Homework #6 Solutions

8.1.10: $\displaystyle \lim_{x\to 0} \frac{e^x - e^{-x}}{2\sin x} \overset{L}{=} \lim_{x\to 0} \frac{e^x + e^{-x}}{2\cos x} = 1.$

8.1.12: $\displaystyle \lim_{x\to 0^+} \frac{7^{\sqrt{x}} - 1}{2^{\sqrt{x}} - 1} \overset{L}{=} \lim_{x\to 0} \frac{7^{\sqrt{x}} \cdot \ln 7 \cdot \frac{1}{2\sqrt{x}}}{2^{\sqrt{x}} \cdot \ln 2 \cdot \frac{1}{2\sqrt{x}}} = \frac{\ln 7}{\ln 2}.$

8.1.22: $\displaystyle \lim_{x\to 0^-} \frac{\sin x + \tan x}{e^x + e^{-x} - 2} \overset{L}{=} \lim_{x\to 0^-} \frac{\cos x + \sec^2 x}{e^x - e^{-x}} = -\infty.$

8.2.4: $\displaystyle \lim_{x\to \infty} \frac{3x}{\ln(100x + e^x)} \overset{L}{=} \lim_{x\to\infty} \frac{3}{\frac{100 + e^x}{100x + e^x}}$

$$= \lim_{x\to\infty} 3 \, \frac{100x + e^x}{100 + e^x}$$

$$\overset{L}{=} \lim_{x\to\infty} 3 \, \frac{100 + e^x}{e^x} = 3.$$

8.2.6: $\displaystyle \lim_{x\to 0^+} \frac{\ln \sin^2 x}{3 \ln \tan x} \overset{L}{=} \lim_{x\to 0} \frac{\frac{2\sin x \cos x}{\sin^2 x}}{3 \, \frac{\sec^2 x}{\tan x}}$

$$= \lim_{x\to 0} \frac{2}{3} \, \frac{\cos x}{\sin x} \cdot \sin x \cos x = \frac{2}{3}.$$

[redacted]

8.2.12: $\lim_{x\to 0} 3x^2 \csc^2 x = \lim_{x\to 0} \frac{3x^2}{\sin x} \overset{L}{=} \lim_{x\to 0} \frac{6x}{\cos x} = 0.$

8.2.14: $\lim_{x\to \pi/2} (\tan x - \sec x) = \lim_{x\to\pi/2} \left(\frac{\sin x - 1}{\cos x} \right)$

$\overset{L}{=} \lim_{x\to\pi/2} \frac{\cos x}{-\sin x} = 0.$

8.2.30: $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$:

Let $y = \left(1+\frac{1}{x}\right)^x$, i.e., $\ln y = x \ln\left(1+\frac{1}{x}\right)$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.$$

So $\lim_{x\to\infty} y = e^1 = e \Rightarrow \boxed{\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e.}$

8.2.36: $\lim_{x\to\infty} [\ln(x+1) - \ln(x-1)] = \lim_{x\to\infty} \ln\left(\frac{x+1}{x-1}\right)$

$$= \ln\left(\lim_{x\to\infty} \frac{x+1}{x-1}\right)$$

$$= \ln 1$$

$$= 0.$$

8.3.2: $\displaystyle\int_{-\infty}^{-5} \frac{dx}{x^4} = \lim_{a\to-\infty} \frac{x^{-3}}{-3}\Bigg|_a^{-5}$

$$= \lim_{a\to\infty}\left(\frac{(-5)^{-3}}{-3} - \frac{a^{-3}}{-3}\right)$$

$$= \boxed{\frac{(-5)^{-3}}{-3}}.$$

8.3.6: $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{\pi x}} = \lim_{b\to\infty} \frac{1}{\sqrt{\pi}}\, 2\sqrt{x}\Big|_1^b$

$$= \frac{1}{\sqrt{\pi}} \lim_{b\to\infty} 2\sqrt{b} - 2\sqrt{1}$$

$\boxed{\text{diverges}}$

8.3.12: $\displaystyle\int_e^{\infty} \frac{\ln x}{x}\,dx = \int_1^{\infty} u\,du = \lim_{b\to\infty} \frac{u^2}{2}\Big|_1^b$

$u = \ln x$

$du = \frac{1}{x}dx$

$\boxed{\text{diverges}}$

8.3.14: $\displaystyle\int_1^{\infty} xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}dx$

$\boxed{\begin{matrix} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x}\end{matrix}}$

$$= -xe^{-x} - e^{-x}\Big|_1^{\infty}$$

$$= \lim_{b\to\infty}\left(-be^{-b} - e^{-b}\right) - \left(-e^{-1} - e^{-1}\right)$$

$$= \boxed{\frac{2}{e}}$$

8.4.4: $$\int_0^9 \frac{dx}{\sqrt{9-x}} = \lim_{t\to 9} -2\sqrt{9-x}\,\Big|_0^t$$

$$= \lim_{t\to 9} -2\sqrt{9-t} + 2\sqrt{9}$$

$$= \boxed{6}$$

8.4.8: $$\int_{-5}^{5} \frac{1}{x^{2/3}}\,dx = \int_{-5}^{0} \frac{1}{x^{2/3}}\,dx + \int_{0}^{5} \frac{1}{x^{2/3}}\,dx$$

$$= \lim_{t\to 0} \overset{\nearrow\infty}{\cancel{3x^{1/3}}}\Big|_{-5}^{t} + \lim_{s\to 0} \overset{\nearrow -\infty}{\cancel{3x^{1/3}}}\Big|_{s}^{5}$$

$\boxed{\text{diverges}}$

8.4.16: $$\int_0^3 \frac{dx}{x^2+x-2} = \int_0^3 \frac{dx}{(x+2)(x-1)} = \int_0^3 \left(\frac{A}{x+2} + \frac{B}{x-1}\right) dx$$

$$\left(\begin{array}{l} 1 = A(x-1) + B(x+2) \\ \Rightarrow A = -\frac{1}{3}, \quad B = \frac{1}{3}. \end{array}\right)$$

$$= -\frac{1}{3}\int_0^3 \frac{1}{x+2}\,dx + \frac{1}{3}\int_0^3 \frac{1}{x-1}\,dx$$

$$= -\frac{1}{3}\ln|x+2|\,\Big|_0^3 + \frac{1}{3}\left[\int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx\right]$$

$$= -\frac{1}{3}\ln 5 + \frac{1}{3}\ln 2 + \frac{1}{3}\left[\lim_{t\to 1} \underset{\searrow -\infty}{\ln|x-1|}\,\Big|_0^t + \lim_{s\to 1} \underset{\searrow \infty.}{\ln|x-1|}\,\Big|_s^1\right]$$

$\boxed{\text{diverges}}$