

**Some Solutions to Review Problems
for Exam #1**

1. Determine whether the following functions are even, odd, or neither.
 $e(x) = x^2 + 4x + 8$: $e(x)$ is neither even nor odd since

$$e(-x) = (-x)^2 + 4(-x) + 8 = x^2 - 4x + 8$$

which is not equal to $e(x)$ and is not equal to $-e(x)$.

$f(x) = x \sin x$: $f(x)$ is even since

$$f(-x) = (-x) \sin(-x) = (-x)(-\sin x) = x \sin x = f(x)$$

$g(x) = \sqrt{x^2 + 1}$: $g(x)$ is even since

$$g(-x) = \sqrt{(-x)^2 + 1} = \sqrt{x^2 + 1} = g(x)$$

$h(x) = 2x - \sqrt[3]{x}$: $h(x)$ is odd since

$$\begin{aligned} h(-x) &= 2(-x) - \sqrt[3]{-x} \\ &= -2x - (-\sqrt[3]{x}) \\ &= -(2x - \sqrt[3]{x}) \\ &= -h(x) \end{aligned}$$

2. For $\ell(t) = t^2 + 1$, compute $\ell(\ell(\ell(1)))$.

$$\ell(\ell(\ell(1))) = \ell(\ell(2)) = \ell(5) = 26$$

3. For $f(x) = \sqrt{x+1}$ and $g(x) = x^2 - 1$ compute $g(f(3))$, and $f(g(3))$.

$$g(f(3)) = g(\sqrt{3+1}) = g(2) = 4 - 1 = 3$$

$$f(g(3)) = f(3^2 - 1) = f(8) = \sqrt{8+1} = 3$$

4. The tide in the Bay of Fundy has a tide cycle of 12.4 hours. Suppose that at a certain measuring point, the depth of the water oscillates between 4 meters and 21 meters. If we model the depth of the water t hours from now by the function

$$D(t) = C + A \sin(at),$$

find the appropriate values for the constants C , A , and a .

Since the depth oscillates between 4 meters and 21 meters, then the vertical shift is the average value of 4 and 21; i.e.,

$$C = \frac{21 + 4}{2} = 12.5$$

To find the amplitude, we compute

$$A = \frac{21 - 4}{2} = 8.5$$

To find the value of a , we recall that the period is $2\pi/a$, so

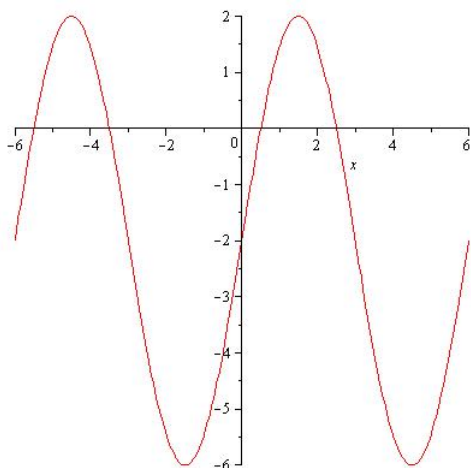
$$12.4 = \frac{2\pi}{a}$$

$$12.4a = 2\pi$$

$$a = \frac{2\pi}{12.4}$$

5. Sketch a graph of $g(x) = -2 + 4 \sin(\frac{\pi}{3}x)$ on the interval $[-6, 6]$.

The sine function is shifted down 2, has an amplitude of 4 and a period of 6.



6. Consider the piecewise function $r(t)$, given below.

$$r(t) = \begin{cases} -2t^2 + 3 & , t \leq -2 \\ 5t - 1 & , -2 < t < 1 \\ 1 & , t = 1 \\ \frac{t^2 + 2t - 3}{t - 1} & , t > 1 \end{cases}$$

Compute the following:

$$\lim_{t \rightarrow -2^-} r(t) = -2(-2)^2 + 3 = -5$$

$$\lim_{t \rightarrow -2^+} r(t) = 5(-2) - 1 = -11$$

$$\lim_{t \rightarrow -2} r(t) \text{ does not exist}$$

$$r(-2) = -5$$

$$\lim_{t \rightarrow 1^-} r(t) = 5(1) - 1 = 4$$

$$\lim_{t \rightarrow 1^+} r(t) = \lim_{t \rightarrow 1} \frac{(t-1)(t+3)}{t-1} = \lim_{t \rightarrow 1} (t+3) = 4$$

$$\lim_{t \rightarrow 1} r(t) = 4$$

$$r(1) = 1$$

7. For the function $f(y) = \frac{2y^2 - 3y - 9}{y^2 - y - 6}$, compute the following:

$$\lim_{y \rightarrow 3} f(y) = \lim_{y \rightarrow 3} \frac{(2y+3)(y-3)}{(y-3)(y+2)} = \lim_{y \rightarrow 3} \frac{2y+3}{y+2} = \frac{9}{5}$$

$$\lim_{y \rightarrow -2^-} f(y) = \lim_{y \rightarrow -2^-} \frac{2y+3}{y+2} = +\infty \text{ since numerator } \rightarrow 7 \text{ and denominator is positive and } \rightarrow 0.$$

$$\lim_{y \rightarrow -2^+} f(y) = \lim_{y \rightarrow -2^+} \frac{2y+3}{y+2} = -\infty \text{ since numerator } \rightarrow 7 \text{ and denominator is negative and } \rightarrow 0.$$

$$\lim_{y \rightarrow \infty} f(y) = \lim_{y \rightarrow \infty} \frac{2y+3}{y+2} = \lim_{y \rightarrow \infty} \frac{2 + \frac{3}{y}}{1 + \frac{2}{y}} = \frac{2+0}{1+0} = 2$$

8. Compute the following limits:

(a)

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{4\sqrt{x}}{\sqrt{2x+1} - \sqrt{x}} &= \lim_{x \rightarrow \infty} \frac{4\sqrt{x}}{\sqrt{2x+1} - \sqrt{x}} \cdot \frac{\sqrt{2x+1} + \sqrt{x}}{\sqrt{2x+1} + \sqrt{x}} \\ &= \lim_{x \rightarrow \infty} \frac{4\sqrt{x(2x+1)} + 4\sqrt{x^2}}{2x+1-x} \\ &= \lim_{x \rightarrow \infty} \frac{4\sqrt{2x^2+x} + 4x}{x+1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\ &= \lim_{x \rightarrow \infty} \frac{4\sqrt{2 + \frac{1}{x}} + 4}{1 + \frac{1}{x}} \\ &= \frac{4\sqrt{2+0} + 4}{1+0} \\ &= 4\sqrt{2} + 4 \end{aligned}$$

(b)

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+x}}{3x} &= \lim_{x \rightarrow -\infty} \frac{1}{3} \cdot \frac{\sqrt{x^2+x}}{x} \\ &= \lim_{x \rightarrow -\infty} \frac{1}{3} \cdot \frac{\sqrt{x^2+x}}{-\sqrt{x^2}} \\ &= \lim_{x \rightarrow -\infty} -\frac{1}{3} \cdot \sqrt{\frac{x^2+x}{x^2}} \\ &= \lim_{x \rightarrow -\infty} -\frac{1}{3} \cdot \sqrt{1 + \frac{1}{x}} \\ &= -\frac{1}{3} \cdot \sqrt{1+0} \\ &= -\frac{1}{3} \end{aligned}$$

(c)

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x}{\tan x} &= \lim_{x \rightarrow 0} \frac{x}{\frac{\sin x}{\cos x}} \\ &= \lim_{x \rightarrow 0} x \cdot \frac{\cos x}{\sin x} \\ &= \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot \cos x \\ &= \lim_{x \rightarrow 0} \frac{1}{\frac{\sin x}{x}} \cdot \cos x \\ &= \frac{1}{1} \cdot 1 \\ &= 1\end{aligned}$$

(d)

$$\lim_{x \rightarrow 0} \frac{1 - \cos 5x}{3x} = \lim_{x \rightarrow 0} \frac{5}{3} \cdot \frac{1 - \cos 5x}{5x} = \frac{5}{3} \cdot 0 = 0$$

9. Let $a(x) = \frac{(x-1)(x^2-1)(3x+2)}{(x-1)(x+3)(x^2+1)}$.

(a) For which values of x is $a(x)$ discontinuous? State whether each discontinuity is a hole or an asymptote and if it is a hole, compute the y -value of the hole.

The function $a(x)$ is discontinuous at $x = 1$ and $x = -3$, since those values make the denominator 0. The discontinuity at $x = 1$ is a hole, since there is at least one factor of $x - 1$ in the numerator. The discontinuity at $x = -3$ is a vertical asymptote since there is no factor of $x + 3$ in the numerator. To find the y -value of the hole, we compute

$$\lim_{x \rightarrow 1} a(x) = \lim_{x \rightarrow 1} \frac{(x^2-1)(3x+2)}{(x+3)(x^2+1)} = \frac{(1^2-1)(3(1)+2)}{(1+3)(1^2+1)} = 0$$

so the hole is located at the point $(1, 0)$.

(b) Find the horizontal asymptote of $a(x)$ by computing $\lim_{x \rightarrow \infty} a(x)$.

$$\lim_{x \rightarrow \infty} a(x) = \lim_{x \rightarrow \infty} \frac{(x^2-1)(3x+2)}{(x+3)(x^2+1)} = \lim_{x \rightarrow \infty} \frac{3x^3 + 2x^2 - 3x - 2}{x^3 + 3x^2 + x + 3} = 3$$

so $a(x)$ has a horizontal asymptote at $y = 3$.

(c) Find the y -intercept of $y = a(x)$.

To find the y -intercept, we set $x = 0$ in $a(x)$.

$$a(0) = \frac{(-1)(-1)(2)}{(-1)(3)(1)} = -\frac{2}{3}$$

so $y = a(x)$ has a y -intercept at $y = -2/3$.

(d) Find the x -intercept(s) of $y = a(x)$.

To find the x -intercepts, we set $y = 0$. But $a(x) = 0$ if and only if the numerator is 0. The points where $(x-1)(x^2-1)(3x+2) = 0$ are $x = 1$, $x = -1$, and $x = -2/3$. But the point $x = 1$ is a discontinuity, so it is not an x -intercept. The x -intercepts are $x = -1$ and $x = -2/3$.

10. Use the Intermediate Value Theorem to show that the function $f(x) = x^5 - 4x^3 + x^2 + 1$ has a zero in the interval $[0, 1]$.

First we check the values of $f(x)$ at the endpoints of the interval.

$$f(0) = 1, \quad f(1) = -1$$

Since $f(0)$ is positive and $f(1)$ is negative, then somewhere between 0 and 1 $f(x)$ must be 0.

11. Let

$$p(y) = \begin{cases} y^3 - 1 & , y \leq 1 \\ 4y + A & , y < 1 \end{cases}$$

For what value of the constant A will $p(y)$ be a continuous function?

The function $p(y)$ is continuous everywhere except possibly at $y = 1$. In order for the function to be continuous at $y = 1$, we need $\lim_{y \rightarrow 1^-} p(y) = \lim_{y \rightarrow 1^+} p(y)$; that is, we need $1^3 - 1 = 4(1) + A$.

This happens when $A = -4$. So for the value $A = -4$, the function $p(y)$ is continuous.

12. Let

$$q(t) = \begin{cases} t^2 + 1 & , t \leq A \\ 3t + 11 & , t > A \end{cases}$$

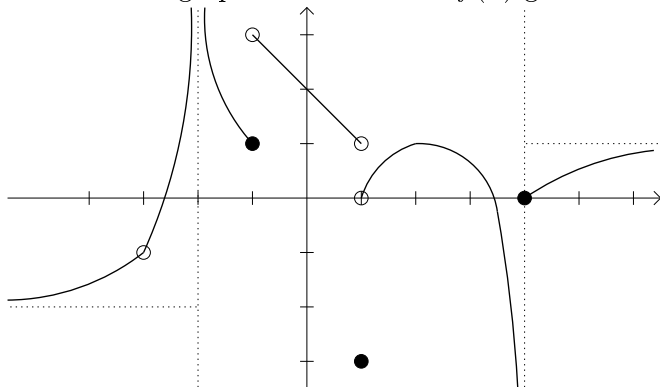
For what value of the constant A will $q(t)$ be a continuous function?

In order for the function $q(t)$ to be at $t = A$, we need $\lim_{t \rightarrow A^-} q(t) = \lim_{t \rightarrow A^+} q(t)$; that is, we need $A^2 + 1 = 3A + 11$, or when

$$0 = A^2 - 3A - 10 = (A - 5)(A + 2)$$

So $q(t)$ will be continuous if $A = 5$ or if $A = -2$.

13. Consider the graph of the function $f(x)$ given below.



Compute the following limits:

$$\begin{array}{ll} \lim_{x \rightarrow -3} f(x) = -1 & \lim_{x \rightarrow -2} f(x) = \infty \\ \lim_{x \rightarrow -1^-} f(x) = 1 & \lim_{x \rightarrow -1^+} f(x) = 3 \\ \lim_{x \rightarrow 1^-} f(x) = 1 & \lim_{x \rightarrow 1^+} f(x) = 0 \\ \lim_{x \rightarrow 4^-} f(x) = -\infty & \lim_{x \rightarrow 4^+} f(x) = 0 \\ \lim_{x \rightarrow -\infty} f(x) = -2 & \lim_{x \rightarrow \infty} f(x) = 1 \end{array}$$