Computing the area of a region in the plane

Instructions: There will be a question on the exam where you are asked to compute the area of a region in the plane. The following 5 questions are similar to what you might see. Solutions are at the end.

Problems

1. Find the area of the region in the plane bounded by the line $y = x - 1$ and the parabola $y = x^2 - 4x + 3$.

2. Find the area of the region in the plane bounded by the line parabolas $y = x^2 - 9$ and $y = -x^2 + 2x + 15$.

3. Find the area of the region in the plane bounded by the parabola $y = x^2 + 2$, the line $y = -x$ and the vertical lines $x = -2$ and $x = 2$.

4. Use integration to find the area of the quadrilateral in the plane bounded by the lines $x = -2$, $y = 10 - x$, $y = \frac{1}{2}x - 9$, and $y = 3x - 14$.

5. Find the area of the region bounded by the curve $x = y^2 - 2y$ and the line $x - y - 4 = 0$.

Solutions

1. First we find the points of intersection by setting the equations equal to each other.

$$\begin{aligned} x-1 &= x^2-4x+3 \\ 0 &= x^2-5x+4 \\ 0 &= (x-4)(x-1) \end{aligned}$$

So the curves intersect at the points $(1,0)$ and $(4,3)$. Since these are the only two points where the curves cross, one of the curves must be higher than the other curve on the interval $(1,4)$. If we pick a value like $x=2$ in the interval, the y-value of the line is $y=2-1=1$ while the y-value of the parabola is $y=4-8+3=-1$. So the line is above the parabola. The total area is found by integrating the distance between the curves from 1 to 4. That is, we integrate the bottom function subtracted from the top function.

$$\begin{aligned} \text{Area} &= \int_1^4 \Big(\text{top function} \;-\; \text{bottom function}\Big)\,dx \\ &= \int_1^4 \Big((x-1)-(x^2-4x+3)\Big)\,dx \\ &= \int_1^4 (-x^2+5x-4)\,dx \\ &= -\frac{x^3}{3}+\frac{5x^2}{2}-4x\Big|_1^4 \\ &= \Big(-\frac{64}{3}+40-16\Big)-\Big(-\frac{1}{3}+\frac{5}{2}-4\Big) \\ &= \frac{9}{2} \end{aligned}$$

2. First we find the points of intersection.

$$\begin{aligned} x^2-9 &= -x^2+2x+15 \\ 2x^2-2x-24 &= 0 \\ 2(x^2-x-12) &= 0 \\ 2(x-4)(x+3) &= 0 \\ x = -3 &\text{ and } x=4 \end{aligned}$$

Next we figure out which function is the above the other. The easiest is to plug in $x=0$. The first parabola given by $y=x^2-9$ has $y=-9$ at $x=0$ while the second parabola given by $y=-x^2+2x+15$ has $y=15$ and $x=0$. So $y=-x^2+2x+15$ is above $y=x^2-9$ on the interval $(-3,4)$. Therefore,

$$\begin{aligned} \text{Area} &= \int_{-3}^4 \Big((-x^2+2x+15)-(x^2-9)\Big)\,dx \\ &= \int_{-3}^4 (-2x^2+2x+24)\,dx \\ &= \frac{-2x^3}{3}+x^2+24x\Big|_{-3}^4 \\ &= \Big(-\frac{128}{3}+16+96\Big)-(18+9-72) \\ &= 168-\frac{16}{3} \end{aligned}$$

3. First we notice that the parabola and the line do not intersect in the interval $(-2, 2)$, by drawing the graph or thinking really hard about it. In fact, the graph of the parabola $y = x^2 + 2$ is always above the line $y = -x$. Therefore,

$$\begin{aligned}
\text{Area} &= \int_{-2}^{2} \Big((x^2 + 2) - (x) \Big) \, dx \\
&= \int_{-2}^{2} (x^2 - x + 2) \, dx \\
&= \frac{x^3}{3} - \frac{x^2}{2} + 2x \Big|_{-2}^{2} \\
&= \left(\frac{8}{3} - 2 + 4 \right) - \left(-\frac{8}{3} - 2 - 4 \right) \\
&= \frac{16}{3} + 8
\end{aligned}$$

4. I don't have access to my program for drawing pictures, so I will try to describe the picture. However, it is very easy to graph these lines. I'll label the lines as follows:

Line #1: $x = -2$
Line #2: $y = 10 - x$
Line #3: $y = 3x - 14$
Line #4: $y = \frac{1}{2}x - 9$

Line #1 and Line #2 intersect at the point $(-2, 12)$. Line #2 and Line #3 intersect when

$$10 - x = 3x - 14, \qquad \text{that is, when } x = 6,$$

so Line #2 and Line #3 intersect at the point $(6, 4)$. Line #3 and Line #4 intersect when

$$\frac{1}{2}x - 9 = 3x - 14, \qquad \text{that is, when } x = 2,$$

so Line #3 and Line #4 intersect at the point $(2, -8)$. Finally, Line #4 and Line #1 intersect at the point $(-2, -10)$.

To find the area of this region, we notice that Line #2 is above Line #4 on the interval $[-2, 2]$ and that Line #2 is above Line #3 on the interval $[2, 6)$. Therefore, the area is the sum of two integrals

$$\begin{aligned}
\text{Area} &= \int_{-2}^{2} \Big(\text{Line \#2} \; - \; \text{Line \#4} \Big) \, dx + \int_{2}^{6} \Big(\text{Line \#2} \; - \; \text{Line \#3} \Big) \, dx \\
&= \int_{-2}^{2} \left((10 - x) - \left(\frac{1}{2}x - 9 \right) \right) \, dx + \int_{2}^{6} \Big((10 - x) - (3x - 14) \Big) \, dx \\
&= \int_{-2}^{2} \left(-\frac{3}{2}x + 19 \right) \, dx + \int_{2}^{6} (-4x + 24) \, dx \\
&= \left(-\frac{3}{4}x^2 + 19x \Big|_{-2}^{2} \right) + \left(-2x^2 + 24x \Big|_{2}^{6} \right) \\
&= \Big((-3 + 38) - (-3 - 38) \Big) + \Big((-72 + 144) - (-8 + 48) \Big) \\
&= 76 + 32 \\
&= 108
\end{aligned}$$

5. This one will be easier to do in terms of y. So we find the points of intersection by setting $x = y^2 - 2y$ equal to $x = y + 4$.

$$\begin{aligned} y^2 - 2y &= y + 4 \\ y^2 - 3y - 4 &= 0 \\ (y-4)(y+1) &= 0 \\ y = -1 \text{ and } y &= 4 \end{aligned}$$

Since we're going to integrate with respect to y, we need to know which function is larger in terms of y. On the graph this corresponds to the right-most curve. But we can do what we did before and plug in $y = 0$ into the two functions to see that the line is the right-most function.

$$\begin{aligned} \text{Area} &= \int_{-1}^{4} \Big((y+4) - (y^2 - 2y)\Big)\, dy \\ &= \int_{-1}^{4} (-y^2 + 3y + 4)\, dy \\ &= -\frac{y^3}{3} + \frac{3y^2}{2} + 4x\Big|_{-1}^{4} \\ &= \left(-\frac{64}{3} + 24 + 16\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right) \\ &= 44 - \frac{3}{2} - \frac{65}{3} \end{aligned}$$