Exam #2 Solutions

1. Compute the following derivative:

$$\begin{aligned}\frac{d}{dt}\left(2t^{-3/2}\sin 5t + 1500\pi\right) &= 2t^{-3/2}\frac{d}{dt}(\sin 5t) + \sin 5t\frac{d}{dt}\left(2t^{-3/2}\right)\\ &= 2t^{-3/2}\cdot 5\cos 5t + \sin 5t \cdot \left(-3t^{-5/2}\right)\\ &= 10t^{-3/2}\cos 5t - 3t^{-5/2}\sin 5t\end{aligned}$$

2. Let $g(x) = 3x^5 + 15x^4 - 40x^3$. Find the interval(s) on which $g(x)$ is concave up and the interval(s) on which $g(x)$ is concave down. Identify the x-values at which $g(x)$ has a point of inflection.

   To find the concavity, we consider the second derivative:

$$\begin{aligned}g(x) &= 3x^5 + 15x^4 - 40x^3\\ g'(x) &= 15x^4 + 60x^3 - 120x^2\\ g''(x) &= 60x^3 + 180x^2 - 240x\\ g''(x) &= 60x(x-1)(x+4)\end{aligned}$$

   So the critical values are $x = -4, 0, 1$. For $x < -4$, $g''(x) < 0$; for $-4 < x < 0$, $g''(x) > 0$; for $0 < x < 1$, $g''(x) < 0$; for $x > 1$, $g''(x) > 0$. Therefore $g(x)$ is

$$\text{concave down: } (-\infty, -4) \cup (0, 1)$$

$$\text{concave up: } (-4, 0) \cup (1, \infty)$$

   and $g(x)$ has points of inflection at $x = -4, 0, 1$.

3. Find the global maximum and global minimum (the y-values) of the function $q(x) = \dfrac{x^2-1}{x^2+1}$ over the interval $[-2, 3]$.

   First we find the critical values of $q'(x)$.

$$q'(x) = \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

   so the only critical value is $x = 0$. Now we compare against the endpoints:

$$q(-2) = \frac{3}{5} \qquad q(0) = -1 \qquad q(3) = \frac{3}{5}$$

   So the global minimum is $y = -1$ and the global maximum is $y = 3/5$.

4. A 75-foot robot dinosaur is moving at 11 feet per second towards a 100-foot lamp. How fast is the tip of the shadow of the robot moving toward the lamp?

   Let x denote the distance from the lamp to the robot dinosaur and let y denote the distance from the lamp to the tip of the shadow. Then $dx/dt = -11$ and we want to find dy/dt. There are similar triangles if you draw a picture of the scenario. The large triangle is the one made with the lamp and the shadow, so it has a height of 100 feet and a base of y feet. The smaller triangle is the one made with the robot dinosaur and the shadow, so it has a height of 75 feet and a base of $y - x$ feet. By similar triangles,

$$\frac{y}{100} = \frac{y-x}{75} \qquad \text{that is,} \qquad 100x = 25y.$$

   Differentiating with respect to t, we get

$$100\frac{dx}{dt} = 25\frac{dy}{dt}$$

   but since $dx/dt = -11$, then

$$\frac{dy}{dt} = -44 \text{ feet per second}$$

5. Find the equation of the tangent line at $(1,3)$ to the curve

$$y^2 - 5x - y\cos(x-1) = 1$$

The equation of the tangent line is given by $y - 3 = m(x-1)$ where m is the slope of the line. To find m, we need to compute dy/dx at the point $(1,3)$. We differentiate implicitly with respect to x and then plug in $x = 1$, $y = 3$:

$$\begin{aligned} \frac{d}{dx}\left(y^2 - 5x - y\cos(x-1)\right) &= \frac{d}{dx}(1) \\ 2y\frac{dy}{dx} - 5 - \left(-y\sin(x-1) + \cos(x-1)\frac{dy}{dx}\right) &= 0 \\ 6\frac{dy}{dx} - 5 + 3\sin(0) - \cos(0)\frac{dy}{dx} &= 0 \\ 6\frac{dy}{dx} - 5 - \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= 1 \end{aligned}$$

so $m = 1$ and the equation is $y = x + 2$.

6. If $\ell(x) = \sqrt{\dfrac{\tan x}{x^3}}$, compute $\ell'(x)$.

$$\ell'(x) = \frac{1}{2}\left(\frac{\tan x}{x^3}\right)^{-1/2} \cdot \frac{x^3\sec^2 x - 3x^2\tan x}{x^6}$$

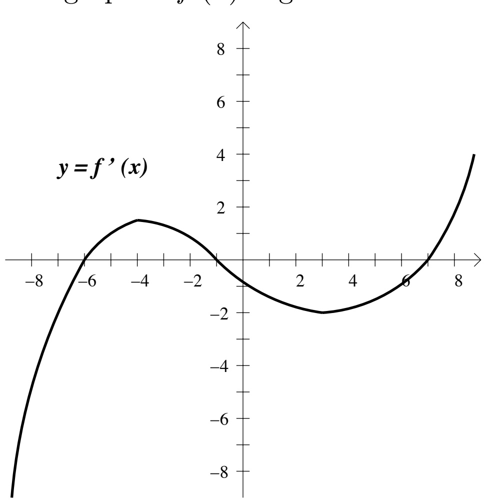
7. For the function $y = x^8 - 2x^4$, find the x-values where the tangent line to the function is a horizontal line.

We need to find when $y' = 0$.

$$y' = 8x^7 - 8x^3 = 8x^3(x^4 - 1) = 8x^3(x^2+1)(x-1)(x+1)$$

so the tangent line is horizontal at $x = -1, 0, 1$.

8. The graph is $f'(x)$ is given below.



(a) On what interval(s) is $f(x)$ increasing? $(-6,-1)\cup(7,\infty)$

(b) For what x-value(s) does $f(x)$ attain a local minimum? $x = -6$ and $x = 7$

(c) For what x-value(s) does $f(x)$ have a point of inflection? $x = -4$ and $x = 3$