Name:

Exam #2 Solutions

Part 1 (20 points) : Choose **ONE** of the following problems to solve:

a. Compute the limit:

$$\lim_{x\to 0}\left(\cos x\right)^{x^{-2}}$$

Solution: Let $y = \left(\cos x\right)^{x^{-2}}$. Then $\ln y = x^{-2}\ln(\cos x) = \dfrac{\ln(\cos x)}{x^2}$. We'll compute $\lim_{x\to 0}\ln y$ first.

$$\begin{aligned}\lim_{x\to 0}\ln y &= \lim_{x\to 0}\frac{\ln(\cos x)}{x^2}\\ \text{Ⓛ} &= \lim_{x\to 0}\frac{\frac{-\sin x}{\cos x}}{2x}\\ &= \lim_{x\to 0}\frac{-\tan x}{2x}\\ \text{Ⓛ} &= \lim_{x\to 0}\frac{-\sec^2 x}{2}\\ &= -\frac{1}{2}\end{aligned}$$

Therefore,

$$\lim_{x\to 0}\left(\cos x\right)^{x^{-2}} = \lim_{x\to 0} y = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}.$$

□

b. Compute the limit of the convergent **sequence**:

$$\frac{9+4}{3-1},\ \frac{27+8}{9-2},\ \frac{81+16}{27-3},\ \frac{243+32}{81-4},\ \ldots$$

Solution: The nth term of the sequence is given by

$$a_n = \frac{3^{n+1}+2^{n+1}}{3^n - n}.$$

To compute the limit of this sequence, we just take the limit as n goes to infinity.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\frac{3^{n+1}+2^{n+1}}{3^n-n}\\ &= \lim_{n\to\infty}\frac{3^n\left(3+2\left(\frac{2}{3}\right)^n\right)}{3^n\left(1-\frac{n}{3^n}\right)}\\ &= \lim_{n\to\infty}\frac{3+2\left(\frac{2}{3}\right)^n}{1-\frac{n}{3^n}}\\ &= \frac{3+0}{1-0}\\ &= 3\end{aligned}$$

so the limit of the sequence is 3. □

Part 2 (20 points): Compute **ONE** of the following indefinite integrals:

a. $\displaystyle\int \frac{v^3}{\sqrt{1-v^2}}\, dv$

Solution: Let $u = 1 - v^2$. Then $du = -2vdv$ and $v^2 = 1 - u$.

$$\begin{aligned}
\int \frac{v^3}{\sqrt{1-v^2}}\, dv &= -\frac{1}{2}\int \frac{v^2}{\sqrt{1-v^2}}\,(-2vdv) \\
&= -\frac{1}{2}\int \frac{1-u}{\sqrt{u}}\, du \\
&= -\frac{1}{2}\int \left(u^{-1/2} - u^{1/2}\right) du \\
&= -\frac{1}{2}\left(\frac{u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2}\right) + C \\
&= -u^{1/2} + \frac{1}{3}u^{3/2} + C \\
&= -\sqrt{1-v^2} + \frac{1}{3}\left(1-v^2\right)^{3/2} + C
\end{aligned}$$

□

b. $\displaystyle\int \frac{1}{\sqrt{4+w^2}}\, dw$

Solution: Let $w = 2\tan t$. Then $dw = 2\sec^2 t dt$ and $\sqrt{4+w^2} = 2\sec t$.

$$\begin{aligned}
\int \frac{1}{\sqrt{4+w^2}}\, dw &= \int \frac{1}{2\sec t}\, 2\sec^2 t\, dt \\
&= \int \sec t\, dt \\
&= \ln\left|\sec t + \tan t\right| + C \\
&= \ln\left|\frac{\sqrt{4+w^2}}{2} + \frac{w}{2}\right| + C
\end{aligned}$$

□

Part 3 (20 points): Compute **ONE** of the following indefinite integrals:

a. $\displaystyle\int \sin^2 x \cos^2 x \, dx$

Solution:

$$\begin{aligned}
\int \sin^2 x \cos^2 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx \\
&= \int \frac{1-\cos^2 2x}{4} \, dx \\
&= \frac{1}{4}\int \left(1 - \frac{1+\cos 4x}{2}\right) dx \\
&= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx \\
&= \frac{1}{8}x - \frac{1}{32}\sin 4x + C
\end{aligned}$$

□

b. $\displaystyle\int \sec^4 y \, dy$

Solution: Let $u = \tan y$. Then $du = \sec^2 y dy$.

$$\begin{aligned}
\int \sec^4 y \, dy &= \int \sec^2 y \sec^2 y \, dy \\
&= \int \left(1+\tan^2 y\right)\sec^2 y \, dy \\
&= \int \left(1+u^2\right) du \\
&= u + \frac{1}{3}u^3 + C \\
&= \tan y + \frac{1}{3}\tan^3 y + C
\end{aligned}$$

□

Part 4 (20 points): Use partial fraction decomposition to solve **ONE** of the following problems:

a. Find the sum of the convergent (collapsing) series:

$$\sum_{k=2}^{\infty} \frac{3k-1}{k^3-k}$$

Solution: Let $\dfrac{3k-1}{k(k-1)(k+1)} = \dfrac{A}{k-1} + \dfrac{B}{k} + \dfrac{C}{k+1}$. Then

$$3k-1 = A(k)(k+1) + B(k-1)(k+1) + C(k-1).$$

If we plug in $k=-1$, we get $C=-2$. If we plug in $k=0$, we get $B=1$. If we plug in $k=1$, we get $A=1$. Therefore,

$$\sum_{k=2}^{\infty} \frac{3k-1}{k^3-k} = \sum_{k=2}^{\infty} \left(\frac{1}{k-1} + \frac{1}{k} - \frac{2}{k+1} \right).$$

The nth partial sum is given by

$$\begin{aligned} S_n =& \sum_{k=2}^{n} \left(\frac{1}{k-1} + \frac{1}{k} - \frac{2}{k+1} \right) \\ =& \left(1 + \frac{1}{2} - \frac{2}{3}\right) + \left(\frac{1}{2} + \frac{1}{3} - \frac{2}{4}\right) + \left(\frac{1}{3} + \frac{1}{4} - \frac{2}{5}\right) + \left(\frac{1}{4} + \frac{1}{5} - \frac{2}{6}\right) + \left(\frac{1}{5} + \frac{1}{6} - \frac{2}{7}\right) + \cdots \\ &+ \left(\frac{1}{n-3} + \frac{1}{n-2} - \frac{2}{n-1}\right) + \left(\frac{1}{n-2} + \frac{1}{n-1} - \frac{2}{n}\right) + \left(\frac{1}{n-1} + \frac{1}{n} - \frac{2}{n+1}\right) \\ =& 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{n} - \frac{2}{n+1} \end{aligned}$$

So the infinite series sums to

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(2 - \frac{1}{n} - \frac{2}{n+1}\right) = 2$$

□

b. Compute the indefinite integral:

$$\int \frac{x-2}{x^3+x}\, dx$$

Solution: Let $\dfrac{x-2}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$. Then

$$x-2 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + Cx + A.$$

So $A=-2$, $B=2$, and $C=1$. Therefore,

$$\begin{aligned} \int \frac{x-2}{x^3+x}\, dx &= \int \left(\frac{-2}{x} + \frac{2x+1}{x^2+1} \right) dx \\ &= -2\int \frac{1}{x}\, dx + \int \frac{2x}{x^2+1}\, dx + \int \frac{1}{x^2+1}\, dx \\ &= -2\ln|x| + \ln(x^2+1) + \tan^{-1} x + C \end{aligned}$$

□

Part 5 (20 points) : Choose **ONE** of the following problems to solve:

a. Determine if the following alternating series converges absolutely, converges conditionally, or diverges. Justify your answer and state which test(s) you use.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\ \ln n}$$

Solution: First, we'll check if the series converges absolutely using the integral test.

$$\sum \left| \frac{(-1)^n}{n \ln n} \right| = \sum \frac{1}{n \ln n}.$$

We'll compare it to the integral, where $u = \ln x$,

$$\begin{aligned} \int_2^{\infty} \frac{1}{x \ln x}\, dx &= \int \frac{1}{u}\, du \\ &= \ln |u| \\ &= \ln(\ln x)\Big|_2^{\infty} \\ &= \infty, \end{aligned}$$

which is a divergent integral. Therefore, the absolute value of the series diverges and the series does **not** converge absolutely. There's still a chance that it converges conditionally, however. We can test it using the alternating series test. The terms are indeed decreasing, since $n \ln n$ gets bigger as n gets bigger. Also,

$$\lim_{n \to \infty} \frac{1}{n \ln n} = 0,$$

and so the alternating series **does** converge. Therefore, the series converges conditionally. □

b. For **BOTH** of the following positive series, determine whether they converge or diverge. Justify your answer and state which test(s) you use.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

Solution: ratio test:

$$\begin{aligned}
\rho &= \lim_{n\to\infty} a_{n+1} \cdot (a_n)^{-1} \\
&= \lim_{n\to\infty} \frac{(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{n!} \\
&= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \\
&= \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} \\
&= 0
\end{aligned}$$

Since $\rho = 0 < 1$, then the series converges. □

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(n^2+1)\sqrt{n}}{(n-1)^3}$

Solution: This series looks overall like $\sum \frac{1}{\sqrt{n}}$. So we will do a limit comparison to this series.

$$\begin{aligned}
L &= \lim_{n\to\infty} a_n \cdot (b_n)^{-1} \\
&= \lim_{n\to\infty} \frac{(n^2+1)\sqrt{n}}{(n-1)^3} \cdot \frac{\sqrt{n}}{1} \\
&= \lim_{n\to\infty} \frac{(n^3+n)}{(n-1)^3} \\
&= 1
\end{aligned}$$

Since $L = 1$, i.e. $0 < L < \infty$, then the series acts like the series $\sum \frac{1}{\sqrt{n}}$, which diverges by p-series test. Therefore, the series in question diverges. □