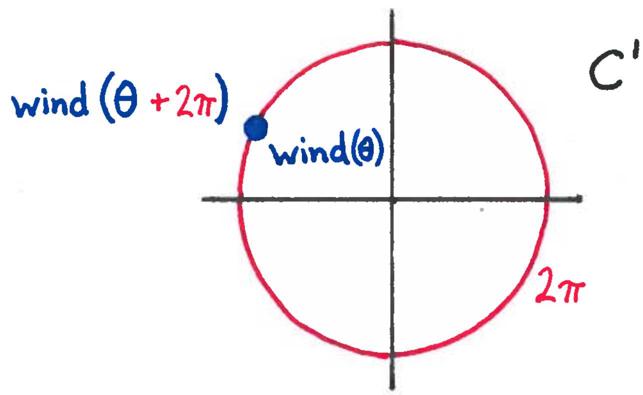
Period of the winding function

Notice that $\text{wind}(2\pi) = (1, 0)$. That is, if you start at the point $(1, 0)$, and travel around the full circumference of the circle, you'll be back where you started, at the point $(0, 1)$.

More generally, if $\theta \in \mathbb{R}$, then $\text{wind}(\theta)$ is a point on the circle, and once at the point $\text{wind}(\theta)$, we could travel the entire circumference of the circle again, a length of 2π, and we'd be back at the point $\text{wind}(\theta)$. That is,

$$\text{wind}(\theta + 2\pi) = \text{wind}(\theta)$$

Functions that behave like this are said to have a *period* of 2π.



Notice that $\text{wind}(\theta + 4\pi)$ is the point obtained by starting at $\text{wind}(\theta)$ and then traveling around the circle twice, a length of $2(2\pi) = 4\pi$. That would put us back at the point $\text{wind}(\theta)$ again, so that $\text{wind}(\theta+4\pi) = \text{wind}(\theta)$. Similarly, $\text{wind}(\theta + 6\pi) = \text{wind}(\theta)$ because $6\pi = 3(2\pi)$, and traveling counterclockwise around the circle three times brings you back to where you started; and $\text{wind}(\theta - 2\pi) = \text{wind}(\theta)$ because traveling the length of the circle clockwise also brings you back to where you started.

Exercises

For #1-6, use Corollary (4) to write an equation for the circle with given radius and center. The equation should have the form

$$(x \pm a)^2 + (y \pm b)^2 = c$$

1.) center: $(0,0)$; radius: 4

2.) center: $(2,3)$; radius: 1

3.) center: $(-5,7)$; radius: $\frac{1}{2}$

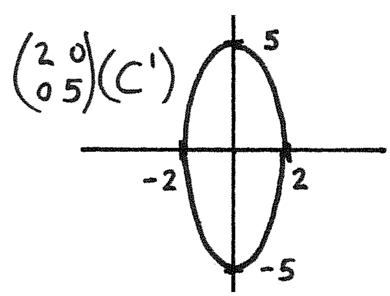
4.) center: $(0,0)$; radius: 3

5.) center: $(4,-2)$; radius: $\frac{1}{3}$

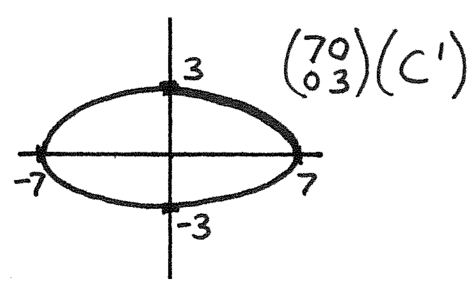
6.) center: $(-5,-7)$; radius: 5

Let C^1 be the unit circle, the set of solutions of the equation $x^2 + y^2 = 1$. For #7-10, use POTS to give an equation for the following subsets of the plane. Write them in the form $ax^2 + by^2 = 1$.

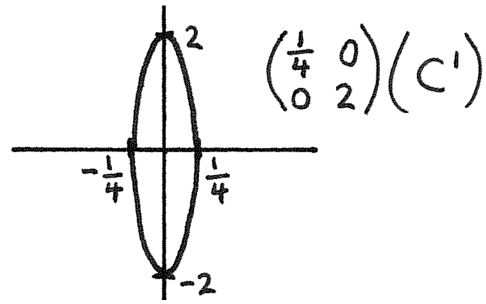
7.) $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}(C^1)$, the unit circle scaled by 2 in the x-coordinate and by 5 in the y-coordinate.



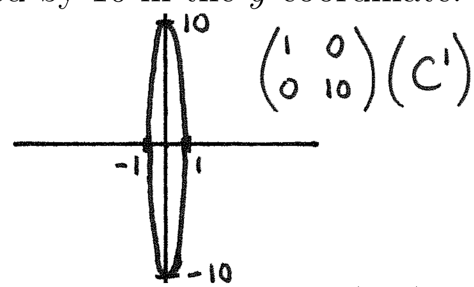
8.) $\begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}(C^1)$, the unit circle scaled by 7 in the x-coordinate and by 3 in the y-coordinate.



9.) $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 2 \end{pmatrix}(C^1)$, the unit circle scaled by $\frac{1}{4}$ in the x-coordinate and by 2 in the y-coordinate.



10.) $\begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}(C^1)$, the unit circle scaled by 10 in the y-coordinate.



For #11-18, write the given point in the plane in the form (a, b) for some numbers a and b. Page 190 will help with these questions.

11.) $\text{wind}(8\pi)$

12.) $\text{wind}\left(\frac{7\pi}{4}\right)$

13.) $\text{wind}\left(-\frac{7\pi}{4}\right)$

14.) $\text{wind}(-\pi)$

15.) $\text{wind}\left(\frac{37\pi}{4}\right)$

16.) $\text{wind}\left(-\frac{13\pi}{2}\right)$

17.) $\text{wind}\left(\frac{125\pi}{4}\right)$

18.) $\text{wind}(-4\pi)$

For #19-22, find the set of solutions of the given equations.

19.) $(3x+4)(2x-1) = x(6x-2)$

20.) $(2\log_e(x) - 3)^2 = 100$

21.) $\sqrt{e^{x-4}+1} = 5$

22.) $\dfrac{\frac{1}{x^2}}{\frac{4}{x}} = 3$