Some Review Problems
for Exam #3

1. Compute the following indefinite integrals:

(a) $\displaystyle\int \frac{r^5-\sqrt{r}}{r^3}\,dr$

$$\begin{aligned}\int \frac{r^5-\sqrt{r}}{r^3}\,dr &= \int (r^2 - r^{-5/2})\,dr\\ &= \frac{r^3}{3} - \frac{r^{-3/2}}{-3/2} + C\\ &= \frac{1}{3}r^3 + \frac{2}{3}r^{-3/2} + C\end{aligned}$$

(b) $\displaystyle\int \sec^2(8y)\ dy$. Let $u = 8y$; then $du = 8dy$, so we can multiply the inside of the integral by 8 and divide by 8 on the outside.

$$\begin{aligned}\int \sec^2(8y)\ dy &= \frac{1}{8}\int \sec^2(8y)\cdot 8\ dy\\ &= \frac{1}{8}\int \sec^2 u\ du\\ &= \frac{1}{8}\tan u + C\\ &= \frac{1}{8}\tan 8y + C\end{aligned}$$

(c) $\displaystyle\int 3x^4\sqrt{x^5-2}\ dx$. Let $u = x^5-2$; then $du = 5x^4dx$, so we can multiply the inside of the integral by 5 and divide by 5 on the outside, and at the same time move the 3 out front.

$$\begin{aligned}\int 3x^4\sqrt{x^5-2}\ dx &= \frac{1}{5}\cdot 3\int (x^5-2)^{1/2}\cdot 5x^4\ dx\\ &= \frac{3}{5}\int u^{1/2}\ du\\ &= \frac{3}{5}\cdot\frac{u^{3/2}}{3/2} + C\\ &= \frac{2}{5}(x^5-2)^{3/2} + C\end{aligned}$$

(d) $\displaystyle\int \frac{1}{w^3}\ \cos\frac{1}{w^2}\ dw = \int w^{-3}\ \cos(w^{-2})\ dw$. Let $u = w^{-2}$; then $du = -2w^{-3}$, so we can multiply the inside of the integral by -2 and divide by -2 on the outside.

$$\begin{aligned}\int w^{-3}\ \cos(w^{-2})\ dw &= -\frac{1}{2}\int \cos(w^{-2})\cdot(-2w^{-3})\ dw\\ &= -\frac{1}{2}\int \cos u\ du\\ &= -\frac{1}{2}\sin u + C\\ &= -\frac{1}{2}\sin(w^{-2}) + C\end{aligned}$$

2. Compute the following definite integrals:

(a) $\int_{1/6}^{1/3} \cos(\pi z)\ dz.$ Let $u = \pi z$; then $du = \pi\ dz$, so we can multiply the inside of the integral by π and divide by π on the outside.

$$\begin{aligned}\int_{1/6}^{1/3} \cos(\pi z)\ dz &= \frac{1}{\pi}\int_{1/6}^{1/3} \cos(\pi z)\ dz \\ &= \frac{1}{\pi}\int_a^b \cos u\ du \\ &= \frac{1}{\pi}\left(\sin u\ \Big|_a^b\right) \\ &= \frac{1}{\pi}\left(\sin \pi z\ \Big|_{1/6}^{1/3}\right) \\ &= \frac{1}{\pi}\Big(\sin(\pi/3) - \sin(\pi/6)\Big) \\ &= \frac{1}{\pi}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)\end{aligned}$$

(b) $\int_0^{\pi/2} \sin v\ \cos v\ dv.$ Let $u = \sin v$; then $du = \cos v\ dv$.

$$\begin{aligned}\int_0^{\pi/2} \sin v\ \cos v\ dv &= \int_a^b u\ du \\ &= \frac{1}{2}u^2\Big|_a^b \\ &= \frac{1}{2}\sin^2 v\Big|_0^{\pi/2} \\ &= \frac{1}{2}\Big(\sin^2(\pi/2) - \sin^2(0)\Big) \\ &= \frac{1}{2}\end{aligned}$$

(c) $\int_1^2 \frac{3t^2+5t}{(4t^3+10t^2-3)^2}\ dt.$ Let $u = 4t^3 + 10t^2 - 3$; then $du = (12t^2 + 20t)dt$, so we can multiply the inside of the integral by 4 and divide by 4 on the outside.

$$\begin{aligned}\int_1^2 \frac{3t^2+5t}{(4t^3+10t^2-3)^2}\ dt &= \frac{1}{4}\int_1^2 \frac{12t^2+20t}{(4t^3+10t^2-3)^2}\ dt \\ &= \frac{1}{4}\int_a^b \frac{1}{u^2}\ du \\ &= \frac{1}{4}\int_a^b u^{-2}\ du \\ &= \frac{1}{4}\left(-u^{-1}\Big|_a^b\right) \\ &= \frac{1}{4}\left(-\frac{1}{4t^3+10t^2-3}\Big|_1^2\right) \\ &= \frac{1}{4}\left(-\frac{1}{32+40-3} + \frac{1}{4+10-3}\right)\end{aligned}$$

(d) $\displaystyle\int_{1/4}^{1/2} \Big(4s + \pi\cos(\pi s)\Big)\, ds = \int_{1/4}^{1/2} 4s\ ds + \int_{1/4}^{1/2} \pi\cos(\pi s)\ ds.$ In the second integral, let $u = \pi s$; then $du = \pi\ ds$.

$$\begin{aligned}
\int_{1/4}^{1/2} 4s\ ds + \int_{1/4}^{1/2} \pi\cos(\pi s)\ ds &= 2s^2\Big|_{1/4}^{1/2} + \int_{1/4}^{1/2} \cos(\pi s)\cdot\pi\ ds \\
&= \left(2\cdot\frac{1}{4} - 2\cdot\frac{1}{16}\right) + \int_a^b \cos u\ du \\
&= \frac{3}{8} + \left(\sin u\Big|_a^b\right) \\
&= \frac{3}{8} + \left(\sin(\pi s)\Big|_{1/4}^{1/2}\right) \\
&= \frac{3}{8} + \Big(\sin(\pi/2) - \sin(\pi/4)\Big) \\
&= \frac{3}{8} + 1 - \frac{1}{\sqrt{2}}
\end{aligned}$$

3. Find the particular solution $x(t)$ of the following differential equation, and compute $x(5)$.

$$\frac{dx}{dt} = \frac{2t}{3x^2}\ , \qquad x(1) = 3$$

This differential equation is separable (meaning we can separate the two variables x, t):

$$\begin{aligned}
\frac{dx}{dt} &= \frac{2t}{3x^2} \\
3x^2\ dx &= 2t\ dt \\
\int 3x^2\ dx &= \int 2t\ dt \\
x^3 &= t^2 + C
\end{aligned}$$

To solve for C, we use the fact that $x = 3$ when $t = 1$.

$$27 = 1 + C, \qquad \text{so}\ \ C = 26$$

Therefore, the particular solution is obtained by taking the cube root of both sides; i.e.,

$$x(t) = \sqrt[3]{t^2 + 26}$$

and

$$x(5) = \sqrt[3]{51}$$

4. Suppose a watermelon is thrown upward at 6 feet per second from the top of an 80-foot building.

(a) Find the equation for the velocity $v(t)$ of the watermelon at all times and compute $v(1)$. The velocity is obtained by integrating the acceleration. The acceleration due to gravity is 32 feet per second per second, so $a(t) = -32$.

$$v(t) = \int -32\ dt = -32t + C$$

To solve for C, we use the fact that an initial velocity of 6 feet per second means that when $t = 0$, $v = 6$; i.e., $6 = -32(0) + C$. So the velocity is given by

$$v(t) = -32t + 6, \qquad \text{so}\ \ v(1) = -26 \text{ feet per second}$$

(b) Find the equation for the position $s(t)$ of the watermelon at all times and compute $s(1)$. The position is obtained by integrating the velocity.

$$s(t) = \int (-32t + 6)\, dt = -16t^2 + 6t + C$$

To solve for C, we use the fact that when $t = 0$, $s = 80$; i.e., $80 = -16(0) + 6(0) + C$. So the position is given by

$$s(t) = -16t^2 + 6t + 80, \qquad \text{so } s(1) = 70 \text{ feet}$$

(c) When is the maximum height attained and what is the maximum height?
The maximum height is attained when $v(t) = 0$; that is, when $0 = -32t + 6$ or

$$t = \frac{6}{32} \text{ seconds.}$$

To find what the maximum height is, we plug this t value into $s(t)$.

$$\text{max height } = s\left(\frac{6}{32}\right) = -16\left(\frac{6}{32}\right)^2 + 6 \cdot \frac{6}{32} + 80 \text{ feet}$$

(d) When does the watermelon hit the ground below and at what speed does it hit?
The watermelon hits the ground when $s(t) = 0$; that is, when $-16t^2 + 6t + 80 = 0$. We will use the positive solution:

$$t = \frac{-6 - \sqrt{36 - 4(-16)(80)}}{-32} = \frac{-6 - \sqrt{5156}}{-32} \text{ seconds}$$

To find the velocity when it hits the ground, we plug this number into $v(t)$:

$$v\left(\frac{-6 - \sqrt{5156}}{-32}\right) = -32\left(\frac{-6 - \sqrt{5156}}{-32}\right) + 6 = -\sqrt{5156} \text{ feet per second}$$

5. Suppose the acceleration of a particle moving along the x-axis is given by $a(t) = 4 - t$, where t is counted in seconds. Further suppose that the particle satisfies $v(1) = 5$ and $s(2) = 10$. Find the equations of the particle's velocity and position at all times.

The velocity is obtained by integrating the acceleration:

$$v(t) = \int 4 - t\, dt = 4t - \frac{1}{2}t^2 + C$$

To solve for C, we plug in 5 for v and 1 for t.

$$5 = 4 - \frac{1}{2} + C, \qquad \text{or } C = \frac{3}{2}$$

So the velocity equation is

$$v(t) = -\frac{1}{2}t^2 + 4t + \frac{3}{2}$$

To obtain the position, we integrate the velocity:

$$s(t) = \int \left(-\frac{1}{2}t^2 + 4t + \frac{3}{2}\right) dt = -\frac{1}{6}t^3 + 2t + \frac{3}{2}t + C$$

To solve for C, we put $s = 10$ when $t = 2$:

$$10 = -\frac{8}{6} + 8 + \frac{6}{2} + C, \qquad \text{or } C = \frac{1}{3}$$

So the position equation is

$$s(t) = -\frac{1}{6}t^3 + 2t + \frac{3}{2}t + \frac{1}{3}$$

6. Suppose the position (in thousands of miles) of an electron in a(n unorthodox) particle accelerator is given by $s(t) = \sqrt{t^2 + 16}$, where t is measured in seconds.

   (a) Find the average velocity of the particle over the time interval $[0, 3]$.

   The average velocity is calculated by dividing the change in position by the change in time.

$$\text{avg. velocity} = \frac{s(3) - s(0)}{3 - 0} = \frac{5 - 4}{3 - 0} = \frac{1}{3} \text{ thousands of miles per second}$$

   (b) Find a time in between $t = 0$ and $t = 3$ at which the particle has instantaneous velocity equal the average velocity.

   The velocity of the particle at any point in time is

$$v(t) = s'(t) = \frac{d}{dt}\left((t^2 + 16)^{1/2}\right) = \frac{t}{\sqrt{t^2 + 16}}$$

   So we're looking for a point in time where $v(t) = 1/3$.

$$\begin{aligned} \frac{1}{3} &= \frac{t}{\sqrt{t^2 + 16}} \\ \sqrt{t^2 + 16} &= 3t \\ t^2 + 16 &= 9t^2 \\ 16 &= 8t^2 \\ \sqrt{2} &= t \end{aligned}$$

   So after $\sqrt{2}$ seconds, the instantaneous velocity is equal to the average velocity.

7. The Mean Value Theorem states that for a function $f(x)$ that is continuous on an interval $[a, b]$ and differentiable on the interval (a, b), there is at least one value $x = c$ between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   For the function $f(x) = x^2$ defined on the interval $[1, 6]$, find the value c guaranteed by the theorem.

   The right hand side of the equation is

$$\frac{f(b) - f(a)}{b - a} = \frac{6^2 - 1^2}{6 - 1} = 7$$

   Since $f'(x) = x^2$, the left hand side of the equation is

$$f'(c) = 2c$$

   So we are trying to solve $2c = 7$; that is,

$$c = \frac{7}{2}$$

8. Find the tangent line to the curve $y = \int_0^{1/x} \frac{1}{\sqrt{1-t^2}}\, dt$ at the point $(2, \pi/6)$.

The equation of the tangent line is $y - \frac{\pi}{6} = m(x-2)$, where m is the slope of the tangent line. To find the slope of the tangent line, we have to find y' and plug in $x = 2$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \int_0^{1/x} \frac{1}{\sqrt{1-t^2}}\, dt \\ &= \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot \frac{-1}{x^2} \\ &= \frac{-1}{x^2\sqrt{1-\frac{1}{x^2}}} \end{aligned}$$

So at $x = 2$,

$$\frac{dy}{dx} = -\frac{1}{4\sqrt{1-\frac{1}{4}}} = -\frac{1}{2\sqrt{3}}$$

and the equation of the tangent line is

$$y = -\frac{1}{2\sqrt{3}}(x-2) + \frac{\pi}{6}$$

9. For $f(x) = \int_0^x t\sqrt{t+1}\, dt$, find intervals on which $f(x)$ is concave up and concave down.

First, we notice that the domain of $f(x)$ is $[-1, \infty)$ since the square root is undefined at negative values. To find the concavity, we need to look at $f''(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx} \int_0^x t\sqrt{t+1}\, dt \\ f'(x) &= x\sqrt{x+1} \\ f''(x) &= \frac{d}{dx}\left(x\sqrt{x+1}\right) \\ f''(x) &= x\frac{d}{dx}\left(\sqrt{x+1}\right) + \sqrt{x+1} \\ f''(x) &= \frac{x}{2\sqrt{x+1}} + \sqrt{x+1} \\ f''(x) &= \frac{x+2(x+1)}{2\sqrt{x+1}} \\ f''(x) &= \frac{3x+2}{2\sqrt{x+1}} \end{aligned}$$

So the critical value is at $x = -\frac{2}{3}$. When we plug in something between $x = -1$ and $x = -\frac{2}{3}$, we get a negative value, so $f(x)$ is concave down in that interval. When we plug in something greater that $x = -\frac{2}{3}$, we get a positive value. Hence, $f(x)$ is

$$\text{concave down on } \left(-1, -\frac{2}{3}\right) \text{ and concave up on } \left(-\frac{2}{3}, \infty\right)$$

10. Compute the integral $\int_0^5 |3x-6|\ dx$ by graphing the function and finding the area under the curve using geometry.

The graph of $y = |3x-6|$ is a v-shape with a slope of -3 on the left of the vertex and a slope of 3 on the right of the vertex. The vertex is the x-intercept and is located at the point $(2,0)$. So the integral of the function from 0 to 5 is the area of the triangle from 0 to 2 of height 6 plus the area of the triangle from 2 to 5 of height 9. Therefore, the area is

$$A = \frac{1}{2}(2)(6) + \frac{1}{2}(3)(9) = \frac{39}{2}$$

11. Approximate the area under the curve $3x^2+1$ from $x=-1$ to $x=2$ by dividing the interval $[-1,2]$ into 6 pieces and using the right endpoints of the subintervals to determine the height of the rectangles.

Subdividing the interval into 6 pieces yields subintervals of width $= .5$. Since we're using right endpoints, we will use $x_1 = -.5$, $x_2 = 0$, $x_3 = .5$, $x_4 = 1$, $x_5 = 1.5$, and $x_6 = 2$. We let $f(x) = 3x^2+1$, and the approximate area is

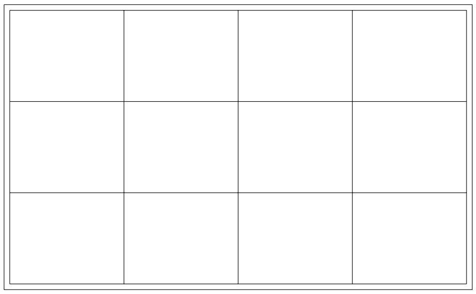
$$\begin{aligned}
Area &\approx f(-.5)(.5) + f(0)(.5) + f(.5)(.5) + f(1)(.5) + f(1.5)(.5) + f(2)(.5) \\
&= \Big(3(-.5)^2+1\Big)(.5) + \Big(3(0)+1\Big)(.5) + \Big(3(.5)^2+1\Big)(.5) \\
&\quad + \Big(3(1)^2+1\Big)(.5) + \Big(3(1.5)^2+1\Big)(.5) + \Big(3(2)^2+1\Big)(.5) \\
&= \frac{7}{8} + \frac{1}{2} + \frac{7}{8} + 2 + \frac{31}{8} + \frac{13}{2} \\
&= \frac{117}{8} \\
&= 14.625
\end{aligned}$$

12. Use the definite integral to find the area under the curve $y = \frac{1}{\sqrt{x}}$ between $x=4$ and $x=64$.

To find the area under the curve, we just integrate the function from 4 to 64.

$$\begin{aligned}
\int_4^{64} \frac{1}{\sqrt{x}}\ dx &= \int_4^{64} x^{-1/2}\ dx \\
&= 2x^{1/2}\Big|_4^{64} \\
&= 2\sqrt{64} - 2\sqrt{4} \\
&= 12
\end{aligned}$$

13. A city planner in a small town wants to put 12 rectangular blocks inside a rectangular region with a total area of 15,000,000 square feet. The city does **not** have to maintain the roads that go around the outside of the rectangular region since they will be state highways. What should the dimensions of one city block be in order to minimize the amount of road to be maintained by the city.



If we label the full vertical piece by x and the full horizontal piece by y, then the amount of road maintained by the city is $R = 3x + 2y$. In addition, the area of the whole town is $xy = 15,000,000$; hence $x = \dfrac{15,000,000}{y}$. We want to minimize R:

$$\begin{aligned} R &= 3\left(\frac{15,000,000}{y}\right) + 2y \\ R &= 45,000,000y^{-1} + 2y \\ R' &= -45,000,000y^{-2} + 2 \\ R' &= \frac{-45,000,000 + 2y^2}{y^2} \end{aligned}$$

So R' has critical values at $y = \pm\sqrt{\frac{45,000,000}{2}}$. The only one we care about is

$$y = \sqrt{\frac{3}{2} \cdot 15,000,000} = \sqrt{22,500,000} = 1000\sqrt{22.5}$$

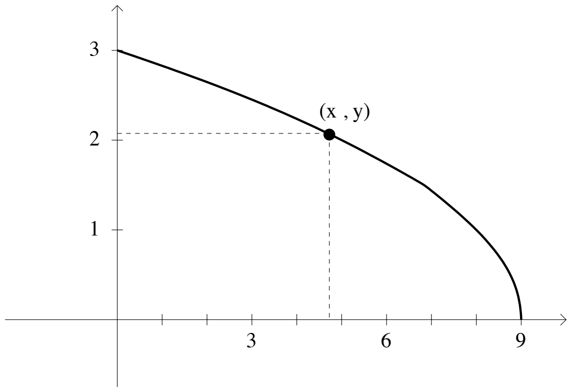
meaning that

$$x = \frac{15,000,000}{\sqrt{\frac{3}{2} \cdot 15,000,000}} = \sqrt{\frac{2}{3} \cdot 15,000,000} = \sqrt{10,000,000} = 1000\sqrt{10}$$

So the dimensions of one city block are

$$\frac{1000}{3}\sqrt{10} \text{ ft} \quad \times \quad 250\sqrt{22.5} \text{ ft}.$$

14. Find the point (x, y) on the curve $y = \sqrt{9-x}$ in the first quadrant such that the rectangle formed has maximal area. Justify (by using the first derivative test or by checking the endpoints of the domain) that the point found yields the maximal area.



Let (x, y) be a point on the curve. Then the area of the rectangle is

$$A = xy.$$

But since the point is on the curve, $y = \sqrt{9-x}$, so the area in terms of x is given by

$$A = x\sqrt{9-x}.$$

This has a natural domain of $[0, 9]$ since values of x outside this domain result in negative area which is nonsensical. We want to maximize this area, so we take the derivative:

$$\begin{aligned} A' &= x\frac{d}{dx}\left(\sqrt{9-x}\right) + \sqrt{9-x}\frac{d}{dx}(x) \\ &= \frac{-x}{2\sqrt{9-x}} + \sqrt{9-x} \\ &= \frac{-x + 2(9-x)}{2\sqrt{9-x}} \\ &= \frac{18-3x}{2\sqrt{9-x}} \end{aligned}$$

So the critical point for A' is $x = 6$. This is a global max since the area for $x = 0$ is 0 and the area for $x = 9$ is 0. Therefore, the point at which maximum area occurs is $(6, \sqrt{3})$ and the maximum area of any rectangle is $A = 6\sqrt{3}$.