

Exam #3

1. Find the area under the curve $y = x\sqrt{4-x^2}$ between $x = 0$ and $x = 2$.

$$\begin{aligned}\text{Area} &= \int_0^2 x\sqrt{4-x^2} dx && \text{let } u = 4-x^2 \text{ and } du = -2x dx \\ &= -\frac{1}{2} \int_0^2 \sqrt{4-x^2} \cdot (-2x) dx \\ &= -\frac{1}{2} \int_a^b u^{1/2} du \\ &= -\frac{1}{2} u^{3/2} \cdot \frac{2}{3} \Big|_a^b = -\frac{1}{3} (4-x^2)^{3/2} \Big|_0^2 \\ &= -\frac{1}{3} (0)^{3/2} - \left(-\frac{1}{3} (4)^{3/2} \right) = \frac{8}{3}\end{aligned}$$

2. Find the equation of the tangent line at the point $(20, 1.5)$ to the curve $y = \int_1^{\sqrt{x}} \frac{1}{t} dt$.

The equation of the tangent line is $y - 1.5 = m(x - 20)$ where m is the slope of the tangent line; that is, $m = y'(20)$.

$$y' = \frac{d}{dx} \int_1^{\sqrt{x}} \frac{1}{t} dt = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}$$

So $y'(20) = 1/40$ and the equation of the tangent line is

$$y = \frac{1}{40}(x - 20) + 1.5$$

3. Find the particular solution $r(t)$ to the following differential equation:

$$\frac{dx}{dt} = 2t x^2, \quad r(0) = -1$$

We separate variables first: $x^{-2} dx = 2t dt$. Then we integrate both sides

$$-x^{-1} = t^2 + C$$

Then we use the fact that $x = -1$ when $t = 0$ to get $C = -(-1)^{-1} = 1$. Then we solve for x :

$$\begin{aligned}-\frac{1}{x} &= t^2 + 1 \\ -x &= \frac{1}{t^2 + 1} \\ x &= -\frac{1}{t^2 + 1}\end{aligned}$$

4. Evaluate the following indefinite integral:

$$\int \sec^2(y^3) \cdot 3y^2 dy.$$

Let $u = y^3$; then $du = 3y^2 dy$, so

$$\int \sec^2(y^3) \cdot 3y^2 dy = \int \sec^2 u du = \tan u + C = \tan(y^3) + C$$

5. Suppose the acceleration (in feet per second per second) is given by the equation $a(t) = 6t - 2$. Further suppose that the velocity satisfies $v(1) = 0$ and the position satisfies $s(0) = 1$.

(a) Find the equations for $v(t)$ and for $s(t)$.

$$v(t) = \int 6t - 2 dt$$

$$v(t) = 3t^2 - 2t + C \quad \text{using } v(1) = 0 \text{ we get } C = -1$$

$$v(t) = 3t^2 - 2t - 1$$

$$s(t) = \int 3t^2 - 2t - 1 dt$$

$$s(t) = t^3 - t^2 - t + C \quad \text{using } s(0) = 1 \text{ we get } C = 1$$

$$s(t) = t^3 - t^2 - t + 1$$

(b) Compute the average velocity over the time interval $[0, 1]$.

$$\text{avg. velocity} = \frac{s(1) - s(0)}{1 - 0} = -1 \text{ ft/sec}$$

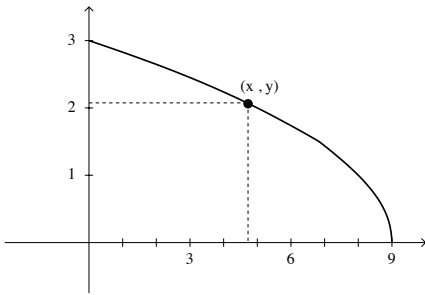
(c) Find the point in time in the open interval $(0, 1)$ where the instantaneous velocity is equal to the average velocity over the time interval $[0, 1]$.

We solve $v(t) = -1$; i.e. $3t^2 - 2t - 1 = -1$;

$$0 = 3t^2 - 2t = t(3t - 2)$$

so at $t = 2/3$ the instantaneous velocity is equal to the average over the whole interval.

6. Find the point (x, y) on the curve $y = \sqrt{9 - x}$ in the first quadrant such that the rectangle formed has maximal area.



The area is given by $A = xy$ and that is the quantity that we want to maximize. We need to eliminate a variable. The easiest one to eliminate is $x = 9 - y^2$, so $A = 9y - y^3$. Then $A' = 9 - 3y^2$ which has critical value in the first quadrant of $y = \sqrt{3}$. A has a local maximum here since $A'' = -6y$ is always negative for $y > 0$. The corresponding x -value to this is $x = 9 - (\sqrt{3})^2 = 9 - 3 = 6$. So the point which gives maximal area is $(6, \sqrt{3})$.

7. (a) Estimate the definite integral $\int_0^1 8x dx$ by using 4 rectangles and by using the right-hand endpoints of the 4 subintervals to determine the heights of the rectangles.

The right endpoints of the subintervals are $x_1 = .25$, $x_2 = .5$, $x_3 = .75$, and $x_4 = 1$. The height of each of these rectangles is the $8x_k$ since the function being integrated is $y = 8x$. The width of each of these rectangles is $.25$. Therefore, the area under the curve is approximately

$$8(.25)(.25) + 8(.5)(.25) + 8(.75)(.25) + 8(1)(.25) = .5 + 1 + 1.5 + 2 = 5$$

- (b) Evaluate the definite integral $\int_0^1 8x dx$.

$$\int_0^1 8x dx = 4x^2 \Big|_0^1 = 4$$