1. (a) Prove or disprove the statement:

If \(d \mid a\) and \(d \mid b\), then \(d \mid (a + b)\).

**Solution.** The statement is true. Suppose that \(d \mid a\) and \(d \mid b\). Then \(a = dm\) for some integer \(m\) and \(b = dn\) for some integer \(n\). Adding them together, we get

\[a + b = dm + dn = d(m + n),\]

which means that \(a + b\) is divisible by \(d\).

(b) Prove or disprove the statement:

If \(d \mid (a + b)\), then \(d \mid a\) and \(d \mid b\).

**Solution.** The statement is false. For example, take \(d = 5\), \(a = 2\) and \(b = 8\). Then \(5 \mid (2 + 8)\), but \(5 \nmid 2\) and \(5 \nmid 8\).

2. (a) Use the Euclidean algorithm to find \(d = \gcd(225, 70)\).

**Solution.** \(\gcd(225, 70) = 5\):

\[
225 = 3 \cdot 70 + 15 \\
70 = 4 \cdot 15 + 10 \\
15 = 1 \cdot 10 + 5 \\
10 = 2 \cdot 5 + 0
\]

(b) Use part (a) to find an integer solution to \(225x + 70y = d\).

**Solution.**

\[
\begin{align*}
5 &= 15 - (70 - 4 \cdot 15) = -70 + 5 \cdot 15 \\
5 &= -70 + 5(225 - 3 \cdot 70) = 5 \cdot 225 - 16 \cdot 70
\end{align*}
\]

So \(x = 5\) and \(y = -16\) is a solution to \(225x + 70y = 5\).

3. Consider the statement \(p(n)\) given by the equation

\[1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}.\]

(a) Verify directly that \(p(n)\) is true for \(n = 1, 2, 3, 4\).

**Solution.**

\[
\begin{array}{c|c|c}
\hline
n & 1^2 = 1 & \frac{1(2)(3)}{6} = 1 \\
\hline
n = 2 & 1^2 + 2^2 = 5 & \frac{2(3)(5)}{6} = 5 \\
\hline
n = 3 & 1^2 + 2^2 + 3^2 = 14 & \frac{3(4)(7)}{6} = 14 \\
\hline
n = 4 & 1^2 + 2^2 + 3^2 + 4^2 = 30 & \frac{4(5)(9)}{6} = 30 \\
\hline
\end{array}
\]
(b) Use mathematical induction to prove that \( p(n) \) is true for all integers \( n \geq 1 \).

**Solution.** The base case is handled in part (a). For the inductive step, we assume the statement is true for \( n = k \); that is, we assume that

\[
1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.
\]

Our goal is to prove the statement is true for \( n = k+1 \), i.e., that

\[
1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.
\]

To do this, we take the equation for \( n = k \) and add \((k+1)^2\) to both sides. The left-hand side becomes

\[
(1^2 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2,
\]

which is exactly the left-hand side for the \( n = k+1 \) equation. The right-hand side becomes

\[
\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k+1}{6} [k(2k+1) + 6(k+1)]
\mathrel{=} \frac{k+1}{6} [2k^2 + 7k + 6]
\mathrel{=} \frac{k+1}{6} [(k+2)(2k+3)]
\mathrel{=} \frac{(k+1)(k+2)(2k+3)}{6}
\]

which is precisely the right-hand side for the \( n = k+1 \) equation. This completes the inductive step, so the statement \( p(n) \) is true for all \( n \geq 1 \).

4. A PIN for an account consists of 5 characters composed of lower-case letters and numbers.

(a) How many PINs are there that consist of all distinct characters?

**Solution.** There are 36 total characters than can be used. Therefore, there are

\[
\frac{36!}{31!} = 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32
\]

possible PINs that have distinct characters.

(b) How many PINs are there that have at least one number.

**Solution.** Here are two ways to do this.

**First method.** There are \( 36^5 \) total PINs including all possible characters. There are \( 26^5 \) total PINs consisting of only letters. So, there are

\[36^5 - 26^5 = 48,584,800\]

total PINs that have at least one number.

**Second method.** If a PIN has exactly one number and four letters, then there are \( \binom{5}{1} = 5 \) locations for the number, 10 choices for the number, and \( 26^4 \) choices for the four letters, so there are a total of \( 5 \cdot 10 \cdot 26^4 \) PINs with exactly one number. We do the similar computation for two numbers,
three numbers, four numbers, and five numbers.

one number, four letters: \( \binom{5}{1} \cdot 10 \cdot 26^4 \)

two numbers, three letters: \( \binom{5}{2} \cdot 10^2 \cdot 26^3 \)

three numbers, two letters: \( \binom{5}{3} \cdot 10^3 \cdot 26^2 \)

four numbers, one letter: \( \binom{5}{4} \cdot 10^4 \cdot 26^1 \)

five numbers, no letters: \( \binom{5}{5} \cdot 10^5 \)

We add these up to get

\[
5(10)(26^4) + 10(10^2)(26^3) + 10(10^3)(26^2) + 5(10^4)(26) + 10^5 = 48,584,800
\]
total PINs that have at least one number.