Homework #10 Solutions

9.6.12: $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Absolute ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right|$$

$$\rho = \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+1)}$$

$$\rho = 0$$

$\rho < 1$ for all x

so $\sum \frac{(-1)^n x^{2n}}{(2n)!}$

converges for $\boxed{-\infty < x < \infty}$

9.6.14: $x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \dots$

$$= \sum_{n=1}^{\infty} n^2 x^n.$$

- Absolute ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{(n+1)^2 x^{n+1}}{n^2 x^n} \right|$$

$$\rho = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} |x|$$

$$\rho = |x|$$

so $\rho < 1$ for $-1 < x < 1$.

- Check Endpoints:

$x = 1$: $\sum_{n=1}^{\infty} n^2$ diverges by n^{th} term test.

$x = -1$: $\sum_{n=1}^{\infty} (-1)^n n^2$ diverges by n^{th} term test.

Therefore

$\sum_{n=1}^{\infty} n^2 x^n$ convergs

for $\boxed{-1 < x < 1}$

9.6.16: $1 + x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \dots$

$$= 1 + \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

Absolute ratio test:

$$p = \lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right|$$

$$p = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \cdot |x|$$

$$p = |x|$$

$p < 1$ for $-1 < x < 1$.

Check Endpoints:

$x = 1$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by alt. series test.

Therefore $1 + \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ converges for

$$\boxed{-1 \le x < 1}$$

9.6.26: $\frac{x-2}{1^2} + \frac{(x-2)^2}{2^2} + \frac{(x-2)^3}{3^2} + \frac{(x-2)^4}{4^2} + \dots$

$$= \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}.$$

Absolute Ratio Test:

$$p = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(x-2)^n} \right|$$

$$p = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \cdot |x-2|$$

$p = |x-2|$, so $p < 1$ for $|x-2| <$

i.e., for $-1 < x-2 < 1$, $1 < x < 3$

Check endpoints:

$x = 3$: $\sum \frac{1}{n^2}$ converges by p-series

$x = 1$: $\sum \frac{(-1)^n}{n^2}$ converges by alt. series test.

Therefore, $\sum \frac{(x-2)^n}{n^2}$ converges for $\boxed{-1 \le x \le 1}$

9.7.6: $f(x) = \dfrac{1}{3+2x} = \dfrac{1}{3\left(1+\frac{2}{3}x\right)} = \dfrac{1}{3} \cdot \dfrac{1}{1+\frac{2}{3}x} = \dfrac{1}{3} \cdot \dfrac{1}{1-\left(-\frac{2}{3}x\right)}$

so $f(x) = \frac{1}{3}\left(1 + \left(-\frac{2}{3}x\right) + \left(-\frac{2}{3}x\right)^2 + \left(-\frac{2}{3}x\right)^3 + \ldots\right)$

$$\boxed{f(x) = \frac{1}{3}\left(1 - \frac{2}{3}x + \left(\frac{2}{3}x\right)^2 - \left(\frac{2}{3}x\right)^3 + \ldots\right)}$$

9.7.16: $f(x) = e^{2x} - 1 - 2x.$

Since $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots,$

then $e^{2x} = 1 + 2x + \dfrac{(2x)^2}{2!} + \dfrac{(2x)^3}{3!} + \ldots$

Therefore $\boxed{f(x) = \dfrac{(2x)^2}{2!} + \dfrac{(2x)^3}{3!} + \ldots}$

9.7.24: $f(x) = \displaystyle\int_0^x \frac{\tan^{-1} t}{t}\, dt.$

Since $\tan^{-1} t = t - \dfrac{t^3}{3} + \dfrac{t^5}{5} - \dfrac{t^7}{7} + \ldots$

then $\dfrac{\tan^{-1} t}{t} = 1 - \dfrac{t^2}{3} + \dfrac{t^4}{5} - \dfrac{t^6}{7} + \ldots$

so $f(x) = \displaystyle\int_0^x \left(1 - \frac{t^2}{3} + \frac{t^4}{5} - \frac{t^6}{7} + \ldots\right) dt$

$f(x) = t - t^3 + t^5 - t^7 + t^9 - \ldots \Big|_0^x$

$$\boxed{f(x) = x - x^3 + x^5 - x^7 + \ldots}$$

9.7.28: Find sum of $\sum_{n=1}^{\infty} n(n+1)x^n$.

Let $f(x) = \sum_{n=1}^{\infty} x^{n+1}$. Then $f'(x) = \sum_{n=1}^{\infty} (n+1)x^n$

and $f''(x) = \sum_{n=1}^{\infty} n(n+1)x^{n-1}$.

So $\sum_{n=1}^{\infty} n(n+1)x^n = x f''(x)$.

Since $f(x) = \sum_{n=1}^{\infty} x^{n+1} = \dfrac{x^2}{1-x}$ for $|x| < 1$,

then $f'(x) = \dfrac{(1-x)(2x) - x^2(-1)}{(1-x)^2} = \dfrac{-x^2+2x}{(1-x)^2}$

and $f''(x) = \dfrac{(1-x)^2(-2x+2) - (-x^2+2x)(2(1-x)(-1))}{(1-x)^4}$

$$f''(x) = \frac{(1-2x+x^2)(-2x+2) - (-x^2+2x)(2x-2)}{(1-x)^4}$$

$$f''(x) = \frac{-2x+2}{(1-x)^4}$$

Therefore,

$$\boxed{\sum_{n=1}^{\infty} n(n+1)x^n = \frac{-2x^2+2x}{(1-x)^4}}$$

9.8.4: $f(x) = e^{-x}\cos x$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$e^{-x}\cos x = 1 + x(-1) + x^2\left(-\frac{1}{2!} + \frac{1}{2!}\right) + x^3\left(-\frac{1}{3!} + \frac{1}{2!}\right)$$
$$+ x^4\left(\frac{1}{4!} - \frac{1}{2!2!} + \frac{1}{4!}\right) + x^5\left(-\frac{1}{5!} + \frac{1}{3!2!} - \frac{1}{4!}\right) + \cdots$$

$$\boxed{e^{-x}\cos x = 1 - x + \frac{x^3}{3} - \frac{x^4}{6} + \frac{x^5}{30} + \cdots}$$

9.8.14: $f(x) = x(\sin 2x + \sin 3x)$

$$\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots$$

$$\sin 3x = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \cdots$$

$$\boxed{x(\sin 2x + \sin 3x) = x^2(2+3) - x^4\left(\frac{2^3}{3!} + \frac{3^3}{3!}\right) + x^6\left(\frac{2^5}{5!} + \frac{3^5}{5!}\right) - \cdots}$$

9.8.20: $\sin x$, $a = \pi/6$.

- $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \dots$$

$f(x) = \sin x$ $\quad$ $f(\pi/6) = \sin \pi/6 = \frac{1}{2}$

$f'(x) = \cos x$ $\quad$ $f'(\pi/6) = \cos \pi/6 = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x$ $\quad$ $f''(\pi/6) = -\sin \pi/6 = -\frac{1}{2}$

$f'''(x) = -\cos x$ $\quad$ $f'''(\pi/6) = -\cos \pi/6 = -\frac{\sqrt{3}}{2}$

$$\boxed{\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \pi/6) - \frac{1}{2}\frac{(x-\pi/6)^2}{2!} - \frac{\sqrt{3}}{2}\frac{(x-\pi/6)^3}{3!} + \dots}$$

9.9.6: $f(x) = \sqrt{1+x} = (1+x)^{1/2}$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

$f(x) = (1+x)^{1/2}$ $\quad$ $f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\quad$ $f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $\quad$ $f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$ $\quad$ $f'''(0) = \frac{3}{8}$

$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2}$ $\quad$ $f^{(4)}(0) = -\frac{15}{16}$

so

$$\boxed{\sqrt{1.12} \approx 1.05829}$$

$$\boxed{\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4}$$