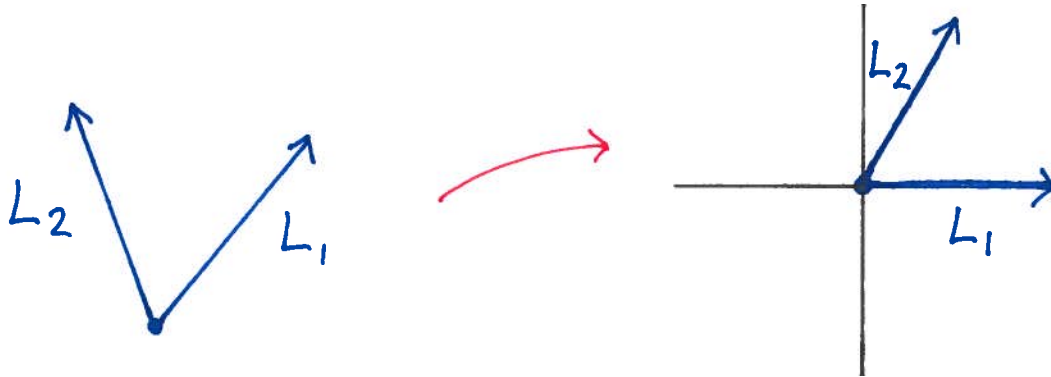


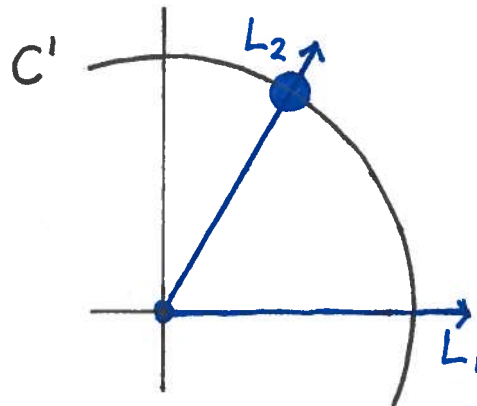
Angles

If two rays, L_1 and L_2 , emanate from the same point in the plane, then their *angle* is a measurement of how close the two lines are to each other.

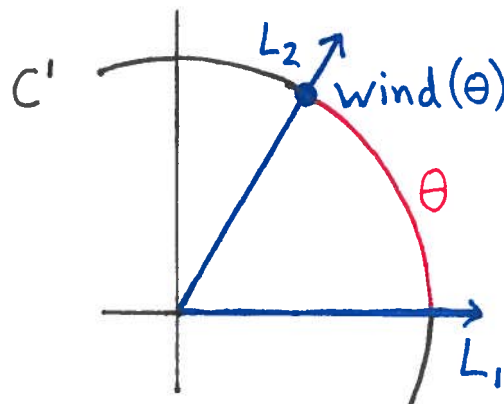
To find the angle between L_1 and L_2 , move them to the point $(0,0)$ in the plane, and so that the ray L_1 matches up with the positive half of the x -axis.



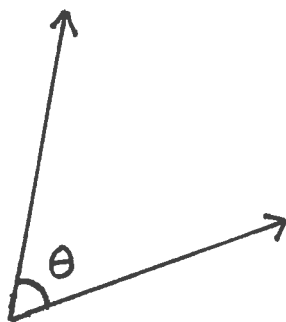
After being moved to the point $(0,0)$, the ray L_2 will intersect the unit circle C^1 in exactly one point.



The *angle* between L_1 and L_2 is the number $\theta \in [0, 2\pi)$ with the property that $\text{wind}(\theta)$ is the point where L_2 intersects the unit circle, C^1 .

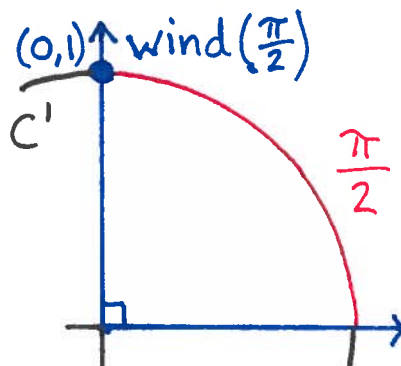


If the angle between two rays equals θ , then we usually denote this by labeling the rays as shown below:

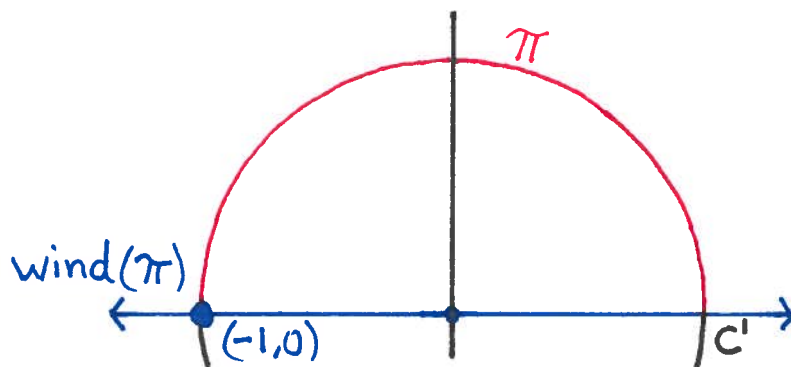


Examples.

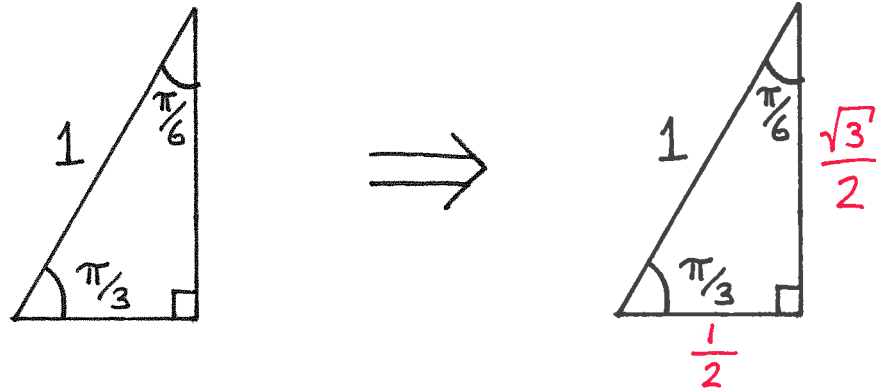
- A *right angle* is an angle resulting from rays that intersect perpendicularly, as the x - and y -axes do. The angle between the positive half of the x -axis and the positive half of the y -axis is $\frac{\pi}{2}$, because the positive half of the y -axis intersects the unit circle at the point $(0, 1)$, and $(0, 1) = \text{wind}(\frac{\pi}{2})$. To summarize, right angles are angles of $\frac{\pi}{2}$.



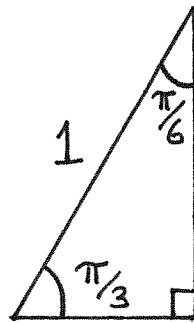
- The positive half of the x -axis and the negative half of the x -axis are two rays that each emanate from the point $(0, 0)$. The angle between the positive ray and the negative ray is π because the negative ray intersects the unit circle in the point $\text{wind}(\pi) = (-1, 0)$.



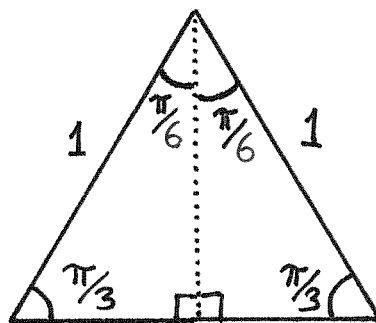
Lemma (5). Suppose a right triangle has angles $\frac{\pi}{6}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. If the length of the side opposite the angle $\frac{\pi}{2}$ has length 1, then the length of the sides opposite the angles $\frac{\pi}{6}$ and $\frac{\pi}{3}$ are $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$, respectively.



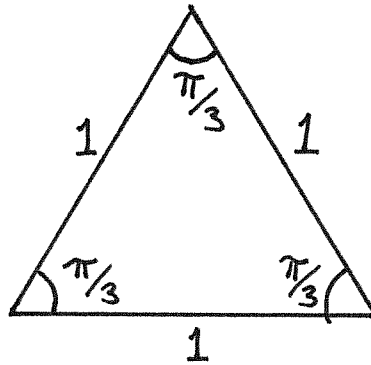
Proof: We begin with the triangle



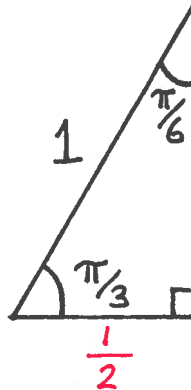
Now let's draw a second copy of the same triangle, flipped over the vertical side.



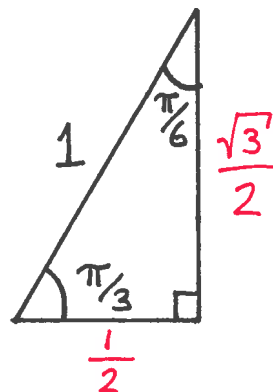
This new triangle is twice as big as the one we started with, and all of its angles equal $\frac{\pi}{3}$. It's what's called a *regular triangle*. All three sides of a regular triangle have the same length, so all three sides of this particular regular triangle have length 1.



The bottom side of our original triangle is half of the bottom side of the larger triangle, so it has length $\frac{1}{2}$.

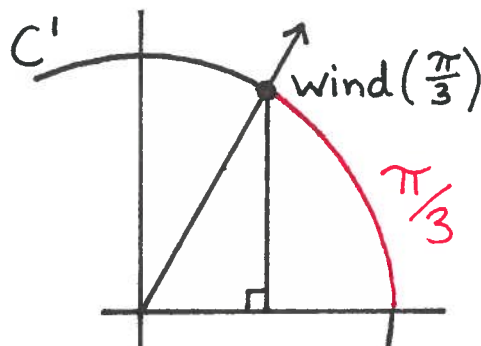


Let's call the length of the third side of our original triangle a . What we have left to do is find what a equals, and for this we can use the Pythagorean Theorem which says that $a^2 + (\frac{1}{2})^2 = 1^2$, which simplifies to $a^2 + \frac{1}{4} = 1$. Subtracting $\frac{1}{4}$ gives us $a^2 = \frac{3}{4}$, and since a is a positive number, we can take the square root of the equation $a^2 = \frac{3}{4}$ to find that $a = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$.

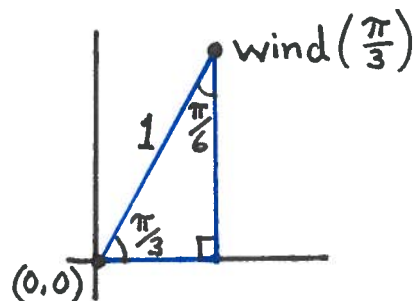


Lemma (6). $\text{wind}(\frac{\pi}{3}) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

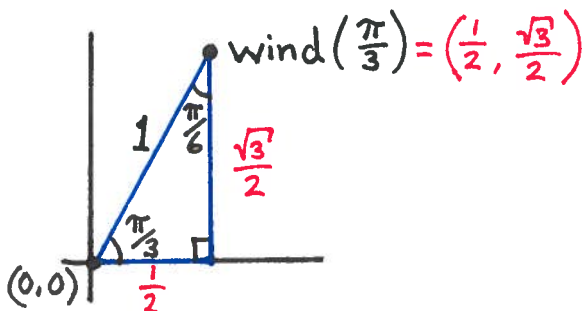
Proof: Let's draw a ray from $(0,0)$ through the point $\text{wind}(\frac{\pi}{3})$ and a vertical line segment from $\text{wind}(\frac{\pi}{3})$ to the x -axis that creates a right angle with the x -axis.



The ray, the vertical line segment, and the x -axis form a right triangle. The distance between the points $(0,0)$ and $\text{wind}(\frac{\pi}{3})$ equals 1, because $\text{wind}(\frac{\pi}{3})$ is a point on the unit circle. The angle at $(0,0)$ is $\frac{\pi}{3}$, the right angle is $\frac{\pi}{2}$, and the third angle must then equal $\frac{\pi}{6}$, because $\frac{\pi}{3} + \frac{\pi}{2} + \frac{\pi}{6} = \pi$, and the sum of the three angles of a triangle must always equal π .

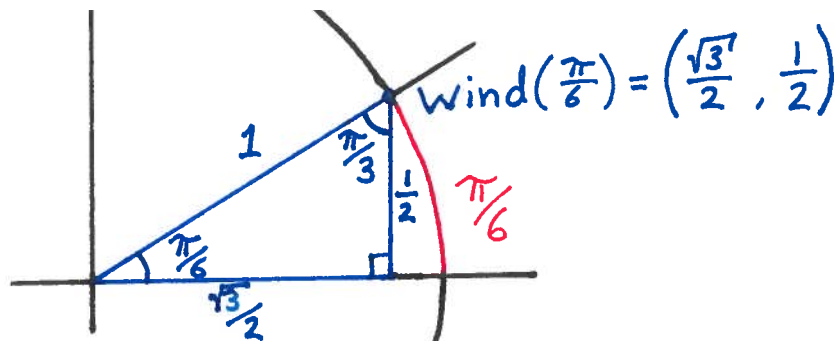


Now Lemma (5) tells us that the length of the other two sides of the triangle are $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$.

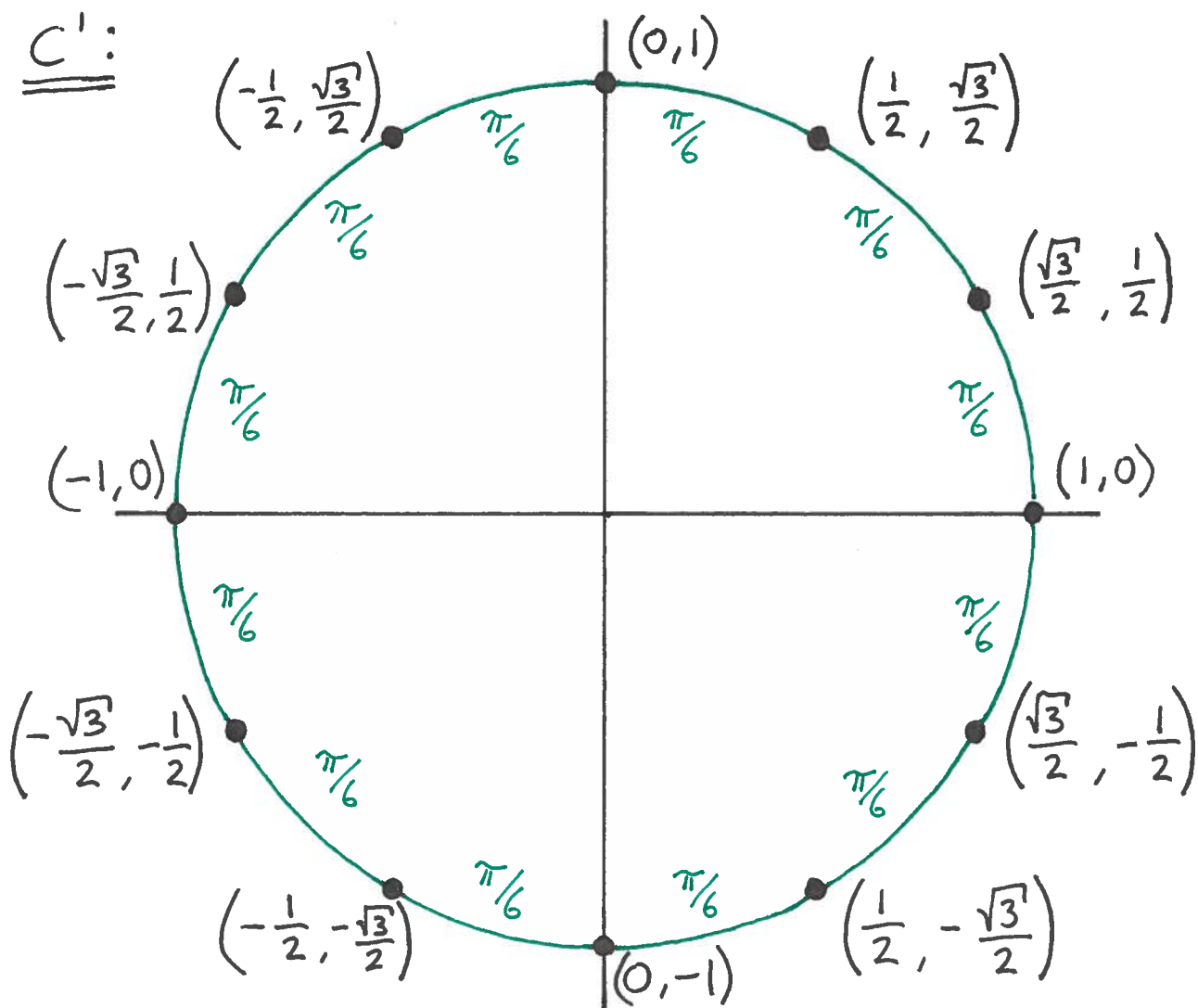


That means that the x -coordinate of $\text{wind}(\frac{\pi}{3})$ equals $\frac{1}{2}$ and the y -coordinate equals $\frac{\sqrt{3}}{2}$. In other words, $\text{wind}(\frac{\pi}{3}) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ which is what we wanted to see. ■

We can use a similar argument to the one from Lemma (6) to show that $\text{wind}(\frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$.

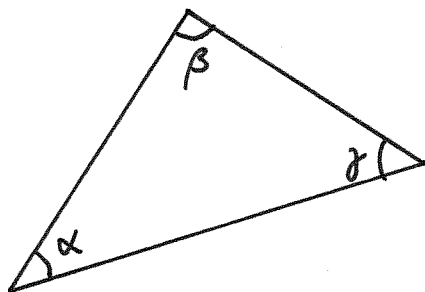


The unit circle can be divided into 12 segments each of length $\frac{\pi}{6}$ as shown below.



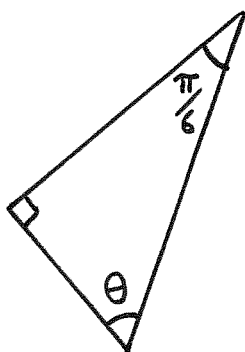
Exercises

The sum of the angles of a triangle equal π . That is, if $\alpha, \beta, \gamma \in \mathbb{R}$ are the three angles of a triangle, then $\alpha + \beta + \gamma = \pi$.

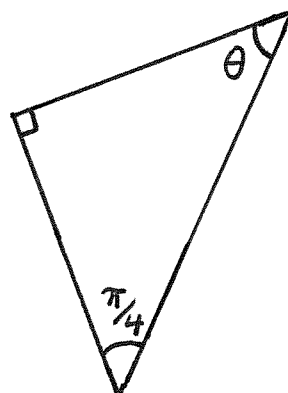


For #1-4, find the unknown angle θ .

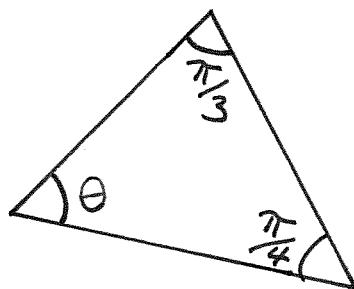
1.)



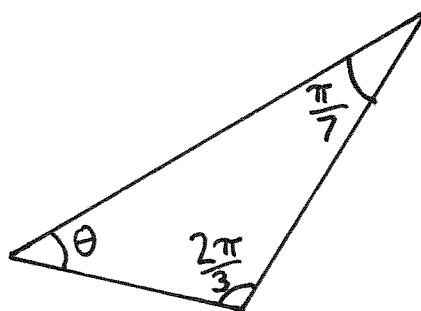
3.)



2.)



4.)



For #5-12, write the given point in the plane in the form (a, b) for some numbers a and b . The circle on page 201 will help with these questions.

5.) $\text{wind}\left(\frac{5\pi}{6}\right)$

9.) $\text{wind}(0)$

6.) $\text{wind}\left(\frac{7\pi}{6}\right)$

10.) $\text{wind}\left(\frac{\pi}{2}\right)$

7.) $\text{wind}\left(-\frac{\pi}{6}\right)$

11.) $\text{wind}\left(\frac{103\pi}{3}\right)$

8.) $\text{wind}\left(-\frac{\pi}{3}\right)$

12.) $\text{wind}\left(\frac{13\pi}{6}\right)$

For #13-16, find the set of solutions of the given equations.

13.) $\log_5(2+x)^2 = 4$

14.) $5^{3x+2} = 7$

15.) $\frac{1}{x} + x = 3$

16.) $(x+1)(x+2) = 1$