Solutions to Review Problems for Exam #2

1. Compute the following derivatives:

(a) $\frac{d}{dt}\left(4t^2\sin t\right) = 4t^2\cdot\frac{d}{dt}(\sin t) + \sin t\cdot\frac{d}{dt}\left(4t^2\right) = 4t^2\cos t + 8t\sin t$

(b) $$\frac{d}{du}\left(\frac{5\tan u}{u^2-3u}\right) = \frac{(u^2-3u)\cdot\frac{d}{du}(5\tan u) - 5\tan u\cdot\frac{d}{du}\left(u^2-3u\right)}{(u^2-3u)^2} = \frac{(u^2-3u)(5\sec^2 u) - 5\tan u(2u-3)}{(u^2-3u)^2}$$

(c) $\frac{d}{dv}\left(6\cos(7v^2+1)\right) = -6\sin(7v^2+1)\cdot\frac{d}{dv}\left(7v^2+1\right) = -6\sin(7v^2+1)(14v)$

(d) $$\frac{d}{dw}\left(\sqrt{\sin^3 w + \frac{8}{w^5}}\right) = \frac{d}{dw}\left(\left(\sin^3 w + 8w^{-5}\right)^{1/2}\right) = \frac{1}{2}\left(\sin^3 w + 8w^{-5}\right)^{-1/2}\cdot\frac{d}{dw}\left(\sin^3 w + 8w^{-5}\right) = \frac{1}{2}\left(\sin^3 w + 8w^{-5}\right)^{-1/2}\left(3\sin^2 w\cdot\cos w - 40w^{-6}\right)$$

(e) $$\frac{d}{dx}\left(\sin 2x\sqrt{x^3-2x}\right) = \sin 2x\cdot\frac{d}{dx}\left(\sqrt{x^3-2x}\right) + \sqrt{x^3-2x}\cdot\frac{d}{dx}(\sin 2x) = \sin 2x\left(\frac{1}{2\sqrt{x^3-2x}}\cdot\frac{d}{dx}\left(x^3-2x\right)\right) + \sqrt{x^3-2x}\cdot\cos 2x\cdot 2 = \sin 2x\left(\frac{3x^2-2}{2\sqrt{x^3-2x}}\right) + \sqrt{x^3-2x}\cdot(2\cos 2x)$$

(f) $$\frac{d}{dy}\left(\frac{2y+4}{(y-1)^5}\right) = \frac{(y-1)^5\cdot\frac{d}{dy}(2y+4) - (2y+4)\cdot\frac{d}{dy}\left((y-1)^5\right)}{(y-1)^{10}} = \frac{(y-1)^5\cdot 2 - (2y+4)\cdot\left(5(y-1)^4\cdot 1\right)}{(y-1)^{10}} = \frac{2(y-1)^5 - 5(2y+4)(y-1)^4}{(y-1)^{10}}$$

(g) $$\frac{d}{dz}\left(\frac{\sqrt{z}\cos z}{z^2+1}\right) = \frac{(z^2+1)\cdot\frac{d}{dz}\left(\sqrt{z}\cos z\right) - \sqrt{z}\cos z\cdot\frac{d}{dz}\left(z^2+1\right)}{(z^2+1)^2} = \frac{(z^2+1)\cdot\left(\sqrt{z}\cdot(-\sin z) + \cos z\cdot\frac{1}{2\sqrt{z}}\right) - \sqrt{z}\cos z\cdot 2z}{(z^2+1)^2}$$

2. Find $\frac{dy}{dx}$ in the following cases:

(a) $x^2y + xy^2 = \tan x$

$$\begin{aligned}
\frac{d}{dx}\left(x^2y\right) + \frac{d}{dx}\left(xy^2\right) &= \frac{d}{dx}(\tan x) \\
\left(x^2 \cdot \frac{dy}{dx} + y \cdot 2x\right) + \left(x \cdot 2y \cdot \frac{dy}{dx} + y^2\right) &= \sec^2 x \\
x^2 \cdot \frac{dy}{dx} + 2xy \cdot \frac{dy}{dx} &= \sec^2 x - 2xy - y^2 \\
\frac{dy}{dx}\left(x^2 + 2xy\right) &= \sec^2 x - 2xy - y^2 \\
\frac{dy}{dx} &= \frac{\sec^2 x - 2xy - y^2}{x^2 + 2xy}
\end{aligned}$$

(b) $\sin(x+y) = \cos(xy)$

$$\begin{aligned}
\frac{d}{dx}(\sin(x+y)) &= \frac{d}{dx}(\cos(xy)) \\
\cos(x+y) \cdot \frac{d}{dx}(x+y) &= -\sin(xy) \cdot \frac{d}{dx}(xy) \\
\cos(x+y) \cdot \left(1 + \frac{dy}{dx}\right) &= -\sin(xy) \cdot \left(x \cdot \frac{dy}{dx} + y\right) \\
\cos(x+y) + \cos(x+y) \cdot \frac{dy}{dx} &= -x\sin(xy) \cdot \frac{dy}{dx} - y\sin(xy) \\
\cos(x+y)\frac{dy}{dx} + x\sin(xy)\frac{dy}{dx} &= -\cos(x+y) - y\sin(xy) \\
\frac{dy}{dx}\big(\cos(x+y) + x\sin(xy)\big) &= -\cos(x+y) - y\sin(xy) \\
\frac{dy}{dx} &= \frac{-\cos(x+y) - y\sin(xy)}{\cos(x+y) + x\sin(xy)}
\end{aligned}$$

(c) $\frac{y^2}{\sqrt{x}} + y = 5$

$$\begin{aligned}
\frac{d}{dx}\left(y^2x^{-1/2}\right) + \frac{d}{dx}(y) &= \frac{d}{dx}(5) \\
\left(y^2 \cdot \left(-\frac{1}{2}x^{-3/2}\right) + x^{-1/2} \cdot \left(2y \cdot \frac{dy}{dx}\right)\right) + \frac{dy}{dx} &= 0 \\
2x^{-1/2}y\frac{dy}{dx} + \frac{dy}{dx} &= \frac{1}{2}x^{-3/2}y^2 \\
\frac{dy}{dx}\left(2x^{-1/2}y + 1\right) &= \frac{1}{2}x^{-3/2}y^2 \\
\frac{dy}{dx} &= \frac{x^{-3/2}y^2}{4x^{-1/2}y + 1}
\end{aligned}$$

3. Find the equation of the tangent line to the following curves at the point $(0, 1)$:

(a) $f(x) = (x+1)^3(x-1)^2$

The equation of the tangent line is of the form $y - 1 = m(x - 0)$ where m is the value of the derivative of $f(x)$ at the point $(0, 1)$.

$$f'(x) = (x+1)^3 \cdot \frac{d}{dx}\left((x-1)^2\right) + (x-1)^2 \cdot \frac{d}{dx}\left((x+1)^3\right) = 2(x+1)^3(x-1) + 3(x+1)^2(x-1)^2$$

so $f'(0) = 2(1)^3(-1) + 3(1)^2(-1)^2 = 1$, and the equation of the tangent line is

$$y = x + 1.$$

(b) $g(x) = \dfrac{x^2+1}{\cos x}$

As in part (a), the equation of the tangent line is $y = mx + 1$ where $m = g'(0)$.

$$g'(x) = \frac{\cos x \frac{d}{dx}\left(x^2+1\right) - (x^2+1)\frac{d}{dx}(\cos x)}{\cos^2 x} = \frac{2x \cos x + (x^2+1)\sin x}{\cos^2 x}$$

so $g'(0) = \dfrac{2(0)(1) + 1(0)}{1^2} = 0$, and the equation of the tangent line is the horizontal line

$$y = 1.$$

(c) $y - \cos(xy) = x$

As in the previous parts, the equation of the tangent line is $y = mx + 1$ where m is $\frac{dy}{dx}$ evaluated at the point $(0, 1)$.

$$\begin{aligned}
\frac{d}{dx}(y) - \frac{d}{dx}(\cos(xy)) &= \frac{d}{dx}(x) \\
\frac{dy}{dx} + \sin(xy) \cdot \frac{d}{dx}(xy) &= 1 \\
\frac{dy}{dx} + \sin(xy)\left(x\frac{dy}{dx} + y\right) &= 1 \\
\frac{dy}{dx} + x\sin(xy)\frac{dy}{dx} &= 1 - y\sin(xy) \\
\frac{dy}{dx} &= \frac{1 - y\sin(xy)}{1 + x\sin(xy)}
\end{aligned}$$

So at $(0, 1)$, $\dfrac{dy}{dx} = \dfrac{1 - 1(0)}{1 + 0(0)} = 1$, and the equation of the tangent line is

$$y = x + 1.$$

4. Recall that the volume of a sphere is given by $V = \dfrac{4}{3}\pi r^3$ and its surface area is given by $SA = 4\pi r^2$. Suppose that a spherical snowball is melting and its volume is decreasing at a rate of 2 cubic centimeters per hour.

   (a) How fast is the radius shrinking when the radius is 5 centimeters?
   We will take the derivative of the equation $V = \frac{4}{3}\pi r^3$ with respect to t and since the volume of the snowball is *decreasing*, we know that $\dfrac{dV}{dt} = -2$ cm^3 per hour.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

   Plugging in $r = 5$ and $\frac{dV}{dt} = -2$ we get $-2 = 4\pi(5)^2\frac{dr}{dt}$; i.e.

$$\frac{dr}{dt} = -\frac{1}{50\pi} = -.00637$$

   So the radius is decreasing at a rate of .00637 centimeters per hour.

   (b) Using the result from part (a), at what rate is the surface area decreasing when the radius is 5 centimeters?
   We will now take the derivative of the equation $SA = 4\pi r^2$ with respect to t.

$$\frac{dSA}{dt} = 8\pi r \frac{dr}{dt}.$$

   Plugging in $r = 5$ and $\frac{dr}{dt} = -\frac{1}{50\pi}$, we get

$$\frac{dSA}{dt} = 8\pi(5)\left(-\frac{1}{50\pi}\right) = -\frac{4}{5}$$

   So the surface area is decreasing at a rate of .8 square centimeters per hour.

5. Oliver is hanging Christmas lights on his house. He is on top of a 10-foot ladder which begins to slip. If the base of the ladder is moving away from the house at a rate of 1 foot per second, how fast is Oliver falling downwards when he is 6 feet in the air?

   Let x be the distance in feet between the base of the ladder and the base of the house and let y be the distance (in feet) between the ground and the top of the ladder where Oliver is. Then x and y are changing in time with $\frac{dx}{dt} = 1$ foot per second. At all points in time, $x^2 + y^2 = 10^2$. If we take the derivative with respect to time, we get

$$\frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right) = \frac{d}{dt}(100)$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

   When Oliver is 6 feet in the air, we have $y = 6$ and $x = \sqrt{100 - 36} = \sqrt{64} = 8$. Plugging these in, we get

$$2(8)(1) + 2(6)\frac{dy}{dt} = 0$$
$$\frac{dy}{dt} = -\frac{16}{12}$$

   so at that point Oliver is *falling* at a rate of $\dfrac{4}{3}$ feet per second.

6. Suppose that the height h (in miles above the atmosphere) of a spacecraft is given by the function $h(t) = t^3 - 6t^2 + 9t$ for $0 \leq t \leq 4$ where t is measured in hours.

   (a) After how many hours does the spacecraft return to 0 miles above the atmosphere?
   Since 0 miles above the atmosphere corresponds to $h = 0$, we need to solve $h(t) = 0$ for t.

   $$h(t) = t(t^2 - 6t + 9) = t(t-3)^2$$

   so the spacecraft returns to 0 miles above the atmosphere after $t = 3$ hours.

   (b) On what time interval(s) is the height of spacecraft increasing? On what time interval(s) is the height of the spacecraft decreasing?
   To find intervals where the height is increasing/decreasing, we need to examine the derivative of $h(t)$.

   $$h'(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t-3)(t-1)$$

   So $h'(t)$ is positive for $t < 1$ and also for $t < 3$, but negative elsewhere. The intervals are then
   increasing: $[0, 1]$, $[3, 4]$
   decreasing: $[1.3]$

   (c) What is the maximum height that the spacecraft attains on the time interval $[0, 4]$?
   There are two critical values for the first derivative: $t = 1$ and $t = 3$. At $t = 1$, the height goes from increasing to decreasing making the local maximum occur at $t = 1$. ($t = 3$ corresponds to a local minimum.) We now just need to check the values of the height at the endpoints versus the local maximum.

   $$h(0) = 0^3 - 6(0)^2 + 9(0) = 0$$
   $$h(1) = 1^3 - 6(1)^2 + 9(1) = 4$$
   $$h(4) = 4^3 - 6(4)^2 + 9(4) = 24$$

   So the maximum height of the spacecraft is 24 miles above the atmosphere, and that max occurs after 4 hours.

7. Let $f(x) = (x+3)^3(x-2)^4$.

   (a) Find the intervals on which $f(x)$ is increasing and on which $f(x)$ is decreasing.
   We need to look at the first derivative.

   $$\begin{aligned} f'(x) &= (x+3)^3 \cdot 4(x-2)^3 + (x-2)^4 \cdot 3(x+3)^2 \\ &= (x+3)^2(x-2)^3\big((4(x+3) + 3(x-2)\big) \\ &= (x+3)^2(x-2)^3(7x+6) \end{aligned}$$

   So $f(x)$ has critical values at $x = -3$, $x = -\frac{6}{7}$, and $x = 2$. For $x < -3$, $f'(x)$ is positive. For $-3 < x < -\frac{6}{7}$, $f'(x)$ is positive; for $-\frac{6}{7} < x < 2$, $f'(x)$ is negative; for $x > 2$, $f'(x)$ is positive. Therefore the intervals are:
   increasing: $\left(-\infty, -\frac{6}{7}\right]$, $[2, \infty)$
   decreasing: $\left[-\frac{6}{7}.2\right)$

(b) Find the local maximums and minimums of $f(x)$.

A local maximum occurs when the derivative changes from positive to negative. So the local maximum of $f(x)$ occurs at $x = -\frac{6}{7}$. The local maximum is

$$f\left(-\frac{6}{7}\right) = \left(-\frac{6}{7}+3\right)^3\left(-\frac{6}{7}-2\right)^4 = 655.703$$

A local minimum occurs when the derivative changes from negative to positive. So the local minimum of $f(x)$ occurs at $x = 2$. The local minimum is

$$f(2) = (2+3)^3(2-2)^4 = 0$$

(c) Find the intervals on which $f(x)$ is concave up and on which $f(x)$ is concave down.

We need to compute the second derivative. We'll use the first-line of the above calculation of $f'(x)$:

$$\begin{aligned}
f'(x) &= 4(x+3)^3(x-2)^3 + 3(x+3)^2(x-2)^4 \\
f''(x) &= \left(4(x+3)^3\frac{d}{dx}\left((x-2)^3\right) + (x-2)^3\frac{d}{dx}\left(4(x+3)^3\right)\right) \\
&\qquad + \left(3(x+3)^2\frac{d}{dx}\left((x-2)^4\right) + (x-2)^4\frac{d}{dx}\left(3(x+3)^2\right)\right) \\
&= 4(x+3)^3\cdot 3(x-2)^2 + (x-2)^3\cdot 12(x+3)^2 + 3(x+3)^2\cdot 4(x-2)^3 + (x-2)^4\cdot 6(x+3) \\
&= 12(x+3)^3(x-2)^2 + 12(x+3)^2(x-2)^3 + 12(x+3)^2(x-2)^3 + 6(x+3)(x-2)^4 \\
&= 6(x+3)(x-2)^2\left(2(x+3)^2 + 2(x+3)(x-2) + 2(x+3)(x-2) + (x-2)^2\right) \\
&= 6(x+3)(x-2)^2(7x^2+12x-2)
\end{aligned}$$

The critical points of $f''(x)$ are $x = -3$, $x = 2$, and (using the quadratic formula)

$$x = \frac{-12 \pm \sqrt{12^2 - 4(7)(-2)}}{2(7)} = \frac{-12 \pm \sqrt{200}}{14} = \frac{-6 \pm 5\sqrt{2}}{7}$$

For $x < -3$, $f''(x)$ is negative, so $f(x)$ is concave down;

for $-3 < x < \dfrac{-6-5\sqrt{2}}{7}$, $f''(x)$ is positive, so $f(x)$ is concave up;

for $\dfrac{-6-\sqrt{2}}{7} < x < \dfrac{-6+\sqrt{2}}{7}$, $f''(x)$ is negative, so $f(x)$ is concave down;

for $\dfrac{-6+\sqrt{2}}{7} < x < 2$, $f''(x)$ is positive, so $f(x)$ is concave up;

for $x > 2$, $f''(x)$ is positive, so $f(x)$ is concave up.

Therefore the intervals are:

concave up: $\left(-3, \dfrac{-6-5\sqrt{2}}{7}\right)$, $\left(\dfrac{-6+5\sqrt{2}}{7}, \infty\right)$

concave down: $(-\infty. -3)$, $\left(\dfrac{-6-5\sqrt{2}}{7}, \dfrac{-6+5\sqrt{2}}{7}\right)$

8. Let $g(x) = 4x^2 - 8x - 1$. Find the global minimum and the global maximum of $g(x)$ on the interval $[0, 3]$.

   First we find the critical values of $g(x)$ by computing $g'(x)$ and finding when $g'(x) = 0$.

   $$g'(x) = 8x - 8 = 8(x - 1)$$

   so $x = 1$ is the only critical value. Now we compare $g(1)$, $g(0)$, and $g(3)$.

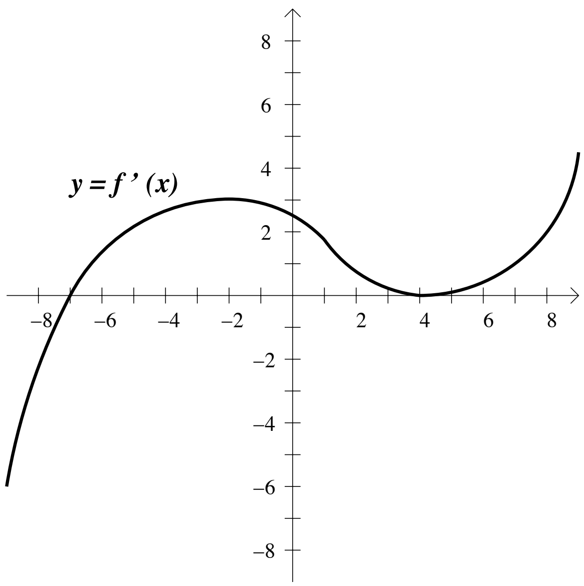
   $$g(0) = 4(0)^2 - 8(0) - 1 = -1$$

   $$g(1) = 4(1)^2 - 8(1) - 1 = -5$$

   $$g(3) = 4(3)^2 - 8(3) - 1 = 11$$

   So the global minimum is -5 and it occurs at $x = 1$, while the global maximum is 11 and it occurs at $x = 3$.

9. Pictured below is the graph of $f'(x)$, the derivative of the function f. Use the graph of $f'(x)$ to answer the following questions.



   (a) On what interval(s) is $f(x)$ increasing?
   $f(x)$ is increasing everywhere $f'(x)$ is positive. $f'(x)$ is positive on the interval $[-7, \infty)$, so $f(x)$ is increasing on $[-7, \infty)$.

   (b) On what interval(s) is $f(x)$ decreasing?
   $f(x)$ is decreasing everywhere $f'(x)$ is negative. $f'(x)$ is negative on the interval $(-\infty, -7]$, so $f(x)$ is decreasing on $(-\infty, -7]$.

   (c) At what x-value(s) does $f(x)$ achieve a local maximum?
   $f(x)$ achieves a local maximum when it changes from increasing to decreasing. This corresponds to the value where $f'(x)$ changes from positive to negative. Therefore, there is no local maximum.

   (d) At what x-value(s) does $f(x)$ achieve a local minimum?
   $f(x)$ achieves a local minimum when it changes from decreasing to increasing. This corresponds to the value where $f'(x)$ changes from negative to positive. Therefore, the local minimum occurs at $x = -7$.