Homework #7 Solutions

9.1.6:

$$a_n = \frac{\sqrt{3n^2+2}}{2n+1} = \frac{\sqrt{3n^2+2}\cdot\sqrt{\frac{1}{n^2}}}{(2n+1)\cdot\left(\frac{1}{n}\right)} = \frac{\sqrt{3+\frac{2}{n^2}}}{2+\frac{1}{n}}$$

so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{3+\frac{2}{n^2}}}{2+\frac{1}{n}} = \boxed{\frac{\sqrt{3}}{2}}$

9.1.8:

$a_n = \frac{n\cos(n\pi)}{2n-1}$. As $n\to\infty$, $\frac{n}{2n-1} \to \frac{1}{2}$, but

$\cos(n\pi)$ bounces back and forth between $-1, 1$

So a_n bounces back and forth close to $-\frac{1}{2}, \frac{1}{2}$.

$\Rightarrow \{a_n\}$ $\boxed{\text{diverges.}}$

9.1.12:

$a_n = \frac{e^{2n}}{4^n} = \left(\frac{e^2}{4}\right)^n$ $\boxed{\text{diverges}}$ since $\frac{e^2}{4} > 1$.

9.1.16:

$$a_n = \frac{n^{100}}{e^n}$$

$$\lim_{x\to\infty} \frac{x^{100}}{e^x} \overset{L}{=} \cdots \overset{L}{=} \lim_{x\to\infty} \frac{100!}{e^x} = 0.$$

So $\boxed{\{a_n\} \longrightarrow 0}$

9.1.22: $\frac{1}{2^2}, \frac{2}{2^3}, \frac{3}{2^4}, \frac{4}{2^5}, \ldots$

$\boxed{a_n = \frac{n}{2^{n+1}}}$ $\boxed{\text{converges to } 0}$

since $\lim_{x\to\infty} \frac{x}{2^{x+1}} \overset{L}{=} \lim_{x\to\infty} \frac{1}{\ln 2 \cdot 2^{x+1}} = 0.$

9.1.26: $\frac{1}{2-\frac{1}{2}}, \frac{2}{3-\frac{1}{3}}, \frac{3}{4-\frac{1}{4}}, \frac{4}{5-\frac{1}{5}}, \ldots$

$$a_n = \frac{n}{n+1 - \frac{1}{n+1}} = \frac{n(n+1)}{(n+1)^2 - 1} = \boxed{\frac{n^2+n}{n^2+2n}}$$

so $\boxed{\{a_n\} \longrightarrow 1.}$

9.1.34: $a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$.

$a_1 = 1$

$a_2 = 1 + \frac{1}{2}$

$a_3 = 1 + \frac{1}{2} + \frac{1}{6}$

$a_4 = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$

$\{a_n\}$ is an increasing sequence since

$$a_{n+1} = a_n + \frac{1}{(n+1)!} > a_n.$$

So If we can find an upper bound for $\{a_n\}$, by the Monotonic Sequence Theorem it will converge.

Notice that

$$k! = k(k-1)(k-2) \cdots (4)(3)(2)$$

$$k! \geq 2(2)(2) \cdots (2)(2)(2)$$

$$k! \geq 2^{k-1} \quad \text{for all } k$$

Therefore

$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 2$$

for all n, and so $\{a_n\}$ converges to $A \leq 2$.