To review for the final, you should study the three reviews and the three midterm exams. Here are some additional exercises you may use to study.

1. (a) Prove the following statement using a direct proof: If \( x \) is an odd integer, then \( x^3 \) is an odd integer.

   **Solution.** If \( x \) is odd, then \( x = 2n + 1 \) for some \( n \), so
   \[
   x^3 = (2n + 1)^3 = 8n^3 + 12n^2 + 6n + 1 = 2(4n^2 + 6n^2 + 3n) + 1,
   \]
   which is an odd integer.

   (b) Prove the following statement using an indirect proof (i.e., proof of contrapositive or proof by contradiction): If \( x^3 \) is an odd integer, then \( x \) is an odd integer.

   **Solution.** The contrapositive of the statement is: “If \( x \) is not odd, then \( x^3 \) is not odd.” We assume that \( x \) is not odd, i.e., that \( x \) is even. Then \( x = 2n \) for some \( n \), so
   \[
   x^3 = (2n)^3 = 8n^3 = 2(4n^3),
   \]
   which is even, i.e., not odd. Since the contrapositive is true, the original is also true.

2. For any integer \( n \), consider the proposition \( p(n) \) given by “\( n \) is even.” Prove or disprove each of the following statements about \( x, y \) in the universal set \( U = \{1, 4, 9, 16, 25\} \).

   (a) \( \forall x \ p(x) \)

   **Solution.** False. Some \( x \)'s are not even, like \( x = 1, 9, 25 \).

   (b) \( \exists x \ p(x) \)

   **Solution.** True. There are even \( x \)'s, namely \( x = 4, 16 \).

   (c) \( \forall x \neg p(x) \)

   **Solution.** False. Some \( x \)'s are even, like \( x = 4, 16 \). (This is the negation of (b), so it has the opposite truth value with counter-examples same as the examples in (b).)

   (d) \( \exists x \neg p(x) \)

   **Solution.** True. There are odd \( x \)'s, namely \( x = 1, 9, 25 \). (This is the negation of (a), so it has the opposite truth value.)

   (e) \( \forall x \exists y \ p(x + y) \)

   **Solution.** True. For any \( x \in U \), we can find a \( y \) such that \( x + y \) is even. If \( x \) is even, we can pick \( y \) to be even. If \( x \) is odd, we can pick \( y \) to be odd. One shortcut is that if we always pick \( y = x \), then \( x + y = x + x = 2x \) is always even.

   (f) \( \exists y \forall x \ p(x + y) \)

   **Solution.** False. There is no \( y \) so that \( x + y \) is even for every single \( x \). For a fixed odd \( y \), some of the \( x \)'s are even, so \( x + y \) could be odd. For a fixed even \( y \), some of the \( x \)'s are odd, so \( x + y \) could be odd. Note that the negation is
   \[
   \forall y \exists x \neg p(x + y),
   \]
   which says that for each \( y \) we can find an \( x \) to make \( x + y \) odd, which is what we described above. The negation is true, so the original is false.

   (g) \( \forall x \exists y \ p(xy) \)

   **Solution.** True. For each \( x \), we can find a \( y \) such that \( xy \) is even. If \( x \) happens to be even, we can choose any \( y \). If \( x \) is odd, we can choose an even \( y \).
(h) \( \exists y \forall x \, p(xy) \)

**Solution.** True. We can fix \( y \) to be an even number (like \( y = 4 \)) so that \( xy \) is even no matter which \( x \) we choose.

3. Build the truth table for the following compound propositions and determine which are tautologies.

(a) \( p \rightarrow [q \rightarrow (p \land q)] \)

**Solution.**

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<tr>
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<th>( q )</th>
<th>( p \land q )</th>
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This is a tautology. In words, it says that \( p \) being true implies that a true \( q \) would imply both \( p \) and \( q \), which is sort of obvious once you think about it.

(b) \( (p \land q) \rightarrow (p \lor q) \).

**Solution.**

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<tr>
<th>( p )</th>
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This is a tautology. In words, it says that if both \( p \) and \( q \) are true, then at least one of \( p \) or \( q \) is true, which is obviously a tautological statement.

4. Use membership tables, Venn diagrams, or definitions to prove or disprove the following statements about generic sets \( A, B, C \).

(a) \( A \setminus (B \setminus C) = (A \setminus B) \setminus C \)

**Solution.** False. For example, take \( A = \{1, 2\} \), \( B = \{2\} \) and \( C = \{2\} \). Then

\[
A \setminus (B \setminus C) = A \setminus (\{2\} \setminus \{2\}) = A \setminus \emptyset = A,
\]

but

\[
(A \setminus B) \setminus C = (\{1, 2\} \setminus \{2\}) \setminus \{2\} = \{1\} \setminus \{2\} = \emptyset.
\]

Here is the membership table that shows the difference between these two sets.

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So the two sets are the same if and only if \( A \cap C \) is empty.

(b) \( A \setminus (B \setminus A) = (B \setminus A) \setminus B \)

**Solution.** False. For example, take \( A = \{1, 2\} \) and \( B = \{2, 3\} \). Then

\[
A \setminus (B \setminus A) = \{1, 2\} \setminus (\{2, 3\} \setminus \{1, 2\}) = \{1, 2\} \setminus \{3\} = \{1, 2\},
\]

but

\[
(B \setminus A) \setminus B = (\{2, 3\} \setminus \{1, 2\}) \setminus \{2, 3\} = \{3\} \setminus \{2, 3\} = \emptyset.
\]
Here is the membership table that shows that $A \setminus (B \setminus A) = A$ and $(B \setminus A) \setminus B = \emptyset$.

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5. Prove or disprove the following statements about generic sets $A, B$.

(a) If $A \cap B = A \cap C$, then $B = C$.

**Solution.** False. It is possible that $B$ and $C$ are different outside of $A$. For a counter-example, if $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{1, 4\}$, then $B \neq C$, but

$$A \cap B = \{1\} = A \cap C.$$ 

(b) If $A \cup B = A \cup C$, then $B = C$.

**Solution.** False. It is possible that $A$ accounts for differences in $B$ and $C$. For a counter-example, if $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{2, 3\}$, then $B \neq C$, but

$$A \cup B = \{1, 2, 3\} = A \cup C.$$ 

6. For each of the following functions from $\mathbb{Z}$ to $\mathbb{Z}$, determine if it is injective, surjective, both or neither. Justify your answer.

(a) $f(x) = 3x - 4$

**Solution.** Injective: if $f(a) = f(b)$, then

$$3a - 4 = 3b - 4 \implies 3a = 3b \implies a = b.$$ 

Not surjective: if $f(x) = 0$, then $3x = 4$, so $x = 4/3$, which is not an integer.

(b) $g(x) = -x + 10$

**Solution.** Injective: if $g(a) = g(b)$, then

$$-a + 10 = -b + 10 \implies -a = -b \implies a = b.$$ 

Surjective: for any $y \in \mathbb{Z}$, choose $x = -y + 10$, so that

$$g(-y + 10) = -(y - 10) + 10 = y - 10 + 10 = y.$$ 

Another way to see that it is bijective is to produce the inverse function. In this case, $g$ is its own inverse since

$$g(g(x)) = g(-x + 10) = -(x - 10) + 10 = x - 10 + 10 = x.$$ 

(c) $h(x) = \sqrt{x^2}$

**Solution.** Not injective: $h(-1) = \sqrt{(-1)^2} = \sqrt{1} = 1$ and $h(1) = \sqrt{1^2} = \sqrt{1} = 1$. 

Not surjective: the square root symbol outputs the nonnegative square root, so $h(x)$ never gives a negative integer.

(d) $p(x) = \left\lfloor \frac{3x}{5} \right\rfloor$.

**Solution.** Not injective: $p(0) = |0| = 0$ and $p(1) = |3/5| = 0$. 

Surjective: Let $y \in \mathbb{Z}$ be any integer. Choose $x$ such that $\frac{3}{5}y \leq x < \frac{3}{5}(y+1)$, which is always possible since this interval has length 5/3. Then $y \leq \frac{3x}{5} < y + 1$, so $p(x) = y$. 


7. Prove the following statements using mathematical induction.

(a) For \( n \geq 1 \), the integer \( n^2 - n \) is divisible by 2.

**Solution.** The base case is that \( 1^2 - 1 = 0 \) is divisible by 2, which is true since 0 = 2(0). For the inductive step, we assume that \( k^2 - k \) is divisible by 2. We look at

\[(k + 1)^2 - (k + 1) = (k^2 + 2k + 1) - (k + 1) = (k^2 - k) + 2k.\]

By hypothesis, \( k^2 - k \) is divisible by 2. By definition, 2k is divisible by 2. Therefore, the sum \( (k^2 - k) + 2k \) is divisible by 2, which proves that \( (k + 1)^2 - (k + 1) \) is divisible by 2.

By the way, here are two alternate proofs that do not use induction.

**Alternate proof #1.** The number \( n^2 - n \) factors as \( n(n-1) \). If \( n \) is even, then \( n(n-1) \) is even. If \( n \) is odd, then \( n-1 \) is even, so \( n(n-1) \) is even. In either case, \( n(n-1) \) is even, so it is divisible by 2.

**Alternate proof #2.** The possible remainders modulo 2 are \( \{0, 1\} \). If we square them modulo 2, we get \( 0^2 \equiv 0 \) and \( 1^2 \equiv 1 \). Therefore, modulo 2, we get \( 0^2 - 0 \equiv 0 \) and \( 1^2 - 1 \equiv 0 \), so \( n^2 - n \) is always congruent to 0 modulo 2, which proves that \( n^2 - n \) is always divisible by 2.

(b) For \( n \geq 1 \), the integer \( n^3 - n \) is divisible by 3.

**Solution.** The base case is that \( 1^3 - 1 = 0 \) is divisible by 3, which is true. For the inductive step, we assume that \( k^3 - k \) is divisible by 3. We look at

\[(k + 1)^3 - (k + 1) = (k^3 + 3k^2 + 3k + 1) - (k + 1) = (k^3 - k) + 3(k^2 + k).\]

We know that \( k^3 - k \) is divisible by 3 (by hypothesis) and we see that \( 3(k^2 + k) \) is divisible by 3, so the sum \( (k^3 - k) + 3(k^2 + k) \) is also divisible by 3, which proves that \( (k + 1)^3 -(k + 1) \) is divisible by 3.

Again, here are two alternate proofs that do not use induction.

**Alternate proof #1.** The number \( n^3 - n \) factors as \( (n-1)(n)(n+1) \). Since \( n-1 \), \( n \), and \( n+1 \) are three consecutive integers, exactly one of them is divisible by 3. Therefore, the product of the three integers is always divisible by 3.

**Alternate proof #2.** The remainders modulo 3 are \( \{0, 1, 2\} \). If we cube these numbers modulo 3, we get \( 0^3 \equiv 0 \), \( 1^3 \equiv 1 \), and \( 2^3 \equiv 8 \equiv 2 \). Therefore, \( n^3 - n \) is always congruent to 0 modulo 3, which means that \( n^3 - n \) is always divisible by 3.

(c) For \( n \geq 1 \), the integer \( 5^n - 2^n \) is divisible by 3.

**Solution.** The base case is that \( 5^1 - 2^1 = 5 - 2 = 3 \) is divisible by 3, which is true. For the inductive step, we assume that \( 5^k - 2^k \) is divisible by 3. In other words, we assume that \( 5^k - 2^k = 3m \) for some integer \( m \). Next, we look at the number \( 5^{k+1} - 2^{k+1} \), which may be rewritten as

\[5^{k+1} - 2^{k+1} = 5 \cdot 5^k - 2 \cdot 2^k.\]

Since \( 5^k = 3m + 2^k \), we get

\[5^{k+1} - 2^{k+1} = 5(3m + 2^k) - 2 \cdot 2^k = 15m + (5 - 2)2^k = 3(5m + 2^k),\]

which is clearly divisible by 3, completing the inductive step.

Here are two alternate proofs that do not use induction.

**Alternate proof #1.** For any \( x, y \in \mathbb{R} \), the expression \( x^n - y^n \) factors as

\[x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1}).\]
To see this, multiply out the right-hand side and observe the cancellation of all but two terms. If we apply this factorization for \(x = 5\) and \(y = 2\), then we get

\[
5^n - 2^n = (3)(5^{n-1} + 5^{n-2}2 + 5^{n-3}2^2 + \cdots + 5^22^{n-3} + 52^{n-2} + 2^{n-1}),
\]

which is clearly divisible by 3.

**Alternate proof #2.** Modulo 3, we have \(5 \equiv 2\), so \(5^n \equiv 2^n\), which means that

\[
5^n - 2^n \equiv 2^n - 2^n \equiv 0 \quad (3),
\]

which proves that \(5^n - 2^n\) is divisible by 3.

(d) For \(n \geq 1\), the sum \(1 + 4 + 4^2 + 4^3 + \cdots + 4^{n-1}\) is equal to \(\frac{4^n - 1}{3}\).

**Solution.** The base case is that \(1\) is equal to \((4^1 - 1)/3\), which is true. For the inductive step, we assume that

\[
1 + 4 + 4^2 + \cdots 4^{k-1} = \frac{4^k - 1}{3}.
\]

We add \(4^k\) to both sides to get

\[
1 + 4 + 4^2 + \cdots 4^{k-1} + 4^k = \frac{4^k - 1}{3} + 4^k.
\]

If we get a common denominator, the right-hand side becomes

\[
\frac{4^k - 1}{3} + \frac{3 \cdot 4^k}{3} = \frac{(1 + 3)4^k - 1}{3} = \frac{4 \cdot 4^k - 1}{3} = \frac{4^{k+1} - 1}{3},
\]

which proves the inductive step.

(c) For \(n \geq 1\), the sum \(1!(1!) + 2!(2!) + 3!(3!) + \cdots + n!(n!)\) is equal to \((n + 1)! - 1\).

**Solution.** The base case is that \(1!\) is equal to \(2! - 1\), which is true. For the inductive step, we assume that

\[
1!(1!) + 2!(2!) + 3!(3!) + \cdots k!(k!) = (k + 1)! - 1.
\]

We add \((k + 1)!((k + 1)!))\) to both sides to get

\[
1!(1!) + 2!(2!) + 3!(3!) + \cdots k!(k!) + (k + 1)!(k + 1)! = (k + 1)! - 1 + (k + 1)!(k + 1)!
\]

If we factor \((k + 1)!\) out of the first and last term of the right-hand side, it becomes

\[
(k + 1)!(1 + (k + 1)) - 1 = (k + 1)! \cdot (k + 2) - 1.
\]

Since \((k + 1)!\) multiplied by \((k + 2)\) is just \((k + 2)\)!, we get that

\[
1!(1!) + 2!(2!) + 3!(3!) + \cdots k!(k!) + (k + 1)!(k + 1)! = (k + 2)! - 1,
\]

which proves the inductive step.

8. Prove or disprove the statement about integers \(a, b, c, d\): If \(a \mid b\) and \(c \mid d\), then \(ac \mid bd\).

**Solution.** The statement is true. Suppose that \(a \mid b\) and \(c \mid d\). Then \(b = am\) and \(d = cn\) for some integers \(m, n\). Therefore,

\[
bd = (am)(cn) = ac(mn),
\]

which proves that \(ac \mid bd\).

9. Prove the statement about integers \(a, b, c\): If \(a \mid bc\) and \(\gcd(a, b) = 1\), then \(a \mid c\).

(Hint: use Bezout’s identity to write \(ax + by = 1\), then multiply both sides by \(c\).)

**Solution.** Suppose that \(a \mid bc\) and \(\gcd(a, b) = 1\). Since \(a \mid bc\), we can write \(bc = ak\) for some integer \(k\). Since \(\gcd(a, b) = 1\), we can use Bezout’s identity to write \(ax + by = 1\) for some integers \(x, y\). If we multiply both sides by \(c\), we get

\[
acx + bcy = c.
\]
Using the fact that \( bc = ak \), we get
\[
acx + aky = c \quad \rightarrow \quad a(cx + ky) = c,
\]
which proves that \( a \mid c \).

10. (A thanksgiving problem.) A storage bin contains frozen turkeys and frozen hams. Each turkey weighs 20 pounds and each ham weighs 13 pounds. If the bin contains 450 total pounds of frozen meat, find the number of turkeys and hams in the bin.

**Solution.** Suppose that there are \( T \) turkeys and \( H \) hams. A total weight of 450 pounds translates to finding nonnegative integer solutions to
\[
20T + 13H = 450.
\]
We begin by solving Bezout’s problem \( 20x + 13y = 1 \). You can use the Euclidean algorithm in reverse to find a first solution, or you can find a first solution by inspection. To me, an obvious solution is \( x_0 = 2 \) and \( y_0 = -3 \), i.e.,
\[
20(2) + 13(-3) = 1.
\]
Multiplying both sides by 450, we get
\[
20(900) + 13(-1350) = 450.
\]
So \( T_0 = 900 \) and \( H_0 = -1350 \) is an integer solution to the original equation. We know that every other solution is of the form
\[
T = 900 - 13k \quad \text{and} \quad H = -1350 + 20k.
\]
In order for both to be nonnegative, we need \( 13k \leq 900 \) and \( 20k \geq 1350 \), which puts \( k \) in the interval
\[
\frac{1350}{20} \leq k \leq \frac{900}{13}, \quad \rightarrow \quad 67.5 \leq k \leq 96.23.
\]
Therefore, the only two solutions are with \( k = 68 \) and \( k = 69 \):
\[
(T, H) = (900 - 13(68), -1350 + 20(68)) = (16, 10)
\]
and
\[
(T, H) = (900 - 13(69), -1350 + 20(69)) = (3, 30).
\]

**Alternate approach #1.** Maybe you notice right off the bat that 10 hams will weigh 130 pounds, leaving 320 pounds split between 16 turkeys. In other words, maybe you see \( T = 16 \) and \( H = 10 \) as a solution just by inspection. The least common multiple of 20 and 13 is 260, which means that 13 turkeys weigh the same as 20 hams. Therefore, to get other solutions we could replace any 13 turkeys with 20 hams or 20 hams with 13 turkeys. From our initial solution \( T = 16 \) and \( H = 10 \), the only other possibility is to subtract 13 turkeys and add 20 hams to get \( T = 3 \) and \( H = 30 \).

**Alternate approach #2.** Multiples of 20 end in a 0, so we definitely need \( H \) to be a multiple of 10, i.e., \( 13H = 130, 260, 390 \). Only \( 13H = 130 \) and \( 13H = 390 \) make \( 20T + 13H = 450 \) possible, giving \( 20T = 320 \) and \( 20T = 60 \), respectively. So, the only two solutions are 10 hams and 16 turkeys, or 30 hams and 3 turkeys.

11. A bag of candy contains 5 orange candies, 3 red candies, and 2 green candies. Zella picks 4 candies out of the bag (without replacing them).

(a) What is the probability that Zella chose at least one of each color?

**Solution.** First of all, there are
\[
\binom{10}{4} = \frac{10!}{4!6!} = \frac{10(9)(8)(7)}{4(3)(2)(1)} = 210
\]
total possibilities.
We will count directly how many ways there are to get one of each. The first option is 2 orange, 1 red, and 1 green:
\[
\binom{5}{2} \binom{3}{1} \binom{2}{1} = 10 \cdot 3 \cdot 2 = 60.
\]
The second option is 1 orange, 2 red, and 2 green:
\[
\binom{5}{1} \binom{3}{2} \binom{2}{1} = 5 \cdot 3 \cdot 2 = 30.
\]
The third option is 1 orange, 1 red, and 2 green:
\[
\binom{5}{1} \binom{3}{1} \binom{2}{2} = 5 \cdot 3 \cdot 1 = 15.
\]
Therefore, the total number of ways to have one of each color is 60 + 30 + 15 = 105, so the probability of choosing one of each color is \(\frac{105}{210} = \frac{1}{2} = 50\%\).

(b) What is the probability that Zella chose at least 2 red candies?

**Solution.** As a first method, we will count directly the number of ways to have at 2 or 3 red candies. The first option is 2 red and 2 other:
\[
\binom{3}{2} \binom{7}{2} = 3 \cdot 21 = 63.
\]
The second option is 3 red and 1 other:
\[
\binom{3}{3} \binom{7}{1} = 7 \cdot 1 = 7.
\]
The total number of ways to get at least 2 red candies is 63 + 7 = 70, so the probability is \(\frac{70}{210} = \frac{1}{3} = 33.33\%\).

As a second method, we will count indirectly by finding the number of ways to have 0 or 1 red candy. The first option is 0 red and 4 other:
\[
\binom{7}{4} = 35.
\]
The second option is 1 red and 3 other:
\[
\binom{3}{1} \binom{7}{3} = 3 \cdot 35 = 105.
\]
So there are 35 + 105 = 140 ways to have less than 2 candies, which means that there are 70 ways to have at least 2, giving the same probability 1/3 as before.

(c) What is the probability that Zella chose at least 2 orange candies?

**Solution.** At least 2 orange means 2, 3, or 4 orange candies. The first option is 2 orange and 2 other:
\[
\binom{5}{2} \binom{5}{2} = 10 \cdot 10 = 100.
\]
The second option is 3 orange and 1 other:
\[
\binom{5}{3} \binom{5}{1} = 10 \cdot 5 = 50.
\]
The last option is 4 orange and 0 other:
\[
\binom{5}{4} = 5.
\]
So there are 100 + 50 + 5 = 155 ways to have at least 2 orange candies.
Alternatively, we could count how many ways to have less than 2 orange candies. The first option is
1 orange and 3 other:
\[
\binom{5}{1} \binom{5}{3} = 5 \cdot 10 = 50.
\]
The second option is 0 orange and 4 other:
\[
\binom{5}{4} = 5.
\]
So there are \(50 + 5 = 55\) ways to have less than 2 orange candies, so there are \(210 - 55 = 155\) ways
to have at least 2 orange candies.

Either way, the probability is \(\frac{155}{210} = \frac{31}{42} = 0.7381\) = 73.81%.

12. There are 3 friends (Hattie, Felix, and Ophelia) in a room with a table containing exactly 5 pieces of fruit
(an orange, a banana, an apple, a peach, and an apricot).
   (a) Each of the 3 friends writes their name and their fruit choice on a piece of paper. How many different
       scenarios are possible?

   \textbf{Solution.} They could each pick the same fruit, so there are 3 people choosing between 5 fruits for a
total of \(5^3 = 125\) possibilities.

(b) After some discussion, the 3 friends decide on 3 distinct pieces of fruit to eat. How many different
    scenarios are possible?

   \textbf{Solution.} They have to pick distinct fruits (and order matters!), so there are \(\text{P}(5, 3) = 5 \cdot 4 \cdot 3 = 60\)
    possibilities.

(c) After taking the 3 pieces of fruit, there are 2 pieces of fruit remaining. How many different scenarios
    are possible?

   \textbf{Solution.} There are 2 fruits remaining (and order doesn’t matter!), so there are \(\text{C}(5, 2) = 10\) possi-
bilities.

13. Find all solutions to the following congruences modulo 14.
   (a) \(5x \equiv 4 \pmod{14}\)

   \textbf{Solution.} \(x \equiv 12 \pmod{14}\) is the unique solution. One way to see this is to notice that 3 is the
       multiplicative inverse of 5 since \(3 \cdot 5 \equiv 15 \equiv 1 \pmod{14}\), so we can solve by multiplying both sides by 3:
       \[
       5x \equiv 4 \pmod{14} \\
       3(5x) \equiv 3(4) \pmod{14} \\
       x \equiv 12 \pmod{14}
       \]
       Another way to see this is to make a table of values: \(x\) can be any remainder modulo 14 and we
       compute \(5x\) modulo 14

       \[
       \begin{array}{c|cccccccccccc}
       x & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
       \hline
       5x & 0 & 5 & 10 & 1 & 6 & 11 & 2 & 7 & 12 & 3 & 8 & 13 & 4 & 9
       \end{array}
       \]
       to see that \(x \equiv 12\) is the only solution to \(5x \equiv 4\) modulo 14.

   (b) \(6x \equiv 5 \pmod{14}\)

   \textbf{Solution.} There is no solution since \(\gcd(6, 14) = 2\) does not divide 5. One way to see this is to
translate the equation into the integers to get
   \[
   6x = 5 + 14y \quad \rightarrow \quad 6x - 14y = 5 \quad \rightarrow \quad 2(3x - 7y) = 5,
   \]
which has no integer solution since 5 is not divisible by 2. Another method is to make a table like
the one below.
(c) \(6x \equiv 6 \pmod{14}\) (14)

**Solution.** There are two solutions since \(\gcd(6, 14) = 2\) and 2 divides 6. One method is to subtract 6 from both sides and factor to get

\[
6x \equiv 6 \pmod{14} \iff 6x - 6 \equiv 0 \pmod{14} \iff 6(x - 1) \equiv 0 \pmod{14}.
\]

You can multiply 6 by 0 or by 7 to get 0 modulo 14, so \(x - 1 \equiv 0, 7\), which means that \(x \equiv 1, 8\) are the two solutions.

Or, we could make a table to see that \(x \equiv 1, 8\) are solutions.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>(6x)</td>
<td>0</td>
<td>6</td>
<td>12</td>
<td>4</td>
<td>10</td>
<td>2</td>
<td>8</td>
<td>0</td>
<td>6</td>
<td>12</td>
<td>4</td>
<td>10</td>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

14. Prove that \(n^3 - n\) is divisible by 3 for all integers \(n\) by showing that

\[
n^3 \equiv n \pmod{3} \quad (3)
\]

for all possible remainders modulo 3.

**Solution.** The possible remainders modulo 3 are \(\{0, 1, 2\}\). If we cube each one modulo 3, we get

\[
0^3 \equiv 0, \quad 1^3 \equiv 1, \quad \text{and} \quad 2^3 \equiv 8 \equiv 2,
\]

which shows that \(n^3 \equiv n \pmod{3}\) for all integers \(n\).

15. Prove that \(n^4 + 2n^3 + 3n^2 + 2n\) is divisible by 8 for all integers \(n\) by showing that

\[
n^4 + 2n^3 + 3n^2 + 2n \equiv 0 \pmod{8} \quad (8)
\]

for all possible remainders modulo 8.

**Solution.** The remainder of \(n\) modulo 8 is one of 0, 1, 2, 3, 4, 5, 6, 7. We compute powers of these remainders modulo 8 and combine them in the polynomial below to get

<table>
<thead>
<tr>
<th>(n)</th>
<th>(n^2)</th>
<th>(n^3)</th>
<th>(n^4)</th>
<th>(n^4 + 2n^3 + 3n^2 + 2n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0 + 2(0) + 3(0) + 2(0) \equiv 0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1 + 2(1) + 3(1) + 2(1) \equiv 8 \equiv 0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0 + 2(0) + 3(2) + 2(2) \equiv 16 \equiv 0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1 + 2(3) + 3(1) + 2(3) \equiv 16 \equiv 0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0 + 2(0) + 3(0) + 2(4) \equiv 8 \equiv 0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>5</td>
<td>1</td>
<td>1 + 2(5) + 3(1) + 2(5) \equiv 24 \equiv 0</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0 + 2(0) + 3(4) + 2(6) \equiv 24 \equiv 0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>7</td>
<td>1</td>
<td>1 + 2(7) + 3(1) + 2(7) \equiv 32 \equiv 0</td>
</tr>
</tbody>
</table>

We see that \(n^4 + 2n^3 + 3n^2 + 2n \equiv 0 \pmod{8}\) for every integer \(n\), so this expression is always divisible by 8.

**Exercise.** Prove this result for \(n \geq 1\) by induction.