3×3 Matrices

Much of this chapter is similar to the chapter on 2×2 matrices. The most substantial difference between 2×2 matrices and 3×3 matrices is that it's harder to write a 3×3 matrix than it is to write a 2×2 matrix.

3×3 matrices have 3 rows and 3 columns. They are a square block of 9 numbers, such as

$$\begin{pmatrix} 2 & 0 & 6 \\ 4 & -5 & 14 \\ -10 & 3 & 4 \end{pmatrix}$$

Two matrices are equal if the entry in any position of the one matrix equals the entry in the same position of the other matrix.

Examples.

$$\begin{pmatrix} 3 & 2 & 7 \\ 5 & 0 & -1 \\ 6 & 9 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 7 \\ 5 & 0 & -1 \\ 6 & 9 & 4 \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 3 & 2 & 7 \\ 5 & 0 & -1 \\ 6 & 9 & 4 \end{pmatrix} \neq \begin{pmatrix} 3 & 2 & 7 \\ 5 & 9 & -1 \\ 6 & 0 & 4 \end{pmatrix}$$

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Matrices as functions

A 3×3 matrix defines a function whose domain is $\mathbb{R}^3$ and whose target is $\mathbb{R}^3$. The function is defined as follows:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} au + bv + cw \\ du + ev + fw \\ gu + hv + iw \end{pmatrix}$$

Notice that the first, second, or third entry in the vector on the right side of the above equation can be found by multiplying the first, second, or third row of the matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and the column

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Example. The matrix

$$\begin{pmatrix} 5 & 3 & 1 \\ -2 & 2 & 4 \\ 7 & 0 & -1 \end{pmatrix}$$

has 3 rows and 3 columns, so it is a function whose domain is $\mathbb{R}^3$, and whose target is $\mathbb{R}^3$.

Because,

$$\begin{pmatrix} 2 \\ 9 \\ -3 \end{pmatrix}$$

is a vector in $\mathbb{R}^3$,

$$\begin{pmatrix} 5 & 3 & 1 \\ -2 & 2 & 4 \\ 7 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \\ -3 \end{pmatrix}$$

is also a vector in $\mathbb{R}^3$. The vector it equals is

$$\begin{aligned} \begin{pmatrix} 5 & 3 & 1 \\ -2 & 2 & 4 \\ 7 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \\ -3 \end{pmatrix} &= \begin{pmatrix} 5 \cdot 2 & + & 3 \cdot 9 & + & 1 \cdot (-3) \\ -2 \cdot 2 & + & 2 \cdot 9 & + & 4 \cdot (-3) \\ 7 \cdot 2 & + & 0 \cdot 9 & + & (-1) \cdot (-3) \end{pmatrix} \\ &= \begin{pmatrix} 10 + 27 - 3 \\ -4 + 18 - 12 \\ 14 + 0 + 3 \end{pmatrix} \\ &= \begin{pmatrix} 34 \\ 2 \\ 17 \end{pmatrix} \end{aligned}$$

Identity matrix

Notice that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \cdot x + 0 \cdot y + 0 \cdot z \\ 0 \cdot x + 1 \cdot y + 0 \cdot z \\ 0 \cdot x + 0 \cdot y + 1 \cdot z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Thus,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the identity function whose domain is $\mathbb{R}^3$. We call this matrix the 3×3 *identity matrix*.

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Matrix multiplication

You can "multiply" two 3×3 matrices to obtain another 3×3 matrix.

Order the columns of a matrix from left to right, so that the 1^{st} column is on the left, the 2^{nd} column is directly to the right of the 1^{st}, and the 3^{rd} column is to the right of the 2^{nd}.

To multiply two matrices, call the columns of the matrix on the right "input columns", and put each of the input columns into the matrix on the left (thinking of it as a function). The column that is assigned to the 1^{st} input column by the matrix function will be the 1^{st} column of the product you are trying to find.

The column that is assigned to the 2^{nd} input column by the matrix function will be the 2^{nd} column of the product, and the column that is assigned to the 3^{rd} input column by the matrix function will be the 3^{rd} column of the product.

Example. To find the product

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}$$

separate the matrix on the right into its three "input columns":

$$\begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix} \mapsto \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

Then the product

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}$$

will be a 3×3 matrix whose first column (when read left-to-right) equals

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 23 \\ 21 \\ 9 \end{pmatrix}$$

whose second column equals

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 13 \\ 21 \\ 6 \end{pmatrix}$$

and whose third column is

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 \\ 33 \\ 4 \end{pmatrix}$$

To repeat the previous sentence,

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 5 & 8 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 13 & 14 \\ 21 & 21 & 33 \\ 9 & 6 & 4 \end{pmatrix}$$

Matrix multiplication is function composition

If A and B are 3×3 matrices, then the result of multiplying the matrices AB would determine the same function $AB : \mathbb{R}^3 \to \mathbb{R}^3$ as the function that results from composition, namely $A \circ B$.

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Inverse matrices

If B is a 3×3 matrix, then B^{-1} is the 3×3 matrix where

$$BB^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$B^{-1}B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example. We can check that the two matrices

$$\begin{pmatrix} 7 & 0 & 2 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & -2 \\ -1 & 1 & 1 \\ -3 & 0 & 7 \end{pmatrix}$$

are inverses of each other by multiplying them in either order and checking to see that their product is the identity matrix:

$$\begin{pmatrix} 7 & 0 & 2 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ -1 & 1 & 1 \\ -3 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & -2 \\ -1 & 1 & 1 \\ -3 & 0 & 7 \end{pmatrix} \begin{pmatrix} 7 & 0 & 2 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Exercises

1.) The matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 3 & -2 & 0 \end{pmatrix}$$

describes a function $A : \mathbb{R}^3 \to \mathbb{R}^3$.

Find the vectors

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 3 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 3 & -2 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

2.) The matrix

$$B = \begin{pmatrix} 3 & 2 & -2 \\ 1 & 0 & 10 \\ 4 & -5 & 7 \end{pmatrix}$$

describes a function $B : \mathbb{R}^3 \to \mathbb{R}^3$.

Find the vectors

$$\begin{pmatrix} 3 & 2 & -2 \\ 1 & 0 & 10 \\ 4 & -5 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 2 & -2 \\ 1 & 0 & 10 \\ 4 & -5 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

3.) Find the product

$$\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & -4 \end{pmatrix}$$

4.) Find the product

$$\begin{pmatrix} 2 & -1 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix}$$

5.) Find the product

$$\begin{pmatrix} 3 & -17 & 5 \\ 17 & 3 & 34 \\ 41 & 3 & 18 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For #6 and #7, determine whether the two matrices given are inverses of each other.

6.) $\begin{pmatrix} 4 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 3 \\ -1 & 0 & 2 \end{pmatrix}$

7.) $\begin{pmatrix} -1 & 1 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Match the functions with their graphs.

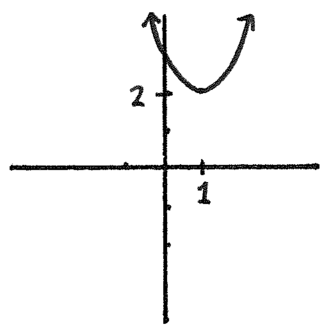
8.) $(x+1)^2+2$

9.) $(x+1)^2-2$

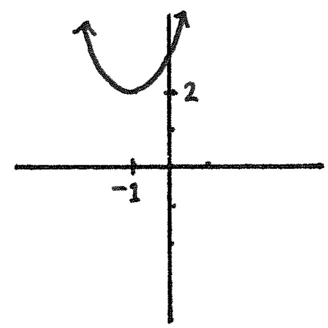
10.) $(x-1)^2+2$

11.) $(x-1)^2-2$

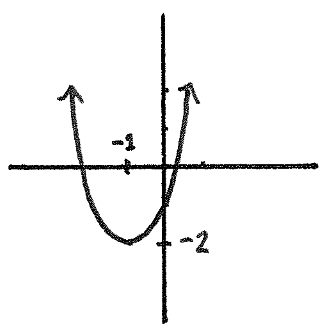
A.)



B.)



C.)



D.)