Practice Second Midterm Exam

Conics

For #1-12, match the numbered quadratic equations in two variables with their lettered sets of solutions. Worth $\frac{1}{2}$ point each.

1.) $y = x^2$ F

2.) $x^2 - y^2 = 0$ D

3.) $x^2 = 0$ K

4.) $xy = 1$ A

5.) $x^2 + y^2 = 0$ I

6.) $x^2 + y^2 = -1$ H

7.) $x^2 = -1$ H

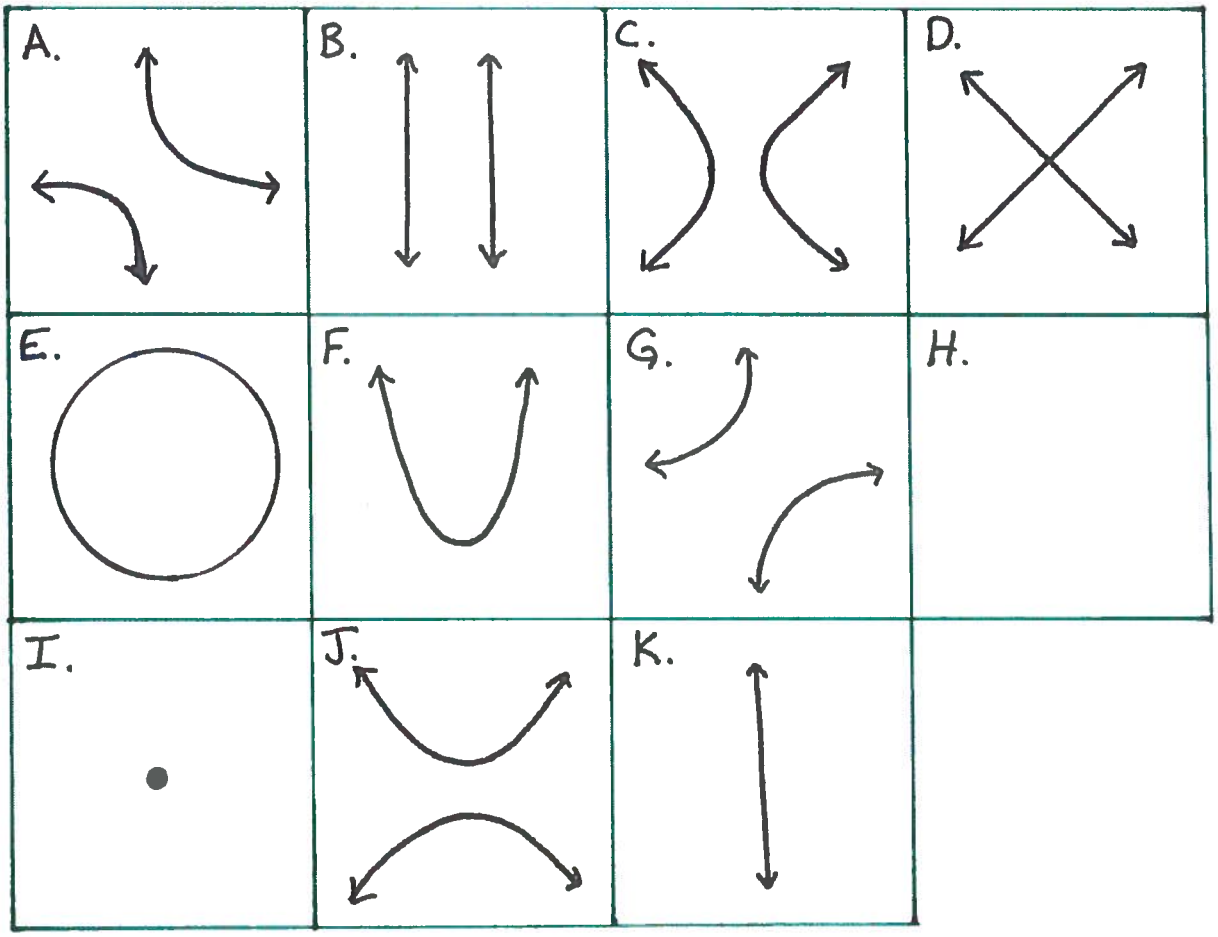
8.) $x^2 = 1$ B

9.) $x^2 - y^2 = 1$ C

10.) $y^2 - x^2 = 1$ J

11.) $xy = -1$ G

12.) $x^2 + y^2 = 1$ E



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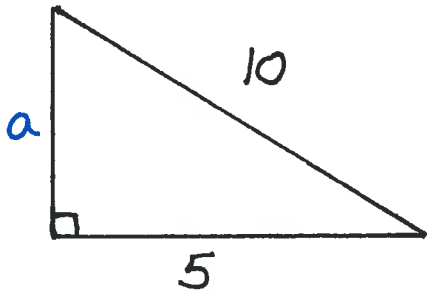
Trigonometry

13.) What is the distance between the points $(4, -1)$ and $(-3, 5)$?

$$\sqrt{(4-(-3))^2 + (-1-5)^2} = \sqrt{7^2 + (-6)^2} = \sqrt{49+36} = \sqrt{85}$$

14.) Find the length of the unlabled side of the triangle below.



Pythagorean Theorem:

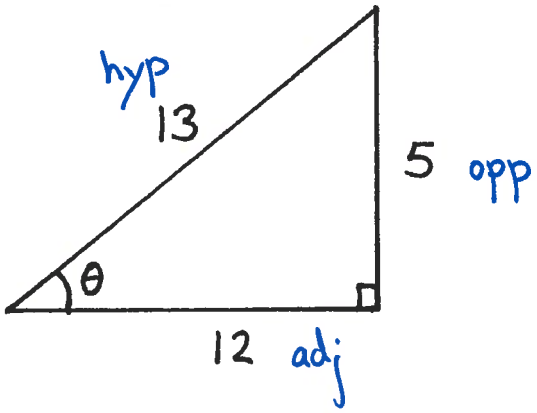
$$10^2 = a^2 + 5^2$$

$$100 = a^2 + 25$$

$$75 = a^2$$

$$a = \sqrt{75} \quad \text{since } a > 0$$

15.) Find $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for the angle θ given below. (3 points.)



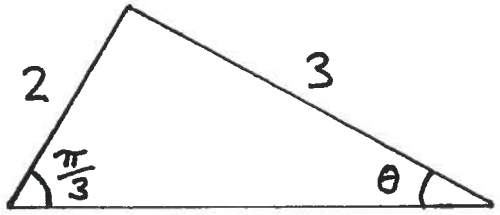
soh-cah-toa:

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{5}{13}$$

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{12}{13}$$

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{5}{12}$$

16.) Find $\sin(\theta)$ for the angle θ given below.

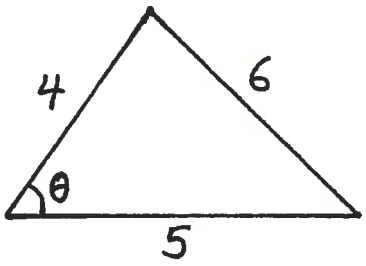


law of sines:

$$\frac{\sin(\theta)}{2} = \frac{\sin(\pi/3)}{3}$$

$$\sin(\theta) = \frac{2\sin(\pi/3)}{3} = \frac{2\,\frac{\sqrt{3}}{2}}{3} = \frac{\sqrt{3}}{3}$$

17.) Find $\cos(\theta)$ for the angle θ given below.



law of cosines:

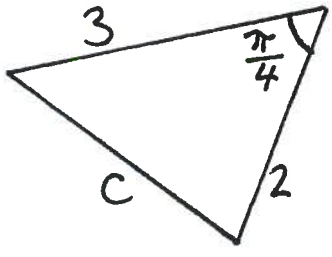
$$6^2 = 4^2 + 5^2 - 2(4)(5)\cos(\theta)$$

$$36 = 16 + 25 - 40\cos(\theta)$$

$$-5 = -40\cos(\theta)$$

$$\cos(\theta) = \frac{-5}{-40} = \frac{1}{8}$$

18.) Find the length c shown below.



law of cosines:

$$c^2 = 2^2 + 3^2 - 2(2)(3)\cos\left(\frac{\pi}{4}\right)$$

$$c^2 = 4 + 9 - 12\cos\left(\frac{\pi}{4}\right)$$

$$c^2 = 4 + 9 - 12\left(\frac{1}{\sqrt{2}}\right)$$

$$c^2 = 13 - \frac{12}{\sqrt{2}}$$

$$c = \sqrt{13 - \frac{12}{\sqrt{2}}} \quad \text{since } c > 0$$

19.) Write the vector $(-2,5)$ in polar coordinates.

$$\|(-2,5)\| = \sqrt{(-2)^2+5^2} = \sqrt{4+25} = \sqrt{29}$$

$$(-2,5) = \sqrt{29}\left(\frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$$

20.) Rotate the point $2(\cos(4), \sin(4))$ counterclockwise by an angle of 5.

$$2\left(\cos(4+5), \sin(4+5)\right) = 2\left(\cos(9), \sin(9)\right)$$

21.) Write the matrix that rotates the plane clockwise by an angle of $\frac{2\pi}{3}$. Simplify your answer so that it does not contain the letters sin or cos.

Clockwise means negative:

$$R_{-\frac{2\pi}{3}} = \begin{pmatrix} \cos\left(-\frac{2\pi}{3}\right) & -\sin\left(-\frac{2\pi}{3}\right) \\ \sin\left(-\frac{2\pi}{3}\right) & \cos\left(-\frac{2\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

22.) Use your answer from #21 to rotate the vector $(2,4)$ clockwise by an angle of $\frac{2\pi}{3}$. Write your answer as a row vector.

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{2}{2} + \frac{4\sqrt{3}}{2} \\ -\frac{2\sqrt{3}}{2} - \frac{4}{2} \end{pmatrix} = \begin{pmatrix} -1+2\sqrt{3} \\ -\sqrt{3}-2 \end{pmatrix} = \left(-1+2\sqrt{3}, -\sqrt{3}-2\right)$$

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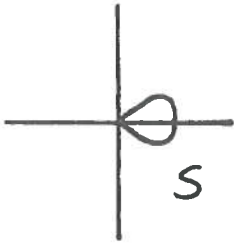
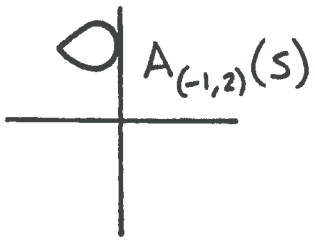
Transformations of Solutions of Equations in Two Variables

The "Pear Shaped Quartic" is the set of solutions, S, of the polynomial equation $x^4 - x^3 + y^2 = 0$.



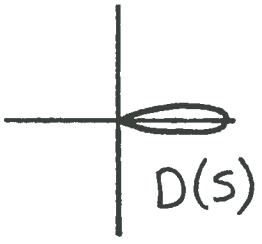
23.) Give an equation for $A_{(-1,2)}(S)$, the Pear Shaped Quartic shifted left 1 and up 2.



$A_{(-1,2)}^{-1} = A_{(1,-2)}$ $\begin{cases} x \longmapsto x+1 \\ y \longmapsto y-2 \end{cases}$

$x^4 - x^3 + y^2 = 0 \xrightarrow[\substack{x \longmapsto x+1 \\ y \longmapsto y-2}]{A_{(1,-2)}} (x+1)^4 - (x+1)^3 + (y-2)^2 = 0$

24.) Let $D = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$. Give an equation for $D(S)$, the Pear Shaped Quartic scaled by 2 in the x-coordinate and $\frac{1}{3}$ in the y-coordinate.



$D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix}$ $\begin{cases} x \longmapsto \frac{x}{2} \\ y \longmapsto 3y \end{cases}$

$x^4 - x^3 + y^2 = 0 \xrightarrow[\substack{x \longmapsto \frac{x}{2} \\ y \longmapsto 3y}]{\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix}} \left(\frac{x}{2}\right)^4 - \left(\frac{x}{2}\right)^3 + (3y)^2 = 0$

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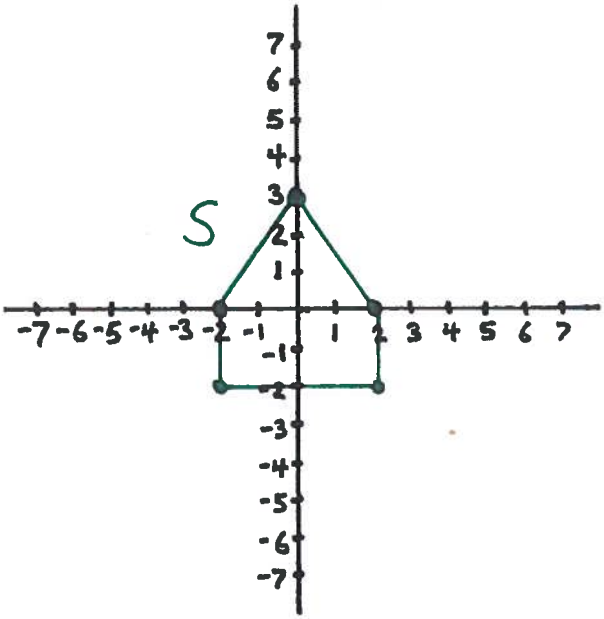
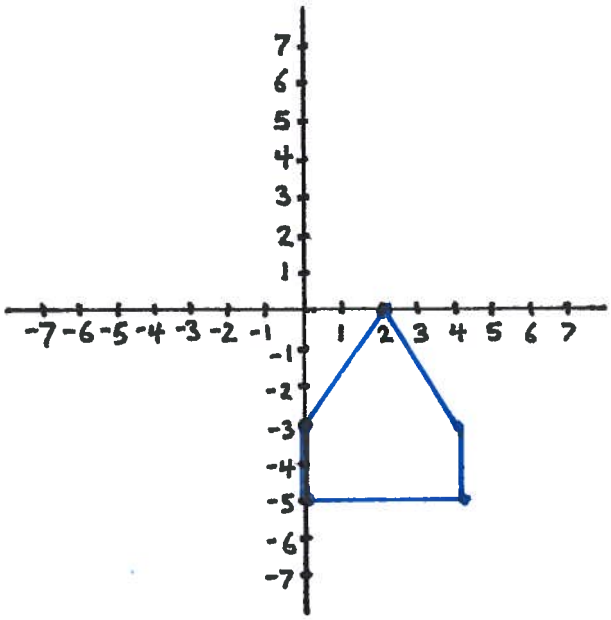
Planar Transformations

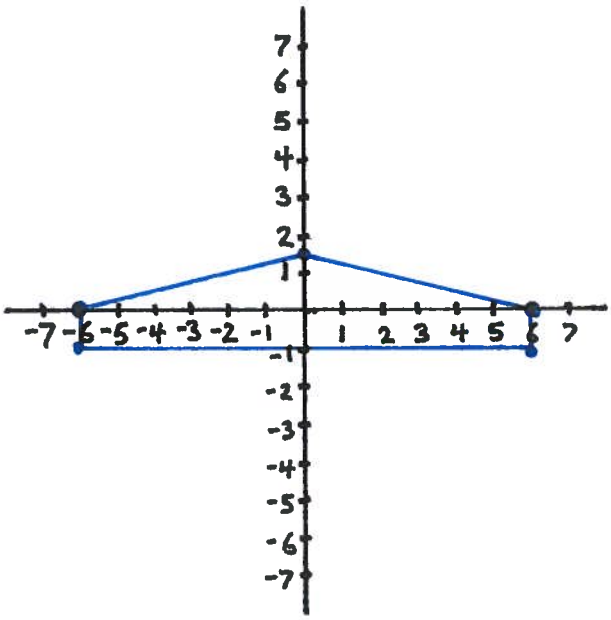
Shown below is a set S in the plane. (The x- and y-axes are not part of S. They are just drawn for perspective.)



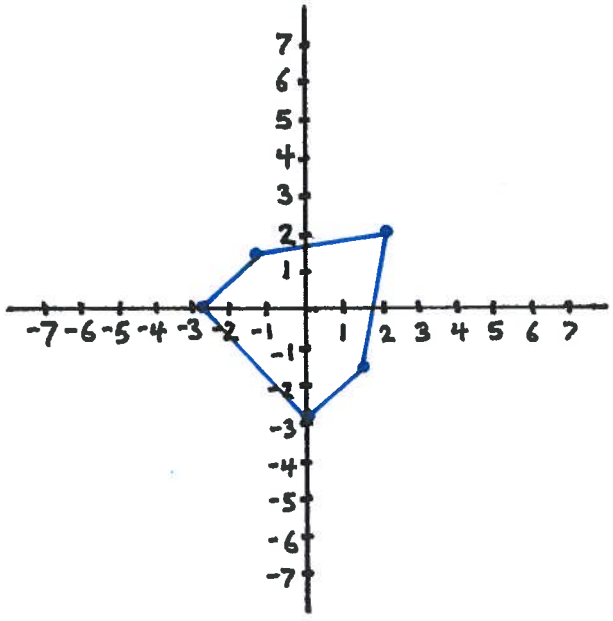
25.) Draw $A_{(2,-3)}(S)$



26.) Draw $\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}(S)$



27.) Draw $R_{\frac{-\pi}{4}}(S)$



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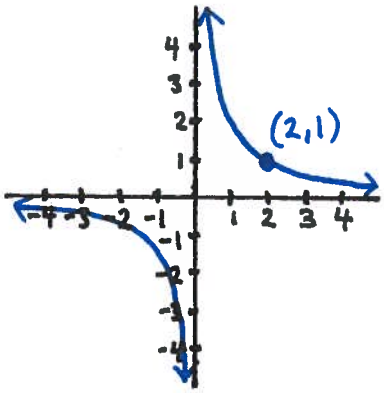
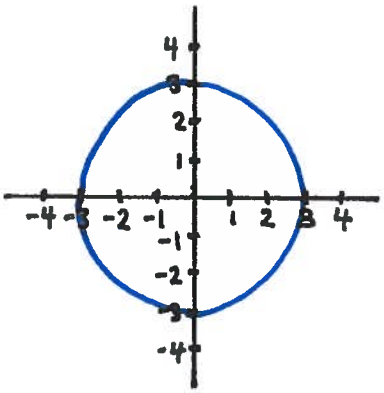
Conics

For #28-30, Draw the set of solutions of the given equation in two variables. (Label at least one point precisely in #28.)

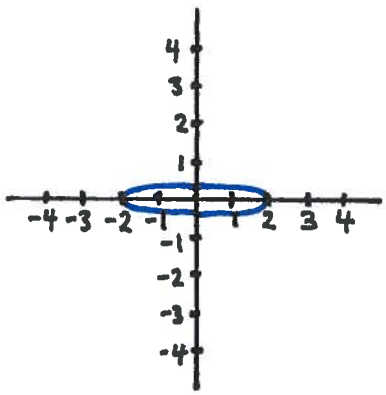
28.) $xy = 2$



29.) $x^2 + y^2 = 9$



30.) $\frac{x^2}{4} + 4y^2 = 1$



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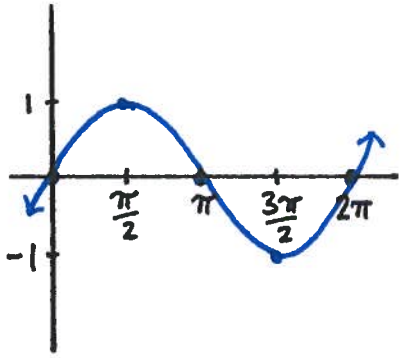
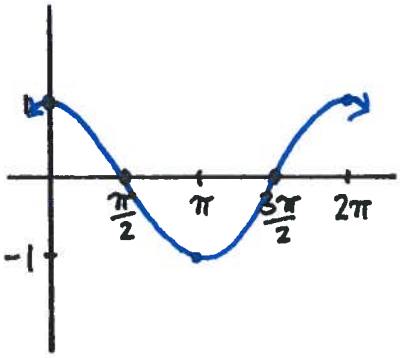
Trigonometric Functions

For #31-36, draw the graphs of the given functions.

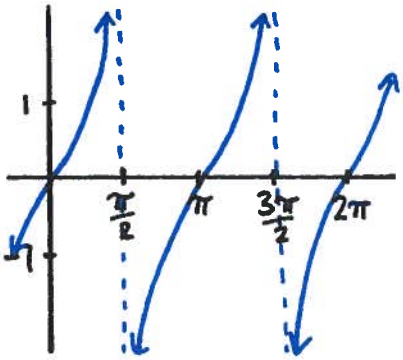
31.) $\sin(\theta)$



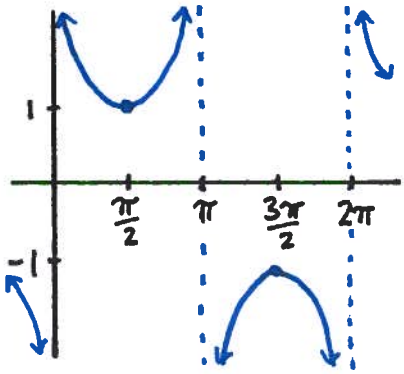
32.) $\cos(\theta)$



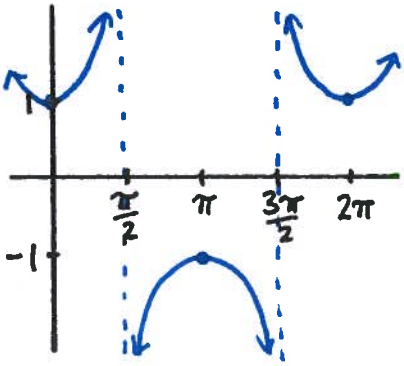
33.) $\tan(\theta)$



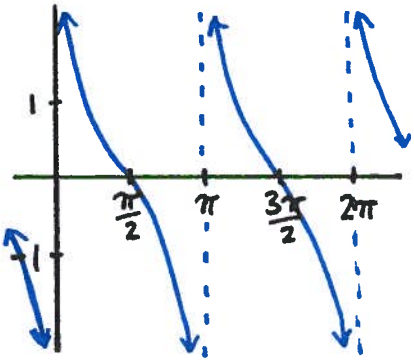
34.) $\csc(\theta)$



35.) $\sec(\theta)$



36.) $\cot(\theta)$



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Equations in One Variable

The remaining questions are each worth 2 points. For #37-42, find the solutions of the given equations, and show your work. If an equation has no solution, explain why.

37.) $\log_3(x-7)=4$

Domain: $x-7>0 \Rightarrow x>7$

$\log_3(x-7)=4$

$x-7=3^4=81$

$x=81+7=88$

$88>7$, so 88 is the solution.

38.) $(2x-5)^2=16$ Domain: $\mathbb{R}$

$2x-5=4$

$2x=4+5=9$

$x=\frac{9}{2}$

or

$2x-5=-4$

$2x=-4+5=1$

$x=\frac{1}{2}$

The solutions are $\frac{1}{2}$ and $\frac{9}{2}$.

39.) $\sqrt{3x^2-2}=-3$

No solution, because a square-root can't be negative.

40.) $\log_e(x) + \log_e(x+1) = \log_e(6)$

Domain: $x>0$ and $x+1>0 \Rightarrow x>0$ and $x>-1 \Rightarrow \underline{x>0}$

$\log_e(x(x+1)) = \log_e(6)$

$x(x+1) = 6$

$x^2 + x = 6$

$x^2 + x - 6 = 0$

$x = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-6)}}{2(1)}$

$= \dfrac{-1 \pm \sqrt{25}}{2} = \dfrac{-1 \pm 5}{2}$

so $x = \dfrac{-6}{2} = -3$ or $x = \dfrac{4}{2} = 2$

$-3 \neq 0$, so -3 is not a solution, but $2 > 0$ so 2 is a solution.

2 is the only solution.

41.) $(e^x)^2 e^{x+4} = 5$

Domain: $\mathbb{R}$

$e^{2x} e^{x+4} = 5$

$e^{2x+x+4} = 5$

$e^{3x+4} = 5$

$3x + 4 = \log_e(5)$

$3x = \log_e(5) - 4$

$x = \dfrac{\log_e(5) - 4}{3}$

42.) $\dfrac{\frac{x}{x+1} + x}{x-2} = 1$

Domain: $x \neq 2$ and $x \neq -1$

$\dfrac{x}{x+1} + x = \dfrac{x}{x+1} + \dfrac{x(x+1)}{x+1} = \dfrac{x + x(x+1)}{x+1} = \dfrac{x^2+2x}{x+1}$

so our equation is $\dfrac{\frac{x^2+2x}{x+1}}{x-2} = 1.$

Multiply by $(x-2)$: $\dfrac{x^2+2x}{x+1} = x-2.$

Multiply by $(x+1)$: $x^2+2x = (x-2)(x+1) = x^2 - x - 2$

Subtract $x^2 - x - 2$:

$3x + 2 = 0$

$3x = -2$

$x = \dfrac{-2}{3}$

First Name:_____ Last Name:_____

1.) F

2.) D

3.) K

4.) A

5.) I

6.) H

7.) H

8.) B

9.) C

10.) J

11.) G

12.) E

13.) $\sqrt{85}$

14.) $\sqrt{75}$

15.) $\sin(\theta) = \frac{5}{13}$

$\cos(\theta) = \frac{12}{13}$

$\tan(\theta) = \frac{5}{12}$

16.) $\frac{\sqrt{3}}{3}$

17.) $\frac{1}{8}$

18.) $\sqrt{13 - \frac{12}{\sqrt{2}}}$

19.) $\sqrt{29}\left(\frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$

20.) $2(\cos(9), \sin(9))$

21.) $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

22.) $(-1+2\sqrt{3}, -\sqrt{3}-2)$

23.) $(x+1)^4 - (x+1)^3 + (y-2)^2 = 0$

24.) $\left(\frac{x}{2}\right)^4 - \left(\frac{x}{2}\right)^3 + (3y)^2 = 0$