Financial Mathematics: A Comprehensive Treatment
(Campolieti-Makarov)
Solutions to selected exercises

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Chapter 4: Primer on Derivative Securities

Exercise 1.1. [CM14, Exercise 4.27] A European binary option is a so-called "all-or-nothing” claim on an underlying asset. For example, one share of a cash-or-nothing binary call has a payoff of exactly one dollar if the asset price ends up above the strike and zero otherwise, i.e. the payoff function is $\Lambda(S) = 1_{S \geq K}(S)$.

Similarly, a cash-or-nothing binary put has payoff $1_{S < K}$. Assume that the asset price process $\{S(t)\}_{t \geq 0}$ is a geometric Brownian motion.

(a) Derive the Black-Scholes exact pricing formulas for both the binary call $C(S,T)$ and the put $P(S,T)$.

(b) Give the relationship between the binary call and put price when both options have the same strike $K$ and maturity $T$.

(c) Derive the exact formula for the Greek delta of the binary call and put: $\Delta_c = \frac{\partial C}{\partial S}$ and $\Delta_p = \frac{\partial P}{\partial S}$.

Solution.

(a) By the risk-neutral pricing formula we have

$$C(t, S) = e^{-r(T-t)} \tilde{E}[\Lambda(S(T)) | S(t) = S]$$

$$= e^{-r(T-t)} \left[ \Lambda \left( S(t)e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma \sqrt{T-t}Z} \right) | S(t) = S \right]$$

$$= e^{-r(T-t)} \tilde{E} \left[ \frac{1}{Z \geq \frac{1}{\sigma \sqrt{T-t}}} \left( \ln \frac{S}{K} - \frac{(r-\frac{\sigma^2}{2})(T-t)}{2} \right) \right]$$

$$= e^{-r(T-t)} \tilde{P} \left( Z \geq -\frac{1}{\sigma \sqrt{T-t}} \left( \ln \frac{S}{K} + \left( r - \frac{\sigma^2}{2} \right) (T-t) \right) \right)$$

$$= e^{-r(T-t)} \tilde{N}(\alpha)$$

where in the last line we denoted $\alpha = \frac{1}{\sigma \sqrt{T-t}} \left( \ln \frac{S}{K} + \left( r - \frac{\sigma^2}{2} \right) (T-t) \right)$.

(b) The put-call parity relationship for European options reads

$$C(t, S) - P(t, S) = S - Ke^{-r(T-t)}$$

whereby

$$P(t, S) = C(t, S) - S + Ke^{-r(T-t)} = e^{-r(T-t)}[\tilde{N}(\alpha) + K] - S$$

(c) ...

□

Exercise 1.2. [CM14, Exercise 4.28] Consider the European-style option with payoff $\Lambda(S) = 1_{K_1 \leq S \leq K_2}(S)$ and assume the geometric Brownian motion model for the stock price process $\{S(t)\}_{t \geq 0}$. 

2
(a) Find the option value by decomposing the payoff in terms of binary options with appropriate indicator functions.

(b) Derive formulas for the following sensitivities of the option value: $\Delta = \frac{\partial V}{\partial S}$, $\Gamma = \frac{\partial^2 V}{\partial S^2}$ and $\Theta = \frac{\partial V}{\partial t}$

**Solution.**

(a) The payoff can be decomposed in terms of binary options

$$\Lambda(S) = 1_{K_1 \leq S \leq K_2}(S) = 1_{S \geq K_1} - 1_{S \geq K_2}$$

so using the pricing formula (2) from the previous exercise we obtain

$$V(t, S) = e^{-r(T-t)} \mathbb{E}[\Lambda(S(T))|S(t) = S] = e^{-r(T-t)} \mathbb{E}[1_{K_1 \leq S(T) \leq K_2}|S(t) = S]$$

$$= e^{-r(T-t)} \mathbb{E}[1_{S(T) \geq K_1}|S(t) = S] - e^{-r(T-t)} \mathbb{E}[1_{S(T) \geq K_2}|S(t) = S]$$

$$= C^{K_1}(t, S) - C^{K_2}(t, S)$$

where $\alpha(K_i) = \frac{1}{\sigma \sqrt{T-t}} \left( \ln \frac{S}{K_i} + \left( r - \frac{\sigma^2}{2} \right) (T-t) \right)$.

(b) ...

$\square$
2 Chapter 10: One-dimensional Brownian motion and related processes.

Exercise 2.1. [CM14, Exercise 10.1]

(a) Show that the PDF of a sum of two continuous random variables $X$ and $Y$ is given by the convolution of the PDF’s $f_X$ and $f_Y$:

$$f_{X+Y}(x) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \, dx$$

(b) Use the result in (a) to show that a sum of two independent standard normal random variables results in a normal random variable and find the PDF of such a sum.

(c) Assuming that $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are correlated with $\text{Corr}(X_1, X_2) = \rho$, find the mean and variance of $aX_1 + a_2X_2$ for $a_1, a_2 \in \mathbb{R}$.

Solution.

(a) We start by computing the CDF of the sum: denoting by $f_{X,Y}(x,y)$ the joint density function of $X$ and $Y$ we have

$$P(X + Y \leq z) = P(X \leq z - Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-Y} f_{X,Y}(x,y) \, dx \, dy$$

and differentiating with respect to $z$ we obtain

$$f_{X+Y}(z) = \frac{d}{dz} P(X + Y \leq z) = \int_{-\infty}^{\infty} f_{X,Y}(z-y,y) \, dy \overset{[1]}{=} \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) \, dy \overset{[2]}{=} \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \, dx$$

where in [2] we used independence of $X$ and $Y$ and in [2] we changed variables to $x = z - y$.

(b) We compute the sum of two standard normal variables by convolving their density functions: if $X, Y \sim \mathcal{N}(0,1)$, then $f_X(u) = f_Y(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$ so

$$f_{X+Y}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{x^2/2} e^{-(z-x)^2/2} \, dx = \frac{1}{2\pi} e^{-z^2/4} \int_{-\infty}^{\infty} e^{-(x-\frac{z}{2})^2} \, dx$$

$$= \frac{1}{2\sqrt{2\pi}} e^{-z^2/4} \int_{-\infty}^{\infty} e^{-u^2/2} \, du$$

$$= \frac{1}{2\sqrt{\pi}} e^{-z^2/2}$$

which shows that $X + Y$ is a normal variable with mean 0 and variance 2.

□

Exercise 2.2. [CM14, Exercise 10.3] Suppose that $X = (X_1, X_2, X_3)^\top$ be a 3-dimensional Gaussian random vector with mean zero and covariance matrix $\Sigma_X = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$. Set $Y = 1 + X_1 - 2X_2 + X_3$ and $Z = X_1 - 2X_3$. 

4
(a) Find the probability distribution of $Y$.

(b) Find the probability distribution of the vector $(Y, Z)\top$.

**Solution.** We start by recalling the following facts:

(i) If $X = (X_1, \ldots, X_n)$ is an $n$-dimensional vector of random variables with covariance matrix $\Sigma_X$ (namely, $(\Sigma_X)_{ij} = \text{Cov}(X_i, X_j)$) and $Y = MX$ where $M$ is an $m \times n$ matrix and $Y = (Y_1, \ldots, Y_m)$ is an $m$-dimensional vector of random variables, then

$$\Sigma_Y = M \Sigma_X M\top$$

(ii) If $Y = b + MX$, where $Y$ is an $m$-dimensional vector of random variables, $b$ is a constant $m$-dimensional vector and $X$ is an $n$-dimensional vector of random variables with mean $\mu_X$ and covariance $\Sigma_X$, then $Y$ has mean vector and covariance matrix given by

$$\mu_Y = b + M\mu_X, \quad \Sigma_Y = M \Sigma_X M\top$$

(iii) An $n$-dimensional random vector has the $n$-dimensional multivariate normal distribution if and only if every combination $\sum_{i=1}^n c_i X_i$ is a normal random variable. In particular, it follows from (ii) that if $X \sim \mathcal{N}(\mu_X, \Sigma_X)$ and $Y = b + MX$, then

$$Y \sim \mathcal{N} \left( b + M\mu_X, M \Sigma_X M\top \right)$$

(iv) If $Z$ has the standard $n$-dimensional normal distribution, $Z \sim \mathcal{N}(0, I_n)$ and $A$ is a symmetric positive definite matrix with Cholesky factorization $A = U\top U$, then $X = U\top Z$ has an $n$-dimensional normal distribution with covariance matrix $\Sigma_X = A$, so $X \sim \mathcal{N}(0, A)$.

We now solve the exercise.

1. $Y$ has a normal distribution by (iii). If $b = (1, -2, 1)\top$, then

$$Y \sim \mathcal{N} \left( 1 + b\top 0, b\top \Sigma_X b \right)$$

with

$$b\top \Sigma_X b = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 11$$

so $Y \sim \mathcal{N}(1, 11)$.

2. Note that

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + A \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

so by (iii) the vector $W := (Y, Z)\top$ has a bivariate normal distribution with mean $\mu_W = (1, 0)\top + A \cdot 0 = (1, 0)\top$ and covariance matrix $\Sigma_W = A \Sigma_X A\top = \begin{pmatrix} 11 & -2 \\ -2 & 8 \end{pmatrix}$. 
Exercise 2.3. [CM14, Exercise 10.8] For a Brownian motion \( \{W(t)\}_{t \geq 0} \) and its natural filtration, calculate \( E_s[W^3(t)] \) for \( 0 \leq s \leq t \).

Solution. Method 1. Taking the differential of \( W^3(t) \) and using It’s lemma we see that
\[
dW^3(t) = 3W^2(t)dW(t) + 3W(t)dt
\]
since \( \int_0^t W^2(u) dW(u) \) is a square-integrable martingale, it follows that that so is \( W^3(t) - 3 \int_0^t W(u) du \), whence
\[
E_s[W^3(t)] = E_s\left[W^3(t) - 3 \int_0^t W(u) du\right] + E_s\left[3 \int_0^t W(u) du\right]
\]
As for the last expectation we have
\[
E_s\left[\int_0^t W(u) du\right] = E_s\left[\int_0^s W(u) du + \int_s^t W(u) du\right] = \int_0^s W(u) du + \int_s^t E_s[W(u)] du
\]
so in conclusion
\[
E_s[W^3(t)] = W^3(s) + 3(t - s)W(s)
\]
Method 2. We can also compute the expectation directly:
\[
E_s[W^3(t)] = E_s[(W(t) - W(s) + W(s))^3] = E_s[(W(t) - W(s))^3] + 3E_s[(W(t) - W(s))^2W(s)] + 3E_s[(W(t) - W(s))W^2(s)] + E_s[W^3(s)]
\]
Now note that
\[
E_s[(W(t) - W(s))^3] = 0
E_s[(W(t) - W(s))^2W(s)] = W(s)E_s[(W(t) - W(s))^2W(s)] = W(s)E[(W(t) - W(s))^2] = (t-s)W(s)
E_s[(W(t) - W(s))W^2(s)] = W^2(s)E_s[W(t) - W(s)] = 0
E_s[W^3(s)] = W^3(s)
\]
In the first equation we used that a normal variable has 0 third central moment (in fact it has 0 odd central moments, as the remark below shows). In the second equation, we used the fact that \( W(s) \) is \( \mathcal{F}_s \)-measurable for the first equality (so it can be taken out of the expectation) and also the fact that \( W(t) - W(s) \) is \( \mathcal{F}_s \)-independent for the second equality, so that the conditional expectation \( E_s \) is in fact unconditional. In the third equation we simply used that \( W^2(s) \) is \( \mathcal{F}_s \)-measurable to take it out of the expectation. □
Remark 2.1 (Central moments of a normal random variable). The $n$-th central moment is by definition $\hat{m}_n = \mathbb{E}((X - \mathbb{E}(X))^n)$. Notice that for the normal distribution $\mathbb{E}(X) = \mu$, and that $Y = X - \mu$ also follows a normal distribution, with zero mean and the same variance $\sigma^2$ as $X$.

Therefore, finding the central moment of $X$ is equivalent to finding the raw moment of $Y$.

\[
\hat{m}_n = \mathbb{E}((X - \mathbb{E}(X))^n) = \mathbb{E}((X - \mu)^n) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma} (x - \mu)^n e^{-(x-\mu)^2/(2\sigma^2)} \, dx
\]

where $\Gamma(n) = \int_{0}^{\infty} u^{n-1} e^{-u} \, du$ stands for the Euler’s Gamma function. Using its properties we obtain

\[
\hat{m}_2n = \sigma^{2n}(2n-1)!! \quad \hat{m}_{2n+1} = 0
\]

Exercise 2.4. [CM14, Exercise 10.9] Find the distribution of $W(1) + \cdots + W(n)$ for $n \in \mathbb{N}$.

Solution. A sum of normal random variables is again normal, so it suffices to compute its mean and variance. Clearly

\[
E[W(1) + \cdots + W(n)] = 0
\]

since each $W(k) \sim \mathcal{N}(0, k)$. As for the variance, recall that

\[
\text{Var}[W(1) + \cdots + W(n)] = \sum_{k=1}^{n} \text{Var}(W(k)) + 2 \sum_{i<j} \text{Cov}(W(i), W(j))
\]

and for $i < j$

\[
\text{Cov}(W(i), W(j)) = E[W(i)W(j)] = E[W(i) \cdot (W(j) - W(i))] + E[W^2(i)] = \text{Var}(W(i)) = i
\]

We hence observe that

\[
\text{Var}[W(1) + W(2)] = (1 + 2) + 2 \cdot 1
\]

\[
\text{Var}[W(1) + W(2) + W(3)] = (1 + 2 + 3) + 2 \cdot [1 + (1+2)]
\]

\[
\text{Var} \left( \sum_{k=1}^{4} W(k) \right) = (1 + 2 + 3 + 4) + 2 \cdot [1 + (1 + 2) + (1+2+3)]
\]

\[
\vdots
\]

\[
\text{Var} \left( \sum_{k=1}^{n} W(k) \right) = \text{Var} \left( \sum_{k=1}^{n-1} W(k) \right) + n + 2 \cdot \frac{(n-1)n}{2} = \text{Var} \left( \sum_{k=1}^{n-1} W(k) \right) + n^2
\]
So
\[ \text{Var} \left( \sum_{k=1}^{n} W(k) \right) = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \]
and hence
\[ \sum_{k=1}^{n} W(k) \sim \mathcal{N} \left( 0, \frac{n(n+1)(2n+1)}{6} \right) \]
\[ \Box \]

**Exercise 2.5.** [CM14, Exercise 10.10] Let \( a_1, \ldots, a_n \in \mathbb{R} \) and \( 0 < t_1 < \cdots < t_n \). Find the distribution of \( \sum_{i=1}^{n} a_i W(t_i) \). Note that the choice \( a_k = \frac{1}{n}, 1 \leq k \leq n \), leads to Asian options.

*Solution.* Arguing as in the previous exercise, we conclude that
\[
\text{Var} \left( \sum_{i=1}^{n} a_i W(t_i) \right) = \text{Var} \left( \sum_{i=1}^{n-1} a_i W(t_i) \right) + a_n^2 t_n + 2a_n \sum_{k=1}^{n-1} a_k t_k
\]
so by induction
\[
\text{Var} \left( \sum_{i=1}^{n} a_i W(t_i) \right) = \sum_{i=1}^{n} \left[ a_i^2 t_i + 2a_i \sum_{k=1}^{i-1} a_k t_k \right]
\]
\[ \Box \]

**Exercise 2.6.** [CM14, Exercise 10.11] Suppose that the processes \{\( X(t) \)\}_{t \geq 0} and \{\( Y(t) \)\}_{t \geq 0} are respectively given by \( X(t) = x_0 + \mu x t + \sigma x W(t) \) and \( Y(t) = y_0 + \mu y t + \sigma y W(t) \), where \( x_0, y_0, \mu_x, \mu_y, \sigma_x > 0, \sigma_y > 0 \) are real constants. Find the covariance \( \text{Cov}(X(t), Y(s)) \), for \( s, t \geq 0 \).

*Solution.*
\[ \Box \]

**Exercise 2.7.** [CM14, Exercise 10.12] Consider the process \( X(t) = x_0 + \mu t + \sigma W(t) \geq 0 \), where \( x_0, \mu \) and \( \sigma \) are real constants. Show that
\[
E[\max(X(t) - K, 0)] = (x_0 + \mu t - K) \mathcal{N} \left( \frac{x_0 + \mu t - K}{\sigma \sqrt{t}} \right) + \sigma \sqrt{t} \ln \left( \frac{x_0 + \mu t - K}{\sigma \sqrt{t}} \right)
\]

*Solution.*
\[ \Box \]

**Exercise 2.8.** [CM14, Exercise 10.13] By directly calculating partial derivatives, verify that the transition PDF
\[
p_0(t; x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}
\]
of standard Brownian motion satisfies the diffusion equation
\[
\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x)
\]
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Solution. We simply compute the derivatives

\[
\frac{\partial p_0}{\partial t} (t, x) = -\frac{1}{2} \frac{1}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} + \frac{x^2}{2t^2 \sqrt{2\pi t}} e^{-\frac{x^2}{2t}}
\]

\[
\frac{\partial p_0}{\partial x} (t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \left( -\frac{x}{t} \right)
\]

\[
\frac{\partial^2 p_0}{\partial x^2} (t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \frac{x^2}{t^2} - \frac{1}{t \sqrt{2\pi t}} e^{-\frac{x^2}{2t}}
\]

whereby clearly

\[
\frac{\partial}{\partial t} p_0(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p_0(t, x)
\]

□

Exercise 2.9. [CM14, Exercise 10.14] Find the transition PDF of \( \{W^n(t)\}_{t \geq 0} \) for \( n \in \mathbb{N} \).

Solution.

□

Exercise 2.10. [CM14, Exercise 10.15] Find the transition PDF of \( \{S^n(t)\}_{t \geq 0} \), where

\[ S(t) = S_0 e^{\mu t + \sigma W(t)}, \quad t \geq 0, \quad S_0 > 0, \quad n \in \mathbb{N} \]

Solution.

□

Exercise 2.11. [CM14, Exercise 10.19] Show that the mean and covariance functions of the Brownian bridge from \( a \) to \( b \) on \([0, T]\) are, respectively

\[
m(t) = a + \frac{b - a}{T} t,
\]

\[
c(s, t) = s \wedge t - \frac{st}{T}
\]

for \( s, t \in [0, T] \).

Solution. Recall that if \( X(t) \equiv X^{[0, a]}_{[0, T]} = W(t) - \frac{t}{T} W(T) \) is the Brownian bridge from 0 to 0, then

\[
X^{[0, a]}_{[0, T]} = a + \frac{b - a}{T} t + X(t)
\]

so

\[
m_{X^{[0, a]}_{[0, T]}}(t) = a + \frac{b - a}{T} t + m_X(t)
\]

\[
c_{X^{[0, a]}_{[0, T]}}(t, s) = c_X(t, s)
\]
and it suffices to compute $m_X(t)$ and $c_X(t,s)$.

$$m_X(t) = E[X(t)] = E \left[ W(t) - \frac{t}{T} W(T) \right] = 0,$$

$$c_X(t,s) = E [X(t)X(s)] = E \left[ \left( W(t) - \frac{t}{T} W(T) \right) \left( W(s) - \frac{s}{T} W(T) \right) \right]$$

$$= E[W(t)W(s)] - \frac{s}{T} E[W(t)W(T)] - \frac{t}{T} E[W(s)W(T)] + \frac{st}{T^2} W^2(T)$$

$$= t \wedge s - \frac{st}{T} \square$$

Exercise 2.12. [CM14, Exercise 10.21] Consider the GBM; process $S(t) = S_0 e^{\mu t + \sigma W(t)}$, $t \geq 0$, $S_0 > 0$. The respective sampled maximum and minimum of this process are defined by

$$M^S(t) = \sup_{0 \leq u \leq t} S(u), \quad m^S(t) = \inf_{0 \leq u \leq t} S(u)$$

and the first hitting time to a level $B > 0$ is defined by

$$T^S_B = \inf \{ t \geq 0 : S(t) = B \}.$$

Derive expressions for the following:

(a) $P(M^S(t) \leq y, S(t) \leq s), t > 0, 0 < x \leq y \leq \infty, S_0 \leq y$.

(b) $P(M^S(t) \leq y), t > 0, S_0 \leq y < \infty$.

(c) $P(T^S_B \leq t), t > 0, S_0 < B$.

Solution.

(a) Let $X(t) = \mu t + \sigma W(t)$. Denoting $M(t) = \sup_{0 \leq s \leq t} W(s)$, We know that

$$f_{M^X(t),X(t)}(w,y) = e^{-\frac{1}{2} \nu^2 + \mu y} f_{M(t),W(t)}(w,y)$$

$$f_{M(t),W(t)}(w,y) = \frac{2(2w-y)}{t \sqrt{2\pi t}} e^{-\frac{(2w-y)^2}{2t}}$$

Note that $S(t) \leq s$ if and only if $X(t) \leq \ln \frac{s}{S_0}$ so

$$P(M^S(t) \leq y, S(t) \leq s) = P \left( M^X(t) \leq \ln \frac{y}{S_0}, X(t) \leq \ln \frac{s}{S_0} \right) \equiv P \left( M^X(t) \leq m, X(t) \leq x \right)$$

and

$$P(M^X(t) \leq m, X(t) \leq x) = \int_0^m \int_0^x f_{M^X(t),X(t)}(w,y) dy dw$$

$$= e^{-\frac{1}{2} \nu^2 t} \int_0^m \int_0^x e^{\mu y} \frac{2(2w-y)}{t \sqrt{2\pi t}} e^{-\frac{(2w-y)^2}{2t}} dy dw$$

And if you have enough patience you can type the computation of the latter integral...
(b) As above note that \( P(M^S(t) \leq y) = P \left( M^X(t) \leq \ln \frac{y}{S_0} \right) \) so it suffices to integrate the joint density

\[
f_{M^X(t),X(t)}(w,y) = e^{-\frac{1}{2}\mu^2+\mu y} \frac{2(2w-y)}{t\sqrt{2\pi t}e^{-(2w-y)^2/2t}}
\]

over the region \(-\infty < y \leq w, 0 \leq w \leq \ln \frac{y}{S_0}\), namely

\[
P \left( M^X(t) \leq \ln \frac{y}{S_0} \right) = \int_0^m \int_{-\infty}^{m} f_{M^X(t),X(t)}(w,y) \, dy \, dw
\]

where the second equality follows from Fubini’s theorem.

(c) To compute that \( P(T^S_B \leq t) \), simply note that

\[
P(T^S_B \leq t) = P(M^S(t) \geq B) = 1 - P(M^S(t) \leq B)
\]

and we computed the latter in part (b).

**Remark 2.2.** We recall briefly how to obtain

\[
f_{M^X(t),X(t)}(w,y) = e^{-\frac{1}{2}\mu^2+\mu y} f_{M(t),W(t)}(w,y) \tag{3}
\]

\[
f_{M(t),W(t)}(w,y) = \frac{2(2w-y)}{t\sqrt{2\pi t}e^{-(2w-y)^2/2t}} \tag{4}
\]

where, recall \( X(t) = \mu t + \sigma W(t), \ M^X(t) = \sup_{0 \leq s \leq t} X(s) \) and \( M(t) = \sup_{0 \leq s \leq t} W(t) \), where \( W(t) \) is a standard Brownian motion.

The joint density of Brownian motion and its maximum (4) follows from the reflection principle

\[
P(W(t) \leq w, T^W_m \leq t) = P(W(t) \geq 2m - w)
\]

Note on the one hand that for the LHS we have

\[
P(T^W_m \leq t, W(t) \leq w) = P(M(t) \geq m, W(t) \leq w) = \int_{-\infty}^m \int_{-\infty}^w f_{M(t),W(t)}(x,y) \, dy \, dx
\]

As for the RHS, since \( W(t) \sim \mathcal{N}(0, t) \) we have

\[
P(W(t) \geq 2m - w) = \int_{2m-w}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-u^2/2t} \, du
\]

Equating both and differentiating with respect to \( m \) and \( w \) yields (4).

To show (3), assume for simplicity that \( \sigma = 1 \). It then suffices to perform a change of measure that renders the Brownian motion driftless. Assuming that the market price of risk is \( \gamma \) and letting \( \hat{W}(t) = W(t) + \gamma(t) \) we get

\[
dX(t) = (\mu - \gamma)dt + d\hat{W}(t)
\]
which is driftless if $\gamma = \mu$. The change of measure is given by the Radon-Nikodym process

$$\rho_t = \left(\frac{d\hat{P}}{dP}\right)_t = \exp\left(-\frac{1}{2}\int_0^t \frac{\mu^2}{\sigma^2} ds - \int_0^t \mu ds\right) = e^{-\frac{1}{2}\mu^2 t - \mu \hat{W}(t)} = e^{\frac{1}{2}\mu^2 t - \mu \hat{W}(t)}$$

Hence

$$P(M^X(t) \leq m, X(t) \leq x) = E\left[I_{M^X(t) \leq m, X(t) \leq x}\right] = \hat{E}\left[\left(\frac{dP}{d\hat{P}}\right)_t I_{M^X(t) \leq m, X(t) \leq x}\right]$$

$$\leq \hat{E}\left[\rho_t^{-1}I_{M^X(t) \leq m, X(t) \leq x}\right]$$

$$\leq e^{-\frac{1}{2}\mu^2 t} \hat{E}\left[e^{\mu \hat{W}(t)} I_{M^\hat{W}(t) \leq m, \hat{W}(t) \leq x}\right]$$

$$\leq e^{-\frac{1}{2}\mu^2 t} E\left[e^{\mu W(t)} I_{M(t) \leq m, W(t) \leq x}\right]$$

$$= e^{-\frac{1}{2}\mu^2 t} \int_{-\infty}^m \int_{-\infty}^x e^{\mu y} f_{M(t), W(t)}(w, y) dy dw$$

where in [1] we used that $X(t) = \hat{W}(t)$ and in [2] we used the fact that the random variables $M^\hat{W}(t)$ and $\hat{W}(t)$ under measure $\hat{P}$ are the same as $M(t)$ and $W(t)$ under measure $P$. This shows (3).

□
3 Chapter 11: Introduction to continuous-time stochastic calculus

Exercise 3.1. [CM14, Exercise 11.4] Evaluate the following double stochastic integral

\[ \int_0^t \left( \int_0^s dW(u) \right) dW(s) \]

Solution. The inner integral is simply \( \int_0^s dW(u) = W(s) - W(0) = W(s) \) and then \( \int_0^t W(s) dW(s) \) can be computed by applying It’s formula to \( df(W(t)) \) with \( f(x) = x^2 \). Indeed,

\[ dW^2(s) = 2W(s)dW(t) + 2ds \]

so integrating over \( 0 \leq s \leq t \) and re-arranging we obtain

\[ \int_0^t W(s) dW(s) = \frac{1}{2} W^2(t) - t \]

□

Exercise 3.2. [CM14, Exercise 11.5] Show that

\[ \int_0^t W^2(s) dW(s) = \frac{1}{3} W^3(t) - \int_0^t W(s) ds \]

by using an appropriate It formula.

Solution. Applying It’s formula to compute \( df(W(s)) \), with \( f(x) = x^3 \) we obtain

\[ dW^3(s) = 3W^2(s)dW(s) + 3W(s)ds \]

so re-arranging and integrating over \( 0 \leq s \leq t \) yields

\[ \int_0^t W^2(s) dW(s) = \frac{1}{3} W^3(t) - \int_0^t W(s) ds \]

as claimed. □

Exercise 3.3. [CM14, Exercise 11.6] Use the It isometry property to calculate the variances of the following It integrals. Also explain why the integrals are well defined.

(a) \( \int_0^t |W(s)|^{1/2} dW(s) \).

(b) \( \int_0^t |W(s) + s|^2 dW(s) \).

(c) \( \int_0^t |W(s) + s|^{3/2} dW(s) \).
Solution. Recall that an It integral \( \int_0^t X(s) \, dW(s) \) is well defined provided that the integrand \( X(s) \) is \( \mathcal{F}(s) \)-measurable (which is obviously satisfied in all three cases) and provided that it satisfies the square integrability condition

\[
E \left[ \int_0^t X^2(s) \, ds \right] = \int_0^t E \left[ X^2(s) \right] \, ds < \infty
\]

and the It isometry property states that

\[
\text{Var} \left[ \int_0^t X(s) \, dW(s) \right] = \int_0^t E \left[ X^2(s) \right] \, ds
\]

(a) By It isometry we have

\[
\text{Var} \left[ \int_0^t |W(s)|^{1/2} \, dW(s) \right] = \int_0^t E[|W(s)|] \, ds = 0
\]

(b) For the integral \( \int_0^t |W(s) + s|^2 \, dW(s) \) we have that

\[
\text{Var} \left[ \int_0^t |W(s) + s|^2 \, dW(s) \right] = \int_0^t E \left[ |W(s) + s|^4 \right] \, ds
\]

\[
\overset{[1]}{=} \int_0^t E \left[ W^4(s) + 4W^3(s)s + 16W^2(s)s^2 + 4W(s)s^3 + s^4 \right] \, ds
\]

\[
\overset{[2]}{=} \int_0^t \left[ 3s^2 + 16s^3 + s^4 \right] \, ds = t^3 + 4t^4 + \frac{1}{5} t^5
\]

where in [2] we used the result of Exercise 11.9, to the effect that \( E[W^3(s)] = 0 \) and \( E[W^4(s)] = 6 \int_0^t E[W^2(s)] \, ds = 3t^2 \). This integral is clearly finite, which shows that the square integrability condition is satisfied. This could have been concluded directly from [1], since we are integrating a continuous function \( E \left[ |W(s) + s|^4 \right] \) over a compact set.

\[ \square \]

Exercise 3.4. [CM14, Exercise 11.8] Using It’s formula, show that the process defined by

\[
X(t) := W^4(t) - 6 \int_0^t W^2(u) \, du, \quad t \geq 0
\]

is a martingale with respect to a filtration for Brownian motion.

Solution. Note that \( X(t) = f(t, W(t)) \), with \( f(t, x) = x^4 - 6 \int_0^t W^2(u) \, du \) so by It’s formula

\[
dX(t) = -6W^2(t) \, dt + 4W^3(t) \, dW(t) + 6W^2(t) \, dt = 4W^3(t) \, dW(t)
\]

so \( X(t) \) is an It integral

\[
X(t) = 4 \int_0^t W^3(s) \, dW(s)
\]

which is a martingale provided that the usual square integrability is satisfied, which trivially is by continuity of Brownian motion

\[
E \left[ \int_0^t (W^3(s))^2 \, ds \right] = E \left[ \int_0^t W^6(s) \, ds \right] < \infty
\]

\[ \square \]
Exercise 3.5. [CM14, Exercise 11.9] Use It’s formula to show that for any integer \( k \geq 2 \)

\[
E[W^k(t)] = \frac{k(k-1)}{2} \int_0^t E\left[W^{k-2}(s)\right] \, ds
\]

and use this to derive a formula for all the moments of the standard normal distribution.

**Solution.** By It’s formula we have

\[
dW^k(t) = kW^{k-1}(t)dW(t) + \frac{1}{2}k(k-1)W^{k-2}(t)dt
\]

whence

\[
W^k(t) = k \int_0^t W^{k-1}(s) \, dW(s) + \frac{k(k-1)}{2} \int_0^t W^{k-2}(s) \, ds
\]

and taking expectations

\[
E[W^k(t)] = \frac{k(k-1)}{2} \int_0^t E[W^{k-2}(s)] \, ds
\]

as claimed. □

Exercise 3.6. [CM14, Exercise 11.10] Show that \( M(t) = e^{t/2} \sin(W(t)), \ t \geq 0 \) is a martingale with respect to a filtration for Brownian motion.

**Solution.** By It’s formula

\[
dM(t) = \frac{1}{2}e^{t/2} \sin(W(t)) + e^{t/2} \cos(W(t))dW(t) - \frac{1}{2}e^{t/2} \sin(W(t))dt = e^{t/2} \cos(W(t))dW(t)
\]

and since \( M(0) = 0 \) we have

\[
M(t) = \int_0^t e^{s/2} \cos(W(s)) \, dW(s)
\]

which is an It integral provided that the square integrability condition

\[
\int_0^t e^s E[\cos^2(W(s))] \, ds < \infty
\]

is satisfied (and this is clearly the case, by continuity). □

Exercise 3.7. [CM14, Exercise 11.11] Use It’s formula to show that for any non-random, continuously differentiable function \( f(t) \), the following formula of integration by parts holds:

\[
\int_0^t f(s) \, dW(s) = f(t)W(t) - \int_0^t f'(s)W(s) \, ds
\]

**Solution.** By It’s formula we have \( d[f(t)W(t)] = f'(t)W(t) \, dt + f(t)dW(t) \) and integrating we conclude

\[
f(t)W(t) = \int_0^t f'(s) \, dW(s) + \int_0^t f(s) \, dW(s)
\]

as claimed. □
Exercise 3.8. [CM14, Exercise 11.15] Let $\mathcal{N}(x)$ be the standard normal CDF and consider the process

$$X(t) \overset{def}{=} \mathcal{N} \left( \frac{W(t)}{\sqrt{T-t}} \right), \quad 0 \leq t < T$$

Express this process as an Itô process and show that it is a martingale with respect to any filtration of BM. Find the limiting value $X(T^-) = \lim_{t \to T^-} X(t)$. 

Solution. Consider the function

$$f(t, x) = \mathcal{N} \left( \frac{x}{\sqrt{T-t}} \right) = \int_{-\infty}^{\frac{x}{\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

By Itô’s formula we have

$$df(t, W(t)) = \frac{\partial f}{\partial t}(t, W(t))dt + \frac{\partial f}{\partial x}(t, W(t))dW(t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, W(t))dt$$

The derivatives in question are given by

$$\frac{\partial f}{\partial t}(t, x) = \frac{1}{2} \frac{1}{\sqrt{2\pi T-t}} e^{-x^2/(T-t)},$$

$$\frac{\partial f}{\partial x}(t, x) = \frac{1}{\sqrt{2\pi (T-t)}} \frac{1}{\sqrt{2\pi (T-t)}},$$

$$\frac{\partial^2 f}{\partial x^2}(t, x) = - \frac{1}{\sqrt{2\pi (T-t)}} \frac{1}{\sqrt{2\pi (T-t)}}$$

whence

$$dX(t) = \frac{1}{\sqrt{2\pi}} e^{-W^2(t)/2T} \frac{1}{\sqrt{T-t}} dW(t)$$

This is a martingale provided that the square integrability condition is satisfied, namely

$$\int_0^t E \left[ \frac{1}{T-u} e^{-W^2(u)/2T} \right] du < \infty$$

However

$$E \left[ \frac{1}{T-u} e^{-W^2(u)/2T} \right] = \frac{1}{T-u} \int_{-\infty}^{\infty} e^{-z^2/2T} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi T-u}} \int_{-\infty}^{\infty} e^{-z^2(1+u/T)} dz < \infty$$

for $u \in [0, T]$. Its integral over $[0, t]$ is thus also finite (being the integral of a continuous function over a compact interval).

Finally, it is clear that

$$\lim_{t \to T^-} X(t) = \mathcal{N} \left( \lim_{t \to T^-} \frac{W(t)}{\sqrt{T-t}} \right) = 1$$

□
Exercise 3.9. [CM14, Exercise 11.16] Suppose that the processes \( \{X(t)\}_{t \geq 0} \) and \( \{Y(t)\}_{t \geq 0} \) have the log-normal dynamics
\[
    dX(t) = X(t) (\mu_X(t) dt + \sigma_X dW(t)) \\
    dY(t) = Y(t) (\mu_Y(t) dt + \sigma_Y dW(t))
\]
Show that the process \( Z(t) = \frac{X(t)}{Y(t)} \) is also log-normal, with dynamics
\[
    dZ(t) = Z(t) (\mu_Z(t) dt + \sigma_Z dW(t))
\]
and determine the coefficients \( \mu_Z \) and \( \sigma_Z \) in terms of those of \( X \) and \( Y \). Solve the same problem now assuming that \( X \) and \( Y \) are governed by two correlated Brownian motions \( W^X \) and \( W^Y \), respectively, where \( \text{Corr}(W^X(t), W^Y(t)) = \rho t \), for a given correlation \(-1 \leq \rho \leq 1\).

Solution.

Exercise 3.10. [CM14, Exercise 11.17] Consider the time-homogeneous diffusion \( X(t) \) have a stochastic differential
\[
    dX(t) = [3X(t) - 1]dt + 2\sqrt{X(t)}dW(t), \quad X(t) \geq 0
\]
Find the stochastic differential for the process \( Y(t) := \sqrt{X(t)} \) and find the generator for \( Y(t) \).

Solution. Note that \( Y(t) = f(X(t)) \) with \( f(t, x) = \sqrt{x} \). By It’s formula we have
\[
    dY(t) = df(X(t)) = \frac{\partial f}{\partial x}(X(t))dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X(t))dX(t)dX(t)
\]
\[
    = \left[ \frac{1}{2\sqrt{X(t)}}dX(t) - \frac{1}{24X(t)^{3/2}}dX(t)dX(t) \right]dX(t)
\]
\[
    = \frac{1}{2\sqrt{X(t)}}[3X(t) - 1]dt + dW(t) - \frac{1}{2\sqrt{X(t)}}dt
\]
\[
    = \frac{1}{2\sqrt{X(t)}}[3X(t) - 2]dt + dW(t)
\]
\[
    = \frac{1}{2} \left[ 3Y(t) - \frac{2}{Y(t)} \right] dt + dW(t)
\]
Recall that given an It process
\[
    X(t) = \mu(t, X(t)) dt + \sigma(t, X(t))dW(t),
\]
its generator is the operator
\[
    \mathcal{G}_{t,x} = \mu(t, x) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2}{\partial x^2}
\]
which acts on functions \( f \in C^{1,2} \). In our case, for the process \( Y(t) \) we have
\[
    \mathcal{G}_{t,x}^Y = \frac{1}{2} \left[ 3x - \frac{2}{x} \right] \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2}
\]

\[\square\]
Exercise 3.11. [CM14, Exercise 11.18] Let \( X(t) = tW^2(t) \) and \( Y(t) = e^{W(t)} \). Find the stochastic differential of \( Z(t) = \frac{X(t)}{Y(t)} \). Compute the mean and variance of \( Z(t) \).

**Solution.** By It’s formula, the differentials of the two processes \( X(t) \) and \( Y(t) \) are

\[
\begin{align*}
    dX(t) &= [W^2(t) + t]dt + 2tW(t)dW(t), \\
    dY(t) &= e^{W(t)}dW(t) + \frac{1}{2}e^{W(t)}dt.
\end{align*}
\]

To compute \( dZ(t) = df(X(t), Y(t)) \) we apply the 2-dimensional It’s formula to the function \( f(x, y) = \frac{x}{y} \). Since

\[
\begin{align*}
    f_x &= \frac{1}{y}, & f_y &= -\frac{x}{y^2}, & f_{xx} &= 0, & f_{xy} &= -\frac{1}{y^2}, & f_{yy} &= \frac{2x}{y^3}
\end{align*}
\]

we have (omitting the dependence \( (X(t), Y(t)) \) in the first line)

\[
\begin{align*}
    dZ(t) &= f_xdX(t) + f_ydY(t) + \frac{1}{2}f_{xx}dX(t)dX(t) + f_{xy}dX(t)dY(t) + \frac{1}{2}f_{yy}dY(t)dY(t) \\
    &= \frac{1}{Y(t)}dX(t) - \frac{X(t)}{Y^2(t)}dY(t) - \frac{1}{Y^2(t)}dX(t)dY(t) + \frac{X(t)}{Y^3(t)}dY(t)dY(t) \\
    &= e^{-W(t)} \left[ (W^2(t) + t)dt + 2tW(t)dW(t) \right] - tW^2(t)e^{-2W(t)} \left[ \frac{1}{2}e^{W(t)}dt + e^{W(t)}dW(t) \right] \\
    &\quad - e^{-2W(t)}2tW(t)e^{W(t)}dt + tW^2(t)e^{-3W(t)}e^{2W(t)}dt \\
    &\quad + W(t)e^{-W(t)}t[2 - W(t)]dW(t)
\end{align*}
\]

and I am very likely to have made a mistake somewhere... □

Exercise 3.12. [CM14, Exercise 11.19] Let \( X(t) \) be a time-homogeneous diffusion process solving an SDE \( dX(t) = cX(t) + \sigma dW(t) \) with initial condition \( X(0) = x \in \mathbb{R} \) and where \( c, \sigma \) are constants. Consider the process defined by \( Y(t) = X^2(t) - 2c \int_0^t X^2(s) \ ds - \sigma^2 t, \ t \geq 0 \).

(a) Represent \( Y(t) \) as an It process and show that it is a martingale with respect to any filtration for Brownian motion.

(b) Compute the mean and variance of \( Y(t) \) for all \( t \geq 0 \).

**Solution.**

(a) By It’s formula we have

\[
\begin{align*}
    dY(t) &= 2X(t)dX(t) + \sigma^2 dt - 2cX^2(t)dt - \sigma^2 dt \\
    &= 2cX^2(t)dt + 2\sigma X(t)dW(t) - 2cX^2(t)dt \\
    &= 2\sigma X(t)dW(t)
\end{align*}
\]

whence

\[
Y(t) = Y(0) + 2\sigma \int_0^t X(s) \ dW(s) = x^2 + 2\sigma \int_0^t X(s) \ dW(s)
\]

which is a martingale provided that the square integrability condition \( \int_0^t E[X^2(s)] \ ds \) is satisfied. We verify this in part (b).
(b) Clearly $E[Y(t)] = x^2$. In order to compute the variance, we start by solving the SDE for $X(t)$ using an integrating factor. Note that
\[
d(e^{-ct}X(t)) = -ce^{-ct}X(t) + ce^{-ct}X(t)dt + \sigma e^{-ct}dW(t) = \sigma e^{-ct}dW(t)
\]
whence
\[
X(t) = xe^{ct} + \sigma \int_0^t e^{c(t-s)} dW(s)
\]
The expected value of $X(t)$ is $E[X(t)] = xe^{ct}$ and by Itô isometry we have
\[
\text{Var}[X(t)] = \int_0^t \sigma^2 e^{2c(t-s)} ds = -\frac{\sigma^2}{2c} e^{2c(t-s)} \bigg|_0^t = \frac{\sigma^2}{2c}[e^{2ct} - 1]
\]
Since we also have $\text{Var}[X(t)] = E[X^2(t)] - E[X(t)]^2$ we conclude that
\[
\text{Var}[Y(t)] = 4\sigma^2 \int_0^t E[X^2(s)] ds = 4\sigma^2 \int_0^t \bigg[ \frac{\sigma^2}{2c}(e^{2cs} - 1) + x^2 e^{2cs} \bigg] ds
\]
\[
= \frac{2\sigma^4}{c} \left[ (e^{2ct} - 1) \frac{x^2 + 1}{2c} - t \right]
\]
\[\square\]

**Exercise 3.13.** [CM14, Exercise 11.20] Use the Itô formula to write down stochastic differentials for the following processes.

(a) $Y(t) = \exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)$. Also find the expectation and variance of the process $X(t) = \int_0^t Y(s) ds$.

(b) $Z(t) = f(t)W(t)$ where $f$ is continuously differentiable.

**Solution.**

(a) By Itô’s formula we have
\[
dY(t) = e^{\sigma W(t)-\frac{1}{2}\sigma^2 t} \left[ -\frac{1}{2} \sigma^2 dt + \sigma dW(t) + \frac{1}{2} \sigma^2 \right] = \sigma e^{\sigma W(t)-\frac{1}{2}\sigma^2 t}dW(t) = \sigma Y(t)dW(t)
\]
If $A(t) = \int_0^t Y(s) ds$, it is clear that
\[
E[A(t)] = \int_0^t E[Y(s)] ds = \int_0^t e^{-\frac{1}{2}\sigma^2 s} E[e^{\sigma W(t)}] ds = \int_0^t e^{-\frac{1}{2}\sigma^2 s} e^{\frac{1}{2}\sigma^2 t} ds = t
\]
As for the variance we have
\[
\text{Var}[A(t)] = E[A^2(t)] - E[A(t)]^2 = E[A^2(t)] - t^2
\]
and
\[
E[A^2(t)] = E \left[ \int_0^t Y(u) du \int_0^t Y(v) dv \right] = \int_0^t \int_0^t E[Y(u)Y(v)] du dv
\]
which is computed as on [CM14, Page 413].
(b) By It’s formula we have
\[ dZ(t) = d[f(t)W(t)] = f'(t)W(t)dt + f(t)dW(t) \]

Exercise 3.14. [CM14, Exercise 11.22] A time-homogeneous diffusion process \( X \) has stochastic differential \( dX(t) = X(t)(1 - X(t))dW(t) \). Assuming that \( 0 < X(t) < 1 \), show that the process \( Y(t) = \ln \left( \frac{X(t)}{1-X(t)} \right) \) has a constant diffusion coefficient.

Solution. Note that \( Y(t) = f(X(t)) \) with \( f(x) = \ln \frac{x}{1-x} \). Clearly \( f'(x) = \frac{1}{x(1-x)} \). By It’s formula we have
\[ dY(t) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))X^2(t)[1 - X(t)]^2dt = dW(t) + O(dt) \]
so that the diffusion term is constant and equal to 1. Since \( f''(x) = \frac{2x-1}{x^2(1-x)^2} \), the drift term will be given by
\[ \frac{1}{2}f''(X(t))X^2(t)[1 - X(t)]^2 = \frac{1}{2}(2X(t) - 1). \]

Exercise 3.15. [CM14, Exercise 11.23] Let \( X(t) = (1 - t) \int_0^t \frac{1}{1-s} dW(s) \), where \( 0 \leq t < 1 \). Provide the stochastic differential equation for \( X(t) \) and check your answer by solving the SDE obtained subject to the initial condition \( X(0) = 0 \).

Solution. Note that \( X(t) = f(t,Y(t)) \) where \( f(t,y) = (1-t)y \) and \( Y(t) = \int_0^t \frac{1}{1-s} dW(s) \). By It’s formula we have
\[
\begin{align*}
   dX(t) &= f_t(t,Y(t))dt + f_y(t,Y(t))dY(t) + \frac{1}{2}f_{yy}(t,Y(t))dY(t)dY(t) \\
   &= -Y(t)dt + (1-t)dY(t) \\
   &= -\left[ \int_0^t \frac{dW(s)}{1-s} \right] dt + (1-t)\left[ \frac{dW(t)}{1-t} \right] \\
   &= -\left[ \int_0^t \frac{dW(s)}{1-s} \right] dt + dW(t) \\
   &= -\frac{X(t)}{1-t} dt + dW(t)
\end{align*}
\]

Exercise 3.16. [CM14, Exercise 11.25] Let \( g(y) \) be a given function of \( y \), and suppose that \( f(x) \) is a solution of \( f'(x) = g(f(x)) \). Show that \( X(t) = f(W(t)) \) is a solution of the SDE
\[ dX(t) = \frac{1}{2}g(X(t))g'(X(t))dt + g(X(t))dW(t) \]
Solution. Since $f'(x) = g(f(x))$, clearly $f''(x) = g'(f(x))f'(x)$. By It’s formula we have that
\[
\begin{align*}
    dX(t) &= df(W(t)) = f'(W(t))dW(t) + \frac{1}{2}f''(W(t))dt \\
         &= g(f(W(t)))dW(t) + \frac{1}{2}g'(f(W(t))f'(W(t)) \\
         &= g(X(t))dW(t) + \frac{1}{2}g'(X(t))g(X(t))dt
\end{align*}
\]
as claimed. □

Exercise 3.17. [CM14, Exercise 11.26] Use Exercise 11.25 to solve the following non-linear SDE
\[
dX(t) = X^3(t)dt + X^2(t)dW(t), \quad X(0) = x_0
\]

Solution. By Exercise 11.25, the solution is given by
\[
X(t) = f(W(t)), \quad \text{where } \begin{cases} 
    f'(x) = g(f(x)) = |f(x)|^2, \\
    f(0) = x_0
\end{cases}
\]

Separating variables one immediately obtains
\[
f(x) = \frac{x_0}{1 - xx_0}
\]
which is singular at $x = \frac{1}{x_0}$. □

Exercise 3.18. [CM14, Exercise 11.32] Consider the boundary value problem for the heat equation
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(t,x) \frac{\partial^2 V}{\partial x^2} = 0, \quad V(T,x) = f(x)
\]

Show that the solution is given by
\[
V(t,x) = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} f(y)e^{-\frac{(y-x)^2}{2(T-t)}} dy
\]

Solution. This is essentially identical to [CM14, Example 11.10]. By the Feynman-Kac theorem [CM14, Theorem 11.7], if a stochastic process $\{X(t)\}_{t \geq 0}$ satisfies the SDE
\[
dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t), \quad (5)
\]
then the $C^{1,2}$ function $V(t,x)$ solving the PDE
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(t,x) \frac{\partial^2 V}{\partial x^2} + \mu(t,x) \frac{\partial V}{\partial x} = 0, \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^+
\]

admits a representation
\[
V(t,x) = E[f(X(T))|X(t)=x] \quad (6)
\]
provided that $E[|f(X(T))|] < \infty$. 

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In our case, \( \mu(t, x) \equiv 0 \) and \( \sigma(t, x) = 1 \) so the SDE (5) becomes \( dX(t) = dW(t) \) and has a solution \( X(T) = X(t) + [W(T) - W(t)] \). The representation (6) then reads

\[
V(t, x) = E[f(X(T))|X(t) = x] = E[f(X(t) + W(T) - W(t)|X(t) = x]
\]

\[
= E[f(x + W(T) - W(t))]|x] = \int_{-\infty}^{\infty} f(y)\varphi_{x,T-t}(y) dy
\]

\[
= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} f(y)e^{-\frac{(y-x)^2}{2(T-t)}} dy
\]

where in \([\ast]\) we used the fact that \( E[\phi(X)] = \int_{-\infty}^{\infty} \phi(y)\varphi_X(y) dy \), where \( \varphi_X(\cdot) \) denotes the PDF of \( X \), and also that \( x + [W(T) - W(t)] \sim \mathcal{N}(x, T-t) \). \( \square \)

**Exercise 3.19.** [CM14, Exercise 11.34] Determine \( f(t, x) \) satisfying the following boundary value problem

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(x, t) \frac{\partial^2 V}{\partial x^2} + \mu(x, t) \frac{\partial V}{\partial x} = 0, \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^+, \quad f(T, x) = 1_{[K_1, K_2]}(x)
\]

Solution. Write \( \phi(x) = 1_{[K_1, K_2]}(x) \) more generally to begin with and we will specialize at the end. By the Feynman-Kac theorem [CM14, Theorem 11.7], if a stochastic process \( \{X(t)\}_{t \geq 0} \) satisfies the SDE

\[
dX(t) = \mu(t, X(t)) dt + \sigma(t, X(t)) dW(t), \quad (7)
\]

then the \( C^{1,2} \) function \( f(t, x) \) solving the PDE

\[
\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 f}{\partial x^2} + \mu(x, t) \frac{\partial f}{\partial x} = 0, \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^+, \quad f(T, x) = \phi(x)
\]

admits a representation

\[
f(t, x) = E[\phi(X(T))|X(t) = x] \quad (8)
\]

provided that \( E[|\phi(X(T))|] < \infty \).

In this case, the drift and volatility are \( \mu(t, x) = \mu x \) and \( \sigma(t, x) = \sigma x \), for real constants \( \mu, \sigma > 0 \) and the SDE (7) reads \( dX(t) = \mu X(t) dt + \sigma X(t) dW(t) \). This is a linear SDE with solution (c.f. [CM14, Equation 11.27])

\[
X(T) = X(t) \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) (T-t) + \sigma(W(T) - W(t)) \right]
\]

The representation (8) then reads

\[
f(t, x) = E[\phi(X(T))|X(t) = x]
\]

\[
= E[\phi(X(t)) \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) (T-t) + \sigma(W(T) - W(t)) \right] |X(t) = x]
\]

\[
= E \left[ \phi \left( x \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) (T-t) + \sigma y \right] \right) \right]
\]

\[
= \int_{-\infty}^{\infty} \phi \left( x \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) (T-t) + \sigma y \right] \right) \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{y^2}{2(T-t)}} dy \quad (9)
\]
We now specialize to $\phi(x) = 1_{[K_1, K_2]}(x)$. Note that

$$K_1 \leq x \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) (T - t) + \sigma y \right] \leq K_2$$

if and only if

$$\frac{1}{\sigma} \ln \left( \frac{K_1}{x} \right) - \left( \mu - \frac{\sigma^2}{2} \right) (T - t) \leq y \leq \frac{1}{\sigma} \ln \left( \frac{K_2}{x} \right) - \left( \mu - \frac{\sigma^2}{2} \right) (T - t)$$

so that $\phi \left( x \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) (T - t) + \sigma y \right] \right) \neq 0$ if and only if $y \in [\alpha_1, \alpha_2]$ whence the integral in (9) becomes

$$f(t, x) = \int_{\alpha_1}^{\alpha_2} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{y^2}{2(T-t)}} dy = N(\alpha_2) - N(\alpha_1)$$

Exercise 3.20. [CM14, Exercise 11.36] Assume that a stock price process $\{S(t)\}_{t \geq 0}$ satisfies the SDE

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t)$$

with constants $r, \sigma > 0$, and where $\{\tilde{W}(t)\}_{t \geq 0}$ is a standard $\tilde{P}$-BM. By using Girsanov’s theorem, find the explicit expression for the Radon-Nikodym derivative process

$$\rho_t = \left( \frac{d\hat{P}}{dP} \right)_t$$

such that the process defined by $\hat{S}(t) = \frac{e^{rt}}{S(t)}$, $t \geq 0$ is a $\hat{P}$-martingale. Give the SDE satisfied by the stock price $S(t)$ with respect to the $\hat{P}$-BM.

Solution. We start by computing the differential of the process $\hat{S}(t)$.

$$d\hat{S} = d \left( \frac{e^{rt}}{S(t)} \right) = \frac{e^{rt}}{S(t)} \left( r \frac{e^{rt}}{S^2(t)} dt - \frac{e^{rt}}{S^2(t)} dS(t) + \frac{1}{2} \frac{2e^{rt}}{S^3(t)} \sigma^2 S^2(t) dt \right) = \sigma \hat{S}(t) \left[ -\sigma dt + d\tilde{W}(t) \right]$$

Define $\hat{W}(t) := \tilde{W}(t) - \sigma t$. By Girsanov’s theorem, $\hat{W}$ is a $\hat{P}$-BM under the measure defined by the Radon-Nikodym derivative process

$$\rho_t = \left( \frac{d\hat{P}}{dP} \right)_t = \exp \left( -\frac{1}{2} \int_0^t \sigma ds + \int_0^t \sigma d\hat{W}(s) \right) = e^{-\frac{1}{2} \sigma^2 t + \sigma \hat{W}(t)}$$

and the SDE satisfied by the stock price $S(t)$ with respect to the $\hat{P}$-BM is

$$d\hat{S}(t) = \sigma d\hat{W}(t)$$
Exercise 3.21. [CM14, Exercise 11.38] Consider a one-dimensional general diffusion process \( \{X(t)\}_{t \geq 0} \) having a transition PDF \( p(s, t; x, y), s < t \), with respect to a given probability measure \( P \), for all \( x, y \) in the state space of the process. Assume a change of measure \( P \rightarrow \hat{P} \) is defined by the Radon-Nikodym derivative process

\[
\rho_t = \left( \frac{d\hat{P}}{dP} \right)_t = h(t, X(t)), \quad \forall t \geq 0
\]

Let \( \hat{p}(s, t; x, y) \) denote the PDF with respect to the measure \( \hat{P} \). Show that the two transition PDF’s are related by

\[
\hat{p}(s, t; x, y) = \frac{h(t, y)}{h(s, x)} p(s, t; x, y)
\]

Solution.

\[
\hat{P}(s, t; x, y) = \hat{P}(X(t) \leq y|X(s) = x) = \hat{E}[1_{X(t) \leq y}|X(s) = x]
\]

\[
= \frac{1}{\rho_s} \hat{E}[\rho_t 1_{X(t) \leq y}|X(s) = x]
\]

\[
= \frac{1}{h(s, X(s))} \hat{E}[h(t, X(t))1_{X(t) \leq y}|X(s) = x]
\]

\[
= \frac{1}{h(s, x)} \int_{-\infty}^{\infty} h(t, u)1_{u \leq y}p(s, t; x, u) \, du
\]

\[
= \frac{1}{h(s, x)} \int_{-\infty}^{y} h(t, u)p(s, t; x, u) \, du
\]

so by the first fundamental theorem of calculus

\[
\hat{p}(s, t; x, y) = \frac{\partial \hat{P}}{\partial y}(s, t; x, y) = \frac{h(t, y)}{h(s, x)} \hat{p}(s, t; x, y)
\]

as claimed. □
Chapter 12: Risk-Neutral pricing in the Black-Scholes Economy: one underlying stock

Exercise 4.1. [CM14, Exercise 12.4] Assume the standard Black-Scholes model in an economy with constant continuously compounded interest rate $r$ and with stock price process $\{S(t)\}_{t \geq 0}$ as a GBM with constant volatility $\sigma$ and constant continuous dividend yield $q$. Let $S(t) = S > 0$ be the spot at time $t < T$, where $T$ is the expiry date. Derive the corresponding arbitrage-free time-$t$ pricing formula $V(t, S)$ for a European option with the following payoffs:

(a) $\Lambda(S(T)) = \sum_{n=0}^{N} a_n S^n(T)$, $N \geq 1$, $a_n \in \mathbb{R}$.

(b) $\Lambda(S(T)) = (S^\alpha(T) - K)1_{S(T) > K}$, $\alpha \in \mathbb{R} \setminus 0$.

Solution.

(a) Denote as usual $\tau = T - t$. Clearly

\[ V(t, S) = e^{-rt} \tilde{E}_{t,S} [\Lambda(S(T))] = e^{-rt} \tilde{E}_{t,S} \left[ \sum_{n=0}^{N} a_n S^n(T) \right] = e^{-rt} \sum_{n=0}^{N} a_n \tilde{E}_{t,S} [S^n(T)] \]

These expectations are computed in [CM14, Equation 12.40]. Writing

\[ S(T) = S(t)e^{(r-q-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{T}Z}, \quad Z = \frac{1}{\sqrt{T}}(W(T) - W(t)) \sim N(0, 1) \]

we have:

\[ \tilde{E}_{t,S} [S^n(T)] = \tilde{E}_{t,S} \left[ S^n e^{(r-q-\frac{1}{2}\sigma^2)\tau + n\sigma\sqrt{T}Z} \right] = S^n e^{(r-q-\frac{1}{2}\sigma^2)\tau} E \left[ e^{n\sigma\sqrt{T}Z} \right] \]

\[ \overset{(*)}{=} S^n e^{(r-q+\frac{1}{2}\sigma^2(n-1))\tau} \]

where in $[*]$ we used [CM14, Formula A.2]. Putting everything together we conclude that

\[ V(t, S) = e^{-rt} \sum_{n=0}^{N} a_n S^n e^{(r-q+\frac{1}{2}\sigma^2(n-1))\tau} \]

(b) In this case we have

\[ V(t, S) = e^{-rt} \tilde{E}_{t,S} [(S^\alpha(T) - K)1_{S(T) > K}] = e^{-rt} \tilde{E}_{t,S} [S^\alpha(T)1_{S(T) > K}] - Ke^{-rt} \tilde{E}_{t,S} [1_{S(T) > K}] \]

Write as usual

\[ S(T) = S(t)e^{(r-q-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{T}Z}, \quad Z = \frac{1}{\sqrt{T}}(W(T) - W(t)) \sim N(0, 1) \]
The second expectation above is simply
\[
\tilde{E}_{t,S}[\mathbb{I}_{S(T)>K}] = \tilde{P}\left(S e^{(r-q+\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}Z} > K\right) = \tilde{P}\left(Z > \frac{\ln{\frac{S}{K}} - (r-q+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)
\]
\[= \tilde{P}\left(Z < -\frac{\ln{\frac{S}{K}} + (r-q+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) = \tilde{P}\left(Z < -\ln{\frac{S}{K}} + \frac{1}{2}\sigma^2\tau\right) = \tilde{P}\left(Z > \ln{K} - \frac{1}{2}\sigma^2\tau\right)\]

The first expectation is computed in [CM14, Equation 12.41]:
\[
\tilde{E}_{t,S}\left[\mathbb{I}_{S(T)>K}\right] = S^n e^{n(r-q+\frac{1}{2}\sigma^2)\tau} \tilde{E}_{t,S}\left[e^{\sigma\sqrt{\tau}Z} \mathbb{I}_{Z > \frac{\ln{\frac{S}{K}} - (r-q+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}\right]
\]
\[= S^n e^{n(r-q+\frac{1}{2}\sigma^2)(n-1)\tau} \mathcal{N}\left(\frac{\ln{\frac{S}{K}} + (r-q + \frac{2n-1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)\]

so in conclusion we have:
\[
V(t,S) = e^{-r\tau} S^n e^{n(r-q+\frac{1}{2}\sigma^2(n-1)\tau)} \mathcal{N}\left(\frac{\ln{\frac{S}{K}} + (r-q + \frac{2n-1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau} \mathcal{N}\left(-d_-\right)
\]
\[d_- = \frac{\ln{\frac{S}{K}} + (r-q+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\]

Exercise 4.2. [CM14, Exercise 12.5] Assume the standard Black-Scholes model in an economy with constant continuously compounded interest rate \(r\) and with stock price process \(\{S(t)\}_{t \geq 0}\) as a GBM with constant volatility \(\sigma\) and constant continuous dividend yield \(q\). A European call spread has payoff
\[
\Lambda(S(T)) \begin{cases} 
0, & S(T) \leq K, \\
S(T) - K, & K < S(T) < K + \epsilon, \\
\epsilon, & S(T) \geq K + \epsilon
\end{cases}
\]
where \(K, \epsilon\) are positive real constants.

(a) Give a sketch of the payoff function.

(b) Derive a formula for the option’s present value \(V(t,S)\) and \(\Delta(t,S) = \frac{\partial V}{\partial S}\).

(c) Find \(\lim_{\epsilon \to 0} V(t,S)\) and \(\lim_{\epsilon \to \infty} V(t,S)\) and explain your results.

Solution.

(a)
(b) The payoff can be decomposed as a combination of call payoffs

\[ \Lambda(S(T)) = (S(T) - K)^+ - (S(T) - K - \epsilon)^+ \]

so

\[
V(t, S) = \tilde{E}_{t, S} [(S(T) - K)^+] - \tilde{E}_{t, S} [(S(T) - K - \epsilon)^+]
\]

\[
= e^{-qt} N\left(d_+\left(\frac{e^{-qt} S}{K}, \tau\right)\right) - e^{-rt} K N\left(d_-\left(\frac{e^{-qt} S}{K}, \tau\right)\right)
\]

\[
- e^{-qt} N\left(d_+\left(\frac{e^{-qt} S}{K + \epsilon}, \tau\right)\right) + e^{-rt} K N\left(d_-\left(\frac{e^{-qt} S}{K + \epsilon}, \tau\right)\right)
\]

As for the delta we have

\[
\frac{\partial V}{\partial S}(t, S) = e^{-qt} \left[ N\left(d_+\left(\frac{e^{-qt} S}{K}, \tau\right)\right) - N\left(d_+\left(\frac{e^{-qt} S}{K + \epsilon}, \tau\right)\right)\right]
\]

(c) Obviously \( \lim_{\epsilon \to 0} V(t, S) = 0 \) since the terms in the second line of equation (10) then equal those in the first one and have opposite sign. Note that for \( \epsilon = 0 \) the option payoff is identically zero.

As \( \epsilon \to \infty \), the option becomes a simple call struck at \( K \) whence

\[
\lim_{\epsilon \to \infty} V(t, S) = e^{-qt} N\left(d_+\left(\frac{e^{-qt} S}{K}, \tau\right)\right) - e^{-rt} K N\left(d_-\left(\frac{e^{-qt} S}{K}, \tau\right)\right)
\]

This can also be seen by taking the limit in equation (10) and noting that

\[
\lim_{\epsilon \to \infty} N\left(d_+\left(\frac{e^{-qt} S}{K + \epsilon}, \tau\right)\right) = N\left(d_+\left(0, \tau\right)\right) = N(-\infty) = 0
\]

since recall \( d_+(x, \tau) = \frac{\ln x - (r + \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}} \).

\[ \square \]

**Exercise 4.3.** [CM14, Exercise 12.7]

*Solution.*

\[ \square \]

**Exercise 4.4.** [CM14, Exercise 12.8]

*Solution.*

\[ \square \]

**Exercise 4.5.** [CM14, Exercise 12.9] A so-called pay-later European option costs the holder nothing (i.e. zero premium) to set up at present time \( t = 0 \). The payoff to the holder is \( (S(T) - K)^+ \). Moreover, the holder must pay out \( X \) dollars to the writer in the case that \( S(T) \geq 0 \). Derive an expression fr the fair value of \( X \). Determine the fair value in the limit of infinite volatility \( \lim_{\sigma \to \infty} X(\sigma) \). Assume the standard Black-Scholes model in an economy with constant continuously compounded interest rate \( r \) and with stock price process \( \{S(t)\}_{t \geq 0} \) as a GBM with constant volatility \( \sigma \) and constant continuous dividend yield \( q \).
**Solution.** The effective payoff of the pay-later option is

\[ \Lambda^\text{eff}(S(T)) = (S(T) - K)^+ - X\mathbb{I}_{\{S(T) \geq K\}} \]

and we need to determine the fair value of \( X \) such that the time-0 expected value of the latter is zero, namely

\[ \tilde{E}_{0,S}[\Lambda^\text{eff}(S(T))] = \tilde{E}_{0,S}[(S(T) - K)^+ - X\mathbb{I}_{\{S(T) \geq K\}}] = 0 \]

Clearly

\[ \tilde{E}_{0,S}[\Lambda^\text{eff}(S(T))] = e^{-rT}\tilde{E}_{0,S}[(S(T) - K)^+] - Xe^{-rT}\tilde{E}_{0,S}[(1 - X)\mathbb{I}_{\{S(T) \geq K\}}] \]

\[ = S e^{-qT}N\left(d_+(\frac{e^{-qT}S}{K}, T)\right) - K e^{-rT}N\left(d_-(\frac{e^{-qT}S}{K}, T)\right) \]

where as usual \( d_\pm = \frac{\ln x + (r - q \pm \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \).

Imposing \( \tilde{E}_{0,S}[\Lambda^\text{eff}(S(T))] = 0 \) and solving for \( X \) yields

\[ X = S e^{(r-q)T} \frac{N\left(d_+(\frac{e^{-qT}S}{K}, T)\right)}{N\left(d_-(\frac{e^{-qT}S}{K}, T)\right)} - K \]

Note that

\[ \lim_{\sigma \to \infty} d_\pm(x, \tau) = \lim_{\sigma \to \infty} \frac{\ln x + (r - q \pm \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \equiv \pm \lim_{\sigma \to \infty} \frac{\sigma \sqrt{T}}{2} = \pm \infty \]

so

\[ \lim_{\sigma \to \infty} N(d_+) = 1, \quad \lim_{\sigma \to \infty} N(d_-) = 0 \]

whence \( \lim_{\sigma \to \infty} X = \infty \). \( \square \)

**Exercise 4.6.** [CM14, Exercise 12.10] Let \( C(S, \tau) \) be the Black-Scholes pricing formula of a standard call option with spot \( S \), strike \( K \), fixed interest rate \( r \), zero stock dividend, constant volatility \( \sigma \) and time to maturity \( \tau > 0 \).

(a) Show that the respective limiting values of the call price for vanishing and infinite volatility are given by

\[ \lim_{\sigma \to 0} C(S, \tau) = (S - Ke^{-r\tau})^+, \quad \lim_{\sigma \to +\infty} C(S, \tau) = S \]

(b) Give a financial interpretation of both limits. Note that the second limit is independent of the strike value \( K \); give a financial intuition for this fact.

**Solution.**
(a) We performed an identical computation in the previous exercise. The Black-Scholes formula for a call option is

\[ C(S, \tau) = SN\left(d_+\left(\frac{S}{K}, \tau\right)\right) - Ke^{-r\tau}N\left(d_-\left(\frac{S}{K}, \tau\right)\right), \quad d_\pm(x, \tau) = \frac{\ln x + \left(r \pm \frac{1}{2} \sigma^2\right) \tau}{\sigma \sqrt{\tau}} \]

Also note that

\[ \lim_{\sigma \to \infty} d_\pm \equiv \pm \lim_{\sigma \to \infty} \frac{\sigma \sqrt{\tau}}{2} = \pm \infty \]

whence

\[ \lim_{\sigma \to \infty} N(d_+) = 1, \quad \lim_{\sigma \to \infty} N(d_-) = 0 \]

so

\[ \lim_{\sigma \to \infty} C(S, \tau) = \lim_{\sigma \to \infty} SN\left(d_+\left(\frac{S}{K}, \tau\right)\right) - Ke^{-r\tau}N\left(d_-\left(\frac{S}{K}, \tau\right)\right) = S \]

On the other hand,

\[ \lim_{\sigma \to 0} d_\pm\left(\frac{S}{K}, \tau\right) \equiv \lim_{\sigma \to 0} \frac{\ln \frac{S}{K} + r\tau}{\sigma \sqrt{\tau}} = \begin{cases} +\infty, & \ln \frac{S}{K} + r\tau \geq 0, \\ -\infty, & \ln \frac{S}{K} + r\tau < 0 \end{cases} \]

Note further that \( \ln \frac{S}{K} + r\tau \geq 0 \) if and only if \( S - e^{-r\tau} \geq 0 \). Hence

\[ \lim_{\sigma \to 0} C(S, \tau) = \lim_{\sigma \to 0} SN\left(d_+\left(\frac{S}{K}, \tau\right)\right) - Ke^{-r\tau}N\left(d_-\left(\frac{S}{K}, \tau\right)\right) = 0 \cdot \mathbb{1}_{\{\ln \frac{S}{K} + r\tau < 0\}} + (S - Ke^{-r\tau}) \cdot \mathbb{1}_{\{\ln \frac{S}{K} + r\tau \geq 0\}} = (S - Ke^{-r\tau})^+ \]

(b) The more volatile the market is, the more expensive an option is. In the absence of arbitrage, time-\( t \) value of a call is bounded above by the spot, so as the volatility grows indefinitely, the call value converges to its no-arbitrage maximum \( S \)...

\[
\square
\]

**Exercise 4.7.** [CM14, Exercise 12.11] Consider the value of a European call option written by an issuer who only has a fraction \( 0 \leq \alpha < 1 \) of the underlying asset. That is, at expiration time \( T \) the payoff of this type of call is given by

\[ V_T = (S(T) - K)^+\mathbb{1}_{\{\alpha S(T) \geq S(T) - K\}} + \alpha S(T)\mathbb{1}_{\{\alpha S(T) < S(T) - K\}} \]

Let \( C_L(S, \tau; K, \alpha) \) denote the value of a European call, where \( \tau = T - t \) is the time to expiry, \( K > 0 \) is the strike, \( S(t) = S \) is the spot of the underlying. Show that

\[ C_L(S, \tau; K, \alpha) = C(S, K, \tau) - (1 - \alpha)C\left(S, \frac{K}{1 - \alpha}, \tau\right) \]

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What is the position held in the stock at time $t$?

The value of an option with payoff $(S(T) - K)^+\mathbb{1}_{\{S(T) \geq K\}} + \alpha S(T)\mathbb{1}_{\{S(T) < K\}}$

so the no-arbitrage price of the options is given by the usual risk-neutral formula

$$C_L(S, \tau; K, \alpha) = e^{-rt} \mathbb{E}[V_T] = e^{-rt} \mathbb{E}\left[(S(T) - K)^+\mathbb{1}_{\{S(T) \geq K\}} + \alpha S(T)\mathbb{1}_{\{S(T) < K\}}\right]$$

We seek to show that $C(S, K, \tau) - (1 - \alpha)C(S, \frac{K}{1-\alpha}, \tau)$ indeed represents the above time-$t$ value (11) of an option with payoff $V_T$. Indeed:

$$C(S, K, \tau) = (1 - \alpha)C \left( S, \frac{K}{1-\alpha}, \tau \right)$$

Note: the option value is dependent on $(S, K, \tau, T, r, \sigma, \alpha)$

and the last expression is precisely (11).

**Exercise 4.9.** [CM14, Exercise 12.20] Assume the stock price $\{S(t)\}_{t \geq 0}$ is a GBM with constant volatility $\sigma$ and zero dividend in an economy with constant interest rate $r$. Let $t < T_0 < T$, i.e. $T_0$ is an arbitrary intermediate time before expiry time $T$, and consider a European-style option with payoff at time $T$ given by

$$V_T = \min\{S(T_0), S(T)\}$$

(a) Show that the value $V$ at time $t \leq T_0$ of this option is given by

$$V = S(t) \left[ \mathcal{N}(-d_+) + e^{-r(T-T_0)}\mathcal{N}(d_-) \right], \quad d_+ = \frac{r + \frac{1}{2}\sigma^2}{\sigma}\sqrt{T-t}$$

Note: the option value is dependent on $T - T_0$ and does not depend on $t$.

(b) What is the position held in the stock at time $t \leq T_0$ in a self-financing replicating strategy?
Solution.

(a) We may write the payoff as

\[ V_T = \min\{S(T_0), S(T)\} = S(T) \mathbb{I}_{\{S(T) < S(T_0)\}} + S(T_0) \mathbb{I}_{\{S(T_0) < S(T)\}} \]

and the option price is given by the usual risk-neutral pricing formula

\[ V(t, S) = e^{-r(T-t)} \tilde{E}_t[V_T] = e^{-r(T-t)} \left[ \tilde{E}_t[S(T) \mathbb{I}_{\{S(T) < S(T_0)\}}] + \tilde{E}_t[S(T_0) \mathbb{I}_{\{S(T_0) < S(T)\}}] \right] \]  

(12)

In order to compute these expectations we condition first on \( S(T_0) \) and use iterated conditioning, namely

\[
\tilde{E}_t[S(T) \mathbb{I}_{\{S(T) < S(T_0)\}}] = \tilde{E}_t\left[ E\left[ S(T) \mathbb{I}_{\{S(T) < S(T_0)\}} | S(T_0) \right] \right] \\
= e^{(r- \frac{1}{2} \sigma^2)(T-T_0) + \sigma \sqrt{T-T_0} Z_1} \begin{cases} \\ \tilde{E}_t[S(T_0) \mathbb{I}_{\{Z_1 < -d_-\}}] \end{cases} \\
= \left[ e^{(r- \frac{1}{2} \sigma^2)(T-T_0) \tilde{E}_t[S(T_0)N(-d_+)]} \right] \\
= \left[ e^{(r(T-T_0)-t) S(t)N(d_+)} \right] \\
= e^{r(T-t)} S(t) N(d_+) 
\]

Here [1] follows from the fact that \( S(T) < S(T_0) \) if and only if

\[
Z < -\frac{(r- \frac{1}{2} \sigma^2)(T-T_0)}{\sigma \sqrt{T-T_0}} = -\frac{r- \frac{1}{2} \sigma^2}{\sigma} \sqrt{T-T_0}. 
\]

In [2] we used [CM14, Formula A.2] to write

\[
E\left[ e^{\sigma \sqrt{T-T_0} Z_1} I_{\{Z_1 < -d_-\}} \right] = e^{\frac{1}{2} \sigma^2 (T-T_0)} N(-d_+) 
\]

and in [3] we used that \( e^{-rS(t)} \) is a \( \tilde{P} \)-martingale. The first summand in equation (12) is thus

\[
e^{-r(T-t)} \tilde{E}_t[S(T) \mathbb{I}_{\{S(T) < S(T_0)\}}] = S(t) N(-d_+) 
\]

An identical computation shows that the second expectation in equation (12) is

\[
\tilde{E}_t[S(T_0) \mathbb{I}_{\{S(T_0) < S(T)\}}] = \tilde{E}_t\left[ E\left[ S(T_0) \mathbb{I}_{\{S(T_0) < S(T)\}} | S(T_0) \right] \right] \\
= \tilde{E}_t[S(T_0)N(d_-)] = e^{r(T-t)} S(t) N(d_-) 
\]

whence

\[
e^{-r(T-T_0)} \tilde{E}_t[S(T_0) \mathbb{I}_{\{S(T_0) < S(T)\}}] = e^{-r(T-T_0)} S(t) N(d_-) 
\]

Adding up both expressions we conclude that

\[
V = S(t) \left[ N(-d_+) + e^{-r(T-T_0)} N(d_-) \right] 
\]

as claimed.
(b) The Delta of this option is
\[ \frac{\partial V}{\partial S} = \mathcal{N}(-d_+) + e^{-r(T-T_0)} \mathcal{N}(d_-) \]
which remains static over time.

Exercise 4.10. [CM14, Exercise 12.22]

Solution.

Exercise 4.11. [CM14, Exercise 12.25] Consider the discrete geometric averaging of a stock price process at evenly distributed discrete times \( t_j = t_0 + j\delta t, \ j = 1, 2, \ldots, n \), with a time step \( \delta t = \frac{T-t_0}{n} \); \( t_n = T \) is the time of expiration. Define the discretely monitored geometric averaging by
\[ G_k = \left( \prod_{j=1}^{k} S(t_j) \right)^{1/k}, \quad k = 1, \ldots, n \]

(a) Assuming that the stock price follows a GBM process, show that \( G_n \) is a log-normal random variable. Find the mean and variance of \( \ln G_n \).

(b) Derive the risk-neutral time-\( t_0 \) prices of the fixed strike Asian call and put options with respective payoff functions \( (G_n - K)^+ \) and \( (K - G_n)^+ \), where \( K > 0 \) is a strike price.

Solution.

(a) Since \( S(t) \) follows a GBM (assume zero dividends) we have
\[ S(t_j) = S(t_0) e^{(r - \frac{1}{2}\sigma^2) t_j + \sigma W(t_j)}, \quad j = 1, \ldots, n \]
whence
\[ G_n = \left( \prod_{j=1}^{n} S(t_j) \right)^{1/n} = S(t_0) e^{(r - \frac{1}{2}\sigma^2) (\frac{1}{n} \sum_{j=1}^{n} t_j) + \frac{\sigma}{n} \sum_{j=1}^{n} W(t_j)} \]  

Recall that from [CM14, Exercise 10.10] we have that\(^1\)
\[ \sum_{j=1}^{n} W(t_j) \sim \mathcal{N}(0, \sum_{j=1}^{n} (2j - 1)t_{n+1-j}) \] \[ \frac{d}{\sqrt{\sum_{j=1}^{n} (2j - 1)t_{n+1-j}}} \cdot Z, \quad Z \sim \mathcal{N}(0, 1) \]

\[^1\] This follow from the transformation rule: if \( X, Y \) are a \( p \)-dimensional and \( m \)-dimensional random vectors, with respective covariance matrices \( \Sigma_X \) and \( \Sigma_Y \), and \( Y = AX \), then \( \Sigma_Y = A \Sigma_X A^\top \). In our case \( X = (W(t_1), \ldots, W(t_n))^\top \) and \( A = (1, \ldots, 1) \), so \( \sigma_{i,j} = \operatorname{Cov}(W(t_i), W(t_j)) = \min\{i,j\} \) and \( \operatorname{Var}(Y) = A \Sigma_X A^\top = \sum_{j=1}^{n} (2j - 1)t_{n+1-j} \).
and therefore

\[ \frac{\sigma}{n} \sum_{j=1}^{n} W(t_j) = \frac{\sigma}{n \sqrt{T}} \sum_{j=1}^{n} (2j - 1)t_{n+1-j} \cdot \sqrt{T}Z \]

with \( Z \sim \mathcal{N}(0, 1) \) and \( T = \frac{1}{n} \sum_{j=1}^{n} t_j \).

With regards to the drift term, note that

\[ (r - \frac{1}{2} \sigma^2) \left( \frac{1}{n} \sum_{j=1}^{n} t_j \right) = (r - \frac{1}{2} \sigma^2)T = \left( r - \frac{\bar{\sigma}^2}{2} + \frac{\bar{\sigma}^2}{2} - \frac{\sigma^2}{2} \right) T = \left( r - \frac{\sigma^2}{2} + \delta \right) T \]

so that equation (13) becomes

\[ G_n(T) = S(t_0)e^{(r-\delta-\frac{1}{2} \bar{\sigma}^2)T + \bar{\sigma} \sqrt{T}Z} \]

and clearly

\[ \ln G_n(T) \sim \mathcal{N} \left( \ln S(t_0) + (r - \delta - \frac{1}{2} \bar{\sigma}^2)T, \bar{\sigma}^2 T \right) \]

There was no need to do all this to conclude normality, but now we have \( G_n \) written as a GBM (with constant dividend \( \delta \)), so that we can apply the Black-Scholes formula directly in order to price fixed strike Asian calls and puts as though they were ordinary calls and puts on the underlying \( G_n(T) \).

(b) Since from part (a) we have that \( G_n(T) \) follows a GBM

\[ G_n(T) = S(t_0)e^{(r-\delta-\frac{1}{2} \bar{\sigma}^2)T + \bar{\sigma} \sqrt{T}Z} \]

with

\[ T = \frac{1}{n} \sum_{j=1}^{n} t_j, \quad \bar{\sigma} = \frac{\sigma}{n \sqrt{T}} \sum_{j=1}^{n} (2j - 1)t_{n+1-j}, \quad \delta = \frac{\sigma^2}{2} - \frac{\bar{\sigma}^2}{2} \]

the Black-Scholes formula yields that time-\( t_0 \) value of a fixed-strike Asian call with payoff \( (G_n(T) - K)^+ \) is

\[ C(t_0, S) = e^{-\delta(T-t_0)} S \mathcal{N}(d_+) - e^{-r(T-t_0)} K \mathcal{N}(d_-) \]

with

\[ d_\pm = d_\pm \left( \frac{e^{-\delta(T-t_0)} S}{K}, T - t_0 \right), \quad d_\pm(x, \tau) = \frac{\ln x - (r - \delta \pm \frac{1}{2} \bar{\sigma}^2)\tau}{\bar{\sigma} \sqrt{\tau}} \]

The time-\( t_0 \) value of a put can be obtained by put-call parity.
Exercise 4.12. [CM14, Exercise 12.26] Let the stock price process follow the GBM model
\[
dS(t) = S(t)[(r-q)dt + \sigma d\tilde{W}(t)],
\]
\[
S(T) = S(t)e^{(r-q-\frac{1}{2}\sigma^2)(T-t) + \sigma(W(T)-W(t))} \stackrel{d}{=} S(t)e^{(r-q-\frac{1}{2}\sigma^2)(T-t) + \sigma \sqrt{T-t}Z}, \quad Z \sim N(0,1)
\]
Define the continuously monitored geometric average of \(S(t)\) over a time period \([0,t]\) by
\[
G(t) = \exp \left( \frac{1}{t} \int_0^t \ln S(u) \, du \right).
\]
(a) Show that the process \(\ln G(t)\) is Gaussian.
(b) Find the mean and variance of \(G(T)\) conditional on \(G(t)\) and \(S(t)\), for \(0 \leq t \leq T\).
(c) Show that \(G(T)\) can be written as
\[
G(T) = G(t)^{\frac{1}{t}} S(t)^{\frac{T-t}{t}} e^{\mu + \sigma \dot{Z}}
\]
for some \(\mu, \sigma \in \mathbb{R}\), and where \(\dot{Z} \sim N(0,1)\) under measure \(\tilde{P}\). Find the values of \(\mu\) and \(\sigma\).
(d) Derive the risk-neutral time-\(t\) pricing functions for the fixed strike Asian call and put options with respective payoff functions \((G(T) - K)^+\) and \((K - G(T))^+\) for \(0 \leq t \leq T\). Express the pricing functions in terms of the spot values \(G(t) = G > 0, S(t) = S > 0\), and times \(t\) and \(T\).
(e) Establish the put-call parity relation for the fixed strike Asian call and put options.

Solution.
(a) Since \(S(t) = S(0)e^{(r-q-\frac{1}{2}\sigma^2)t + \sigma W(t)}\), we have that
\[
\ln G(t) = \frac{1}{t} \int_0^t \ln S(u) \, du = \frac{1}{t} \int_0^t \left[ \ln S(0) + (r-q-\frac{1}{2}\sigma^2)u \right] \, du + \sigma \int_0^t W(u) \, du
\]
From [CM14, Proposition 11.1] we know that \(\int_0^t W(u) \, du \sim \mathcal{N}(0, \frac{t^3}{3})\) whence
\[
\ln G(t) \sim \mathcal{N} \left( \ln S(0) + \frac{1}{2}(r-q-\frac{1}{2}\sigma^2)t, \frac{\sigma^2t}{3} \right)
\]
(b) First note that
\[
\ln G(T) = \frac{1}{T} \int_0^T \ln S(u) \, du = \frac{1}{T} \int_0^t \ln S(u) \, du + \frac{1}{T} \int_t^T \ln S(u) \, du
\]
\[
= \frac{1}{T} \ln G(t) + \frac{1}{T} \int_t^T \ln S(u) \, du
\]
(14)
Now write \(S(u) = S(t)e^{(r-q-\frac{1}{2}\sigma^2)(u-t) + \sigma(W(u)-W(t))}\) whence
\[
\frac{1}{T} \int_t^T \ln S(u) \, du = \frac{1}{T} \int_t^T \left[ \ln S(t) + (r-q-\frac{1}{2}\sigma^2)(u-t) \right] \, du + \frac{\sigma}{T} \int_t^T [W(u)-W(t)] \, du
\]
For the first integral we have
\[
\frac{1}{T} \int_t^T \left[ \ln S(t) + (r - q - \frac{1}{2} \sigma^2)(u - t) \right] du = \ln S(t) \frac{T - t}{T} + \frac{1}{2} (r - q - \frac{1}{2} \sigma^2) \frac{(T - t)^2}{T}
\]
and as for the second one
\[
\int_t^T [W(u) - W(t)] du = \int_0^T W(u) du - \int_0^t W(u) du - W(t)(T - t)
\]
Note that
\[
\text{Var} \left( \int_0^T W(u) du - \int_t^T W(u) du \right) = \frac{T^3}{3} + \frac{t^3}{3} - 2 \text{Cov} \left( \int_0^t W(u) du, \int_0^T W(u) du \right)
\]
\[
= \frac{T^3}{3} + \frac{t^3}{3} - 2 \left( \frac{t^3}{3} - (T - t) \frac{t^2}{2} \right)
\]
\[
= \frac{T^3}{3} - \frac{t^3}{3} + t^2(T - t)
\]
where the covariance computation can be found on [CM14, Page 413]. Since \( W(u), u \in [t, T] \) is independent of \( W(t) \) we conclude that
\[
\text{Var} \left( \int_t^T [W(u) - W(t)] du \right) = \text{Var} \left( \int_t^T W(u) du \right) + \text{Var} (W(t)(T - t))
\]
\[
= \frac{T^3}{3} - \frac{t^3}{3} + t^2(T - t) + t(T - t)^2
\]
= \[
\frac{1}{3}(T - t)^3
\]
Putting everything together, from (14) we obtain
\[
\ln G(T) = \frac{t}{T} \ln G(t) + S(t) \frac{T - t}{T} + \frac{1}{2} (r - q - \frac{1}{2} \sigma^2) \frac{(T - t)^2}{T} + \frac{1}{3} \sigma^2 \frac{(T - t)^3}{T^2} \tilde{Z}
\]
for \( \tilde{Z} \sim \mathcal{N}(0, 1) \). In conclusion
\[
\ln G(T) | \ln G(t) \sim \mathcal{N} \left( \frac{t}{T} \ln G(t) + \ln S(t) \frac{T - t}{T} + \frac{1}{2} (r - q - \frac{1}{2} \sigma^2)(T - t)^2, \frac{1}{3} \sigma^2 \frac{(T - t)^3}{T^2} \right)
\]
(c) Exponentiating (15) we obtain
\[
G(T) = G(t)^\frac{1}{T} S(t)^\frac{T - t}{T} \exp \left( \frac{1}{2} (r - q - \frac{1}{2} \sigma^2)(T - t)^2 + \frac{1}{3} \sigma^2 \frac{(T - t)^3}{T^2} \tilde{Z} \right)
\]
\[
= G(t)^\frac{1}{T} S(t)^\frac{T - t}{T} e^{\mu + \sigma \tilde{Z}}
\]
where
\[
\mu = \frac{1}{2} (r - q - \frac{1}{2} \sigma^2)(T - t)^2,
\]
\[
\sigma = \frac{1}{3} \sigma^2 \frac{(T - t)^3}{T^2}
\]
(d) In order find the time-t value of an Asian call with payoff \((G(T) - K)^+\) we proceed as usual:

\[
V(t, S(t), G(t)) = e^{-r(T-t)} \tilde{E}_{t, S, G} \left[ (G(T) - K)^+ \right]
\]

\[
= e^{-r(T-t)} \tilde{E}_{t, S, G} \left[ \left( G(t)^+ S(t) \frac{T-t}{\sigma} e^{\bar{\mu} + a \tilde{Z}} - K \right) 1_{G(t) > K} \right]
\]

\[
\leq e^{\bar{\mu} - r(T-t)} G(t)^+ S(t) \frac{T-t}{\sigma} \tilde{E}_{t, S, G} \left[ e^{\bar{\mu} + a \tilde{Z}} 1_{\tilde{Z} > \alpha(t, S, G)} \right] - K e^{-r(T-t)} \tilde{P}_{t, S, G} \left( \tilde{Z} > \alpha \right)
\]

\[
= e^{\bar{\mu} + \frac{1}{2} r(T-t)} G(t)^+ S(t) \frac{T-t}{\sigma} N \left( \frac{1}{\sigma} \left( \sigma^2 + \ln \frac{G(t)^+ S(t)}{K} \right) + \bar{\mu} \right)
\]

\[
-K e^{-r(T-t)} N \left( \frac{1}{\sigma} \left( \ln \frac{G(t)^+ S(t)}{K} \right) + \bar{\mu} \right)
\]

where in [1] we denoted

\[
\alpha(t, S, G) = \frac{\ln K - \frac{1}{2} \ln G - \frac{T-t}{\sigma} \ln S - \bar{\mu}}{\sigma} = \frac{1}{\sigma} \left[ -\bar{\mu} + \ln \left( \frac{K}{G(t)^+ S(t)} \right) \right]
\]

(e) To derive a put-call parity relation (and the time-t value of a put from it) note that

\[
(G(T) - K)^+ - (K - G(T))^+ = G(T) - K
\]

Discounting and taking expectations yields

\[
\underbrace{e^{-r(T-t)} \tilde{E}_{t, S, G} [(G(T) - K)^+]}_{C(t, S, G)} - \underbrace{e^{-r(T-t)} \tilde{E}_{t, S, G} [(K - G(T))^+]}_{P(t, S, G)} = e^{-r(T-t)} \tilde{E}_{t, S, G} [G(T)] - K e^{-r(T-t)}
\]

whence

\[
P(t, S, G) = C(t, S, G) - e^{-r(T-t)} \tilde{E}_{t, S, G} [G(T)] + K e^{-r(T-t)}
\]

and where

\[
e^{-r(T-t)} \tilde{E}_{t, S, G} [G(T)] = e^{\bar{\mu} - r(T-t)} G(t)^+ S(t) \frac{T-t}{\sigma} \tilde{E} [e^{\bar{\mu} + a \tilde{Z}}] = e^{\bar{\mu} + \frac{1}{2} \sigma^2 - r(T-t)} G(t)^+ S(t) \frac{T-t}{\sigma}
\]
Chapter 13: Risk-Neutral pricing in a Multi-Asset Economy

Exercise 5.1. [CM14, Exercise 13.4] A plain currency call option on a foreign exchange rate has payoff

\[ C_T = (X(T) - \kappa)^+ \]

(a) Derive the pricing function \( C(t, x), t < T \) for this call by evaluating the risk-neutral expectation formula

\[ C(t, x) = e^{r(T-t)} \tilde{E} \left[ (X(T) - \kappa)^+ | X(t) = x \right] \]

(b) Give the BSPDE for the pricing function \( C(t, x) \).

Solution.

(a) Suppose that the exchange rate process is a GBM driven by a \( d \)-dimensional Brownian motion, namely (c.f. [CM14, Equation 13.102])

\[
X(T) = X(t) e^{(r - r_f - \frac{1}{2} \sigma_X^2)(T-t) + \sigma_X \sqrt{T-t} \tilde{Z}}
\]

\[ \tilde{Z} = \frac{1}{\sigma_X \sqrt{T-t}} \sigma(X) \cdot (\tilde{W}(T) - \tilde{W}(t)) \]

where \( \tilde{W}(t) = (W_1(t), \ldots, W_d(t)) \) and \( \sigma(X) = (\sigma_X, \ldots, \sigma_X) \).

The pricing function

\[ C(t, x) = e^{r(T-t)} \tilde{E} \left[ (X(T) - \kappa)^+ | X(t) = x \right] \]

\[ = e^{r(T-t)} \tilde{E} \left[ \left( X(t) e^{(r - r_f - \frac{1}{2} \sigma_X^2)(T-t) + \sigma_X \sqrt{T-t} \tilde{Z}} - \kappa \right)^+ | X(t) = x \right] \]

can be obtained using the standard Black-Scholes formula for the price of a call, with interest \( r - r_f \), volatility \( \sigma_X \), time-to-expiry \( T - t \) and strike \( \kappa \), namely

\[ C(t, x) = e^{r(T-t)} \left[ X(t) N \left( d_+ \left( \frac{X(t)}{\kappa}, T-t \right) \right) - \kappa N \left( d_- \left( \frac{X(t)}{\kappa}, T-t \right) \right) \right] \]

where as usual

\[ d_\pm(x, \tau) = \frac{\ln x + (r - r_f \pm \frac{1}{2} \sigma_X^2) \tau}{\sigma_X \sqrt{\tau}} \]

\[ \square \]

Exercise 5.2. [CM14, Exercise 13.5] Assume that a foreign stock price process \( \{S^f(t)\}_{t \geq 0} \) and the exchange rate \( \{X(t)\}_{t \geq 0} \) are correlated geometric Brownian motions with respective log-volatility vectors \( \sigma(S) = [\sigma_S, 0] \) and \( \sigma(X) = [\rho \sigma_X, \sqrt{1 - \rho^2} \sigma_X] \). Assume the domestic and foreign interest rates \( r \) and \( r_f \) are constants an that foreign stock pays no dividend.

(a) Derive a formula for the current time \( t < T \) price \( C_t \) of a call option on foreign stock denominated in domestic currency with domestic payoff

\[ C_T = \left( X(T) S^f(T) - K \right)^+ \]
(b) Similarly, derive a formula for the current time $t < T$ price $P_t$ of a put option with domestic payoff

$$C_T = \left(K - X(T)S^f(T)\right)^+$$

(c) Derive a put-call parity formula relating the call and put prices $C_t$ and $P_t$.

Solution.

(a) We know (c.f. [CM14, Equations 13.1000 and 13.104]) that the process $(X(t)S^f(t))_{t \geq 0}$ satisfies the SDE

$$d[X(t)S^f(t)] = (r - q_S)dt + (\sigma^{(X)} + \sigma^{(S)}) \cdot dW(t)$$

with solution

$$X(T)S^f(T) = X(t)S^f(t)e^{(r - q_S - \frac{1}{2}\sigma^2_{XS})\tau + (\sigma^{(X)} + \sigma^{(S)}) \cdot (W(T) - W(t))}$$

where

$$\sigma_X = \|\sigma^{(X)}\|, \quad \sigma_S = \|\sigma^{(S)}\|, \quad \sigma_{XS} = \|\sigma^{(X)} + \sigma^{(S)}\| = \sqrt{\sigma^2_X + \sigma^2_S + 2\rho\sigma_X\sigma_S}$$

and

$$\hat{X} = \frac{1}{\sigma_{XS}\sqrt{T - t}}(\sigma^{(X)} + \sigma^{(S)}) \cdot (W(T) - W(t)) \sim N(0, 1)$$

The call option can thus be priced by using the standard Black-Scholes formula $C_{BS}(S, K, \tau; r, q, \sigma)$ with spot $S = XS^f$, strike $K$, time to maturity $\tau = T - t$, interest rate $r$, no dividends $q = q_S = 0$ and volatility $\sigma = \sigma_{XS}$, whence

$$C(t, S^f(t), X(t)) = e^{-r(T-t)} \left[ S^f(t)X(t)N \left( d_+ \left( \frac{X(t)S^f(t)}{K}, T - t \right) \right) - K N \left( d_- \left( \frac{X(t)S^f(t)}{K}, T - t \right) \right) \right]$$

with

$$d_\pm(x, \tau) = \frac{\ln x + (r - r^f \pm \frac{1}{2}\sigma^2_X)\tau}{\sigma X \sqrt{\tau}}$$

(b) Since we are using the standard Black-Scholes formula, the standard put-call parity relation holds, namely

$$C(t, S^f(t), X(t)) - P(t, S^f(t), X(t)) = S^f(t)X(t) - Ke^{-r(T-t)}$$

□

Exercise 5.3. [CM14, Exercise 13.6]
Solution. This call option can be priced as a standard call using the Black-Scholes formula.

Exercise 5.4. [CM14, Exercise 13.7] Consider the foreign equity call struck in foreign currency with payoff

\[ C_T = X(T) \left( S^f(T) - K_f \right)^+. \]

Assume the foreign stock price is a GBM with constant log-volatility \( \sigma^{(S)} \) and having a dividend yield \( q_S \) and the exchange rate is a GBM with constant log-volatility \( \sigma^{(X)} \). Derive its pricing function \( C(t, S, S) \).

Solution. This exercise is identical to [CM14, Example 13.4], with the roles of \( S^f(T) \) and \( X(T) \) exchanged. We will be reproducing most of the argument in [CM14, Exercise 13.9(c)] below, so we don’t duplicate it here.

Exercise 5.5. [CM14, Exercise 13.9] Assume that a foreign stock price \( \{S^f(t)\}_{t \geq 0} \), a foreign exchange rate process \( \{X(t)\}_{t \geq 0} \) and a domestic asset price process \( \{A(t)\}_{t \geq 0} \) are all geometric Brownian motions with respective constant log-volatility vectors \( \sigma^{(S)}, \sigma^{(X)} \) and \( \sigma^{(A)} \):

\[
\begin{align*}
    dA(t) &= \mu_A dt + \sigma^{(A)} \cdot dW(t), \\
    dS^f(t) &= \mu_{S^f(t)} dt + \sigma^{(S)} \cdot dW(t), \\
    dX(t) &= \mu_X dt + \sigma^{(X)} \cdot dW(t)
\end{align*}
\]

\( W(t) \) is a 3-dimensional standard \( P \)-Brownian motion in the physical measure \( P \) and the assets are correlated, where \( \sigma_S = \|\sigma^{(S)}\|, \sigma_X = \|\sigma^{(X)}\|, \sigma_A = \|\sigma^{(A)}\|, \sigma^{(S)} \cdot \sigma^{(A)} = \rho_{S^f S^A} \sigma_S \sigma_A, \sigma^{(X)} \cdot \sigma^{(A)} = \rho_{X^A X^S} \sigma_X \sigma_S, \sigma^{(X)} \cdot \sigma^{(S)} = \rho_{X^S X^S} \sigma_X \sigma_S \). Assume a domestic and foreign economy with respective interest rates \( r \) and \( r^f \) as constants, zero dividends on all assets, and let \( A(t) = A, S^f(t) = S \) and \( X(t) = X \) be the spot values.

(a) Derive the time \( t < T \) pricing function for a domestic European option with payoff

\[ V_T = \max\{X(T)S^f(T), A(T)\} \]

(b) Derive the time \( t < T \) pricing function for a domestic European-style option with payoff

\[ V_T = X(T)S^f(T)1_{X(T) \geq X_0} + A_T1_{X(T) < X_0} \]

where \( X(0) = X_0 \) is a fixed positive initial exchange rate.

(c) Derive the time \( t < T \) pricing function for a domestic European-style option with payoff

\[ V_A = \left( aX(T)S^f(T) - bA(T) \right)^+ \]

with positive constants \( a, b \).

Solution.
(a) Recalling that \( \max\{x, y\} = (x - y)^+ + y \) we can rewrite the payoff as

\[
V_T = \max\{X(T)S^f(T), A(T)\} = \left( X(T)S^f(T) - A(T) \right)^+ + A(T)
\]

so (see [CM14, Equation 13.67])

\[
V(t, S, X, A) = e^{-r(T-t)} E_{t,S,X,A}\left[ \left( X(T)S^f(T) - A(T) \right)^+ \right] + e^{-r(T-t)} E_{t,S,X,A}[A(T)]
\]

and these two expectations have been computed in the exchange option example on [CM14, Section 13.2.2.1] with \( X(T)S^f(T) \leftrightarrow S_2(T) \) and \( A(T) \leftrightarrow S_1(T) \). We simply need to establish the correlation coefficient between both processes. Recall that the processes \( X(t)S^f(t) \) and \( A(t) \) satisfy

\[
X(T)S^f(T) = X(t)S^f(t) e^{(r-\frac{1}{2}\sigma^2_X)(T-t) + \sigma_X W(T) - W(t)}
\]

\[
A(T) = A(t) e^{(r-\frac{1}{2}\sigma^2_A)(T-t) + \sigma_A W(T) - W(t)}
\]

where

\[
\sigma_{XS} = \|\sigma^{(X)} + \sigma^{(S)}\| = \sqrt{\sigma_X^2 + \sigma_S^2 + 2\rho\sigma_X\sigma_S}
\]

The correlation between these two processes is given by

\[
\frac{\sigma^{(A)} \cdot (\sigma^{(X)} + \sigma^{(S)})}{\sigma_A \cdot \sigma_{XS}} = (\rho_{AX} + \rho_{AS}) \frac{\sigma_S}{\sigma_{XS}} = \rho
\]

so we have the following equivalent expressions:

\[
X(T)S^f(T) = X(t)S^f(t) e^{(r-\frac{1}{2}\sigma^2_X)(T-t) + \sigma_X \sqrt{T-t} Z_{XS}}
\]

\[
A(T) = A(t) e^{(r-\frac{1}{2}\sigma^2_A)(T-t) + \sigma_A \sqrt{T-t} \left[ \rho Z_{XS} + (1-\rho^2) Z_A \right]}
\]

where \( Z_{XS} \) and \( Z_A \) are independent standard normal variables.

Now we are in a position to apply the expression obtained in the aforementioned example: concretely, [CM14, Equation 13.70] reads in our case

\[
C(t, S, X, A) = A N\left( d_+ \left( \frac{XS}{A}, T - t \right) \right) + X S N\left( d_- \left( \frac{XS}{A}, T - t \right) \right)
\]

where

\[
d_\pm(x, \tau) = \frac{\ln x \pm \frac{1}{2} \nu^2 \tau}{\nu \sqrt{\tau}}, \quad \nu = \|\sigma^{(A)} - (\sigma^{(S)} + \sigma^{(X)})\| = \sqrt{\sigma_A^2 + \sigma_X^2 - 2\rho\sigma_A\sigma_X}
\]

(c) By the risk-neutral pricing formula we have that

\[
V(t, X(t), S^f(t), A(t)) = e^{-r(T-t)} \tilde{E}_{t} \left[ X(T)S^f(T) 1_{X(T) \geq X_0} \right] + e^{-r(T-t)} \tilde{E}_{t} \left[ A_T 1_{X(T) < X_0} \right]
\]

We compute each expectation separately using an appropriate change of numeraire. For the first one, using \( X(t)S^f(t) \) as numeraire we are reduced to

\[
e^{-r(T-t)} \tilde{E}_{t} \left[ X(T)S^f(T) 1_{X(T) \geq X_0} \right] = X(t)S^f(t) \tilde{E} \left[ 1_{X(T) \geq X_0} \right] = X(t)S^f(t) \tilde{P}(X(T) \geq X_0)
\]
In conclusion, the first expectation we needed to compute is a GBM random variable involving the $P$-BM, exactly as it is done in [CM14, Example 13.4].

Recall that by [CM14, Equation 13.102] we have

$$X(T) = X(t)e^{(r-r^f-\frac{1}{2}\sigma^2)(T-t)+\sigma^X(t)(\hat{W}(T)-\hat{W}(t))}$$

Recall that the Brownian increments in the risk-neutral measure and the measure under numraire $g$ are related by

$$\hat{W}(T) - \hat{W}(t) = \tilde{W}^{(g)}(T) - \tilde{W}^{(g)}(t) + \sigma^{(g)}(T-t)^{\text{not}} \equiv \hat{W}(T) - \hat{W}(t) + \sigma^{(XSf)}(T-t)$$

Substituting this in the representation for $X(T)$ above we obtain

$$X(T) = X(t)e^{(r-r^f-\frac{1}{2}\sigma^2)(T-t)+\sigma^X(t)(\hat{W}(T)-\hat{W}(t)) + \sigma^{(XSf)}(T-t)}$$

where recall that $\sigma^{(XSf)} = \sigma^X + \sigma^{Sf}$ and where $\hat{Z} \sim \mathcal{N}(0,1)$. Therefore we have that

$$X(T) \geq X_0 \iff X(t)e^{(r-r^f+\sigma^X(t)\sigma^X(t)+\frac{1}{2}\sigma^2)(T-t)+\sigma_X \sqrt{T-t}\hat{Z}} \geq X_0$$

$$\iff \hat{Z} \geq \frac{\ln \frac{X_0}{X(t)} - (r-r^f+\sigma^X(t)\sigma^X(t)+\frac{1}{2}\sigma^2)(T-t)}{\sigma_X \sqrt{T-t}}$$

In conclusion, the first expectation we needed to compute is

$$e^{-r(T-t)}\hat{E}_t[X(T)S^f(T)\mathbb{I}_{X(T)\geq X_0}] = X(t)S^f(t)\hat{P}(X(T) \geq X_0) = X(t)S^f(t)\mathcal{N}\left(\frac{\ln \frac{X_0}{X(t)} + (r-r^f+\sigma^X(t)\sigma^X(t)+\frac{1}{2}\sigma^2)(T-t)}{\sigma_X \sqrt{T-t}}\right)$$

The second expectation we compute analogously, but now using $A(t)$ as numraire, namely:

$$e^{-r(T-t)}\hat{E}_t[A_T\mathbb{I}_{X(T)\geq X_0}] = A(t)\hat{E}_t^{(A)}[\mathbb{I}_{X(T)\geq X_0}] \text{not} \equiv A(t)\hat{P}(X(T) < X_0)$$

Proceeding as above, we may relate the Brownian motion increments in the risk-neutral measure and our new measure $\hat{P}$ in numraire $A(t)$ to obtain (we skip the details)

$$X(T) = X(t)e^{(r-r^f+\frac{1}{2}\sigma^2)(T-t)+\sigma^X(t)(\hat{W}(T)-\hat{W}(t))}$$

whereby

$$e^{-r(T-t)}\hat{E}_t[A_T\mathbb{I}_{X(T)\geq X_0}] = A(t)\hat{P}(X(T) < X_0) = A(t)\mathcal{N}\left(\frac{\ln \frac{X_0}{X(t)} - (r-r^f+\frac{1}{2}\sigma^2)(T-t)}{\sigma_X \sqrt{T-t}}\right)$$
In conclusion, adding up both expectation we have shown that

\[ V(t, X(t), S^f(t), A(t)) = e^{-r(T-t)} \hat{E}_t \left[ X(T)S^f(T)1_{X(T) \geq X_0} \right] + e^{-r(T-t)} \hat{E}_t \left[ A_T1_{X(T) < X_0} \right] \]

\[ = X(t)S^f(t)N \left( \frac{\ln \frac{X(t)}{X_0} + (r - r^f + \sigma^{(S)} \cdot \sigma^{(S)} + \frac{1}{2} \sigma^2_X)(T - t)}{\sigma_X \sqrt{T - t}} \right) \]

\[ + A(t)N \left( \frac{\ln \frac{X(t)}{X_0} - (r - r^f + \frac{1}{2} \sigma^2_X)(T - t)}{\sigma_X \sqrt{T - t}} \right) \]

(d) This exercise is just [CM14, Exercise 13.16] with \( X(T)S^f(T) \leftrightarrow S_2(T) \) and \( A(T) \leftrightarrow S_1(T) \) so we may just apply the formulas obtained therein after computing the correlation between the processes \( X(t)S^f(t) \) and \( A(t) \) as in part (a).

\[ \square \]

**Exercise 5.6.** [CM14, Exercise 13.11]

*Solution.*

\[ \square \]

**Exercise 5.7.** [CM14, Exercise 13.12] Assume as in [CM14, Exercise 13.5] that a foreign stock price process \( \{S^f(t)\}_{t \geq 0} \) and the exchange rate \( \{X(t)\}_{t \geq 0} \) are correlated geometric Brownian motions with respective log-volatility vectors \( \sigma^{(S)} = [\sigma_S, 0] \) and \( \sigma^{(X)} = [\rho \sigma_X, \sqrt{1 - \rho^2} \sigma_X] \) and with time-0 (spot) values \( S^f(0) = S, X(0) = X \). Assume the domestic and foreign interest rates \( r \) and \( r^f \) are constants an that foreign stock pays no dividend. Derive the time-0 pricing formula as a function of \( S, T, X \), for a domestic European option having payoff at maturity \( T > 0 \) given by

\[ V_T = X(T)S^f(T)1_{\{M(T) < K\}} \]

where \( M(T) \) is the maximum realized value of the exchange rate up to time \( T \):

\[ M(T) = \max_{0 \leq t \leq T} X(t) \]

*Solution.* Using \( g(t) = X(t)S^f(t) \) as numeraire, we need to compute

\[ e^{-r(T-t)} \hat{E}_t \left[ X(T)S^f(T)1_{\{M(T) < K\}} | \mathcal{F}(t) \right] = X(t)S^f(t)\hat{P}(M(T) < K | \mathcal{F}(t)) \]

and it hence suffices to obtain the PDF of the sampled maximum \( M(T) \) in measure \( \hat{P} \).

Recall that by [CM14, Equation 13.102] we have

\[ X(T) = X(t)e^{(r - r^f - \frac{1}{2} \sigma^2_X)r + \sigma^{(X)} \cdot (W(T) - W(t))} \]

The Brownian motion increments \( W(T) - W(t) \) and \( \hat{W}(T) - \hat{W}(t) \) are related via

\[ W(T) - W(t) = \hat{W}(T) - \hat{W}(t) - \sigma^{(XS)}(T - t), \quad \sigma^{(XS)} = \sigma^{(X)} + \sigma^{(S)} \]

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so the expression for $X(t)$ becomes

$$X(T) = X(t)e^{(r-r_f+\sigma(X),\sigma(S)+\frac{1}{2}\sigma_X^2)\tau+\sigma_X\sqrt{\tau}\bar{Z}}$$

where $\bar{Z} = \frac{1}{\sigma_X\sqrt{\tau}}\sigma(X) \cdot (\bar{W}(T) - \bar{W}(t)) \sim \mathcal{N}(0, 1)$. We can further write

$$X(t) = X(0)e^{\sigma_X X(t)} \text{ where } X(t) = \mu t + \sqrt{t}\bar{Z}, \quad \mu = \frac{r - r_f + \sigma(X) \cdot \sigma(S) + \frac{1}{2}\sigma_X^2}{\sigma_X}$$

and

$$M^X(t) = \max_{0 \leq u \leq t} X(u) = X(0)e^{\sigma_M X(t)}, \quad M^X(t) = \max_{0 \leq u \leq t} X(u)$$

Similarly, since we will be conditioning on $\mathcal{F}(t)$ we have

$$M^X(T) = \max_{0 \leq u \leq T} X(u) = \max \left\{ M^X(t), X(t)e^{\sigma_M X(t)} \right\}, \quad M^X(\tau) = \max_{t \leq u \leq \tau} X(u)$$

We are now in the context of [CM14, Section 12.3.1], whereby the joint density function of $M^X(\tau)$ and $X(\tau)$ in measure $\hat{P}$ is given by [CM14, Equation 12.108], namely

$$f_{M^X(\tau), X(\tau)}(w, x) = \frac{2(2w - x)}{\tau\sqrt{2\pi}\tau}\exp\left(-\frac{1}{2}\frac{\ln^2 r + \ln \tau - (2w - x)^2}{\tau}\right), \quad \mu = \frac{r - r_f + \sigma(X) \cdot \sigma(S) + \frac{1}{2}\sigma_X^2}{\sigma_X}$$

Our expectation is hence either 0 if $M^X(t) \geq K$ or else is given by

$$X(t)S^f(t)\hat{P}(M(T) < K | \mathcal{F}(t)) = X(t)S^f(t)\hat{P}\left(X(t)e^{\sigma_M X(t)} < K | \mathcal{F}(t)\right)$$

$$= X(t)S^f(t)\hat{P}\left(M^X(\tau) < \frac{1}{\sigma_X}\ln\frac{K}{X(t)} | \mathcal{F}(t)\right)$$

$$= X(t)S^f(t)\int_0^\infty \frac{1}{\tau}\ln\frac{K}{X(t)} \int_{-\infty}^w f_{M^X(\tau), X(\tau)}(w, x) \, dx \, dw$$

Exercise 5.8. [CM14, Exercise 13.13] Consider a domestic economy with constant interest rate $r$ and two correlated GB stock price processes given by

$$S_1(T) = S_1(t)e^{(\mu_1 - \frac{1}{2}\sigma_1^2)\tau + \sigma_1\sqrt{\tau}Z_1},$$

$$S_2(T) = S_2(t)e^{(\mu_2 - \frac{1}{2}\sigma_2^2)\tau + \sigma_2\sqrt{\tau}(\rho Z_1 + \sqrt{1-\rho^2}Z_2)}$$

where $Z_1, Z_2$ are independent standard normal random variables. let $S_1(t) = S_1$, $S_2(t) = S_2$, $0 \leq t \leq T$, be the stock spot values. Derive the pricing function $V(t, S_1, S_2)$ for a European chooser max call with payoff

$$V_T = (\max\{S_1(T), S_2(T)\} - K)^+$$

Solution. Re-write the payoff as

$$\Lambda_T = (S_1(T) - K)^+1_{\{S_1(T) > S_2(T)\}} + (S_2(T) - K)^+1_{\{S_2(T) > S_1(T)\}}$$

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Then

\[ V(t, S_1, S_2) = e^{-rt} \hat{E} \left[ (S_1(T) - K)^+ \mathbb{1}_{S_1(T) > S_2(T)} + (S_2(T) - K)^+ \mathbb{1}_{S_2(T) > S_1(T)} \right] \]

We illustrate how to compute

\[ \hat{E} \left[ (S_1(T) - K)^+ \mathbb{1}_{S_1(T) > S_2(T)} \right] \]

We have that

\[ \hat{E} \left[ (S_1(T) - K)^+ \mathbb{1}_{S_1(T) > S_2(T)} \right] = e^{(r - q_1 - \frac{1}{2} \sigma_1^2)T} \hat{E} \left[ (e^{\sigma_1 \sqrt{T} Z_1} - K)^+ \mathbb{1}_{Z_1 > \alpha, f(Z_1) > Z_2} \right] \]

\[ = \int_{\alpha}^{\infty} \int_{-\infty}^{f(x)} \left[ e^{\sigma_1 \sqrt{T} x} - K \right] \eta_2(x, y; \rho) \, dy \, dx \]

\[ = \cdots \]

where

\[ \alpha = \frac{\ln K}{\sigma_1} - (r - q_1 - \frac{1}{2} \sigma_1^2)T \]

\[ f(Z_1) = \cdots \]

\[ \eta_2(x, y; \rho) = \frac{1}{2\pi \sqrt{1 - \rho^2}} e^{-(x^2 + y^2 - 2\rho xy)/2(1 - \rho^2)} \]

\[ \square \]

**Exercise 5.9. [CM14, Exercise 13.15]** Consider three stocks with GBM price dynamics as in [CM14, Exercise 13.1] with constant interest rate \( r \) and constant dividend yield \( q_i \) on each stock \( i = 1, 2, 3 \).

(a) Derive the pricing function \( V(t, S_1, S_2, S_3), t < T \) for a European option with payoff

\[ V_T = S_3(T)\mathbb{1}_{S_3(T) > S_1(T), S_3(T) > S_2(T)} \]

(b) Derive the pricing function for a European option with payoff

\[ V_T = \max\{S_1(T), S_2(T), S_3(T)\} \]

*Solution.* For a derivation using the change of numeraire technique, see [OW06, Section 6.1].

(b) Part (b) is straightforward once we have established (a), since

\[ \max\{S_1(T), S_2(T), S_3(T)\} = S_1(T)\mathbb{1}_{S_1(T) > S_2(T), S_1(T) > S_3(T)} \\
+ S_2(T)\mathbb{1}_{S_2(T) > S_1(T), S_2(T) > S_3(T)} \\
+ S_3(T)\mathbb{1}_{S_3(T) > S_1(T), S_3(T) > S_2(T)} \]
(a) Part (a) is a generalization of [CM14, Equation 13.72], whereby if
\[ S_1(T) = S_1(t) e^{\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)\tau + \sigma_1^2}, \quad S_2(T) = S_2(t) e^{\left(\mu_2 - \frac{1}{2}\sigma_2^2\right)\tau + \sigma_2^2}, \]
with
\[ \sigma_1 = \|\sigma^{(1)}\|, \quad \sigma^{(1)} \cdot \sigma^{(2)} = \rho \sigma_1 \sigma_2, \quad \nu^2 = \|\sigma^{(1)} - \sigma^{(2)}\|^2 = \sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2 \]
then
\[ E \left[ S_1(T) I_{\{S_1(T) > S_2(T)\}} | S_1(t) = S_1, S_2(t) = S_2 \right] = S_1 e^{\mu_1 \tau} \mathcal{N} \left( \frac{\ln \frac{S_1}{S_2} + (\mu_1 - \mu_2 + \frac{1}{2}\nu^2)\tau}{\nu \sqrt{\tau}} \right) \]  

We first show this 2-dimensional case and we will then indicate how to extend it to 3 stocks (even though computations are tedious). Start by writing
\[ S_1(T) = S_1(t) e^{\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)\tau + \sigma_1 \sqrt{\tau} Z_1}, \]
\[ S_2(T) = S_2(t) e^{\left(\mu_2 - \frac{1}{2}\sigma_2^2\right)\tau + \sigma_2 \sqrt{\tau} (\rho Z_1 + \sqrt{1 - \rho^2} Z_2)} \]
where \( Z_1, Z_2 \) are independent standard normal random variables. Then we have
\[ E \left[ S_1(T) I_{\{S_1(T) > S_2(T)\}} | S_1(t) = S_1, S_2(t) = S_2 \right] = S_1 e^{\mu_1 \tau} \mathcal{N} \left[ e^{\sigma_1 \sqrt{\tau} Z_1} I_{\{Z_1 > f(Z_2)\}} \right] \]
where
\[ f(Z_2) = \frac{\ln \frac{S_1}{S_2} + \left[\mu_2 - \mu_1 - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)\right] \tau + \sigma_2 \sqrt{\tau} \sqrt{1 - \rho^2} Z_2}{(\sigma_1 - \rho \sigma_2) \sqrt{\tau}} \]
We now use iterated conditioning
\[ E \left[ S_1(T) I_{\{S_1(T) > S_2(T)\}} | S_1(t) = S_1, S_2(t) = S_2 \right] = S_1 e^{\mu_1 \tau} E \left[ \mathcal{N} \left( \frac{\ln \frac{S_1}{S_2} + (\mu_1 - \mu_2 + \frac{1}{2}\nu^2)\tau - \sigma_2 \sqrt{\tau} \sqrt{1 - \rho^2} Z_2}{(\sigma_1 - \rho \sigma_2) \sqrt{\tau}} \right) I_{\{Z_1 > f(Z_2)\}} \right] \]
Recalling that for \( X \sim \mathcal{N}(0, 1) \) we have \( E[e^{BX} I_{X > A}] = e^{B^2/2} \mathcal{N}(B - A) \) we have
\[ E \left[ S_1(T) I_{\{S_1(T) > S_2(T)\}} | S_1(t) = S_1, S_2(t) = S_2 \right] = S_1 e^{\mu_1 \tau} E \left[ \mathcal{N} \left( \frac{\sigma_1 \sqrt{\tau} - f(Z_2)}{(\sigma_1 - \rho \sigma_2) \sqrt{\tau}} \right) \right] \]
Recalling that for \( X \sim \mathcal{N}(0, 1) \) we have \( E[\mathcal{N}(AX + C)] = \mathcal{N} \left( \frac{C}{\sqrt{1 + A^2}} \right) \) we conclude that
\[ E \left[ S_1(T) I_{\{S_1(T) > S_2(T)\}} | S_1(t) = S_1, S_2(t) = S_2 \right] = S_1 e^{\mu_1 \tau} \mathcal{N} \left( \frac{\ln \frac{S_1}{S_2} + (\mu_1 - \mu_2 + \frac{1}{2}\nu^2)\tau}{\nu \sqrt{\tau}} \right) \]
In dimension 3 we start by computing the lower Cholesky factorization \( U \) of the correlation matrix
\[
\begin{pmatrix}
1 & \rho_{12} & \rho_{13} \\
\rho_{12} & 1 & \rho_{23} \\
\rho_{13} & \rho_{23} & 1
\end{pmatrix} = U U^T, \quad U = \begin{pmatrix}
1 & 0 & 0 \\
\rho_{12} & \sqrt{1 - \rho_{12}^2} & 0 \\
\rho_{13} & \alpha & \beta
\end{pmatrix}
\]
where
\[ \alpha = \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1 - \rho_{12}^2}}, \quad \beta = \sqrt{1 - \rho_{12}^2 - \frac{(\rho_{23} - \rho_{12}\rho_{13})^2}{1 - \rho_{12}^2}} \]

Write
\[
S_1(T) = S_1(t) e^{(\mu_1 - \frac{1}{2}\sigma_1^2) T + \sigma_1 \sqrt{T} Z_1}, \\
S_2(T) = S_2(t) e^{(\mu_2 - \frac{1}{2}\sigma_2^2) T + \sigma_2 \sqrt{T} (\rho_{12} Z_1 + \sqrt{1 - \rho_{12}^2} Z_2)}, \\
S_3(T) = S_3(t) e^{(\mu_3 - \frac{1}{2}\sigma_3^2) T + \sigma_3 \sqrt{T} (\rho_{13} Z_1 + \rho_{12}\rho_{13} Z_2 + \beta Z_1)}
\]

where \( Z_1, Z_2, Z_3 \) are independent standard normal random variables. Now we proceed as in the 2-dimensional case by iterated conditioning
\[
e^{-\tau T} E_{t,S_1,S_2,S_3} \left[ S_3(T) I_{\{S_3(T) > S_1(T), S_3(T) > S_2(T)\}} \right] \\
= e^{-\tau T} S_3(t) e^{(\mu_3 - \frac{1}{2}\sigma_3^2) T} E \left[ e^{\sigma_3 \sqrt{T} (\rho_{13} Z_1 + \rho_{12}\rho_{13} Z_2 + \beta Z_3)} I_{\{Z_3 > f(Z_1), Z_3 > f(Z_2)\}} \right] \\
= e^{-\tau T} S_3(t) e^{(\mu_3 - \frac{1}{2}\sigma_3^2) T} E \left[ e^{\sigma_3 \sqrt{T} (\rho_{13} Z_1 + \rho_{12}\rho_{13} Z_2)} E \left[ e^{\beta Z_3 I_{\{Z_3 > f(Z_1), Z_3 > f(Z_2)\}}} | Z_1, Z_2 \right] \right] \\
= e^{-\tau T} S_3(t) e^{(\mu_3 - \frac{1}{2}\sigma_3^2) T} E \left[ e^{\beta Z_3 I_{\{Z_3 > \max\{f(Z_1), f(Z_2)\}\}}} | Z_1, Z_2 \right]
\]

The inner expectation
\[ E \left[ e^{\beta Z_3 I_{\{Z_3 > \max\{f(Z_1), f(Z_2)\}\}}} | Z_1, Z_2 \right]
\]
can be computed using once again that \( E[e^{B^2/2} 1_{X > A}] = e^{B^2/2} N(B - A) \) for \( X \sim N(0,1) \).

Alternatively: if we instead use \( S_3(t) \) as numéraire, so are reduced to computing
\[
e^{-\tau(T-t)} \hat{E}_t \left[ S_3(T) I_{\{S_3(T) > S_1(T), S_3(T) > S_2(T)\}} \right] = S_3(t) \hat{P} \left( 1 > \frac{S_1(T)}{S_3(T)}, 1 > \frac{S_2(T)}{S_3(T)} \right)
\]

and it now suffices to express \( \frac{S_1(T)}{S_3(t)} \) and \( \frac{S_2(T)}{S_3(T)} \) as GBM’s generated by two correlated standard normal variables and to integrate the corresponding bivariate normal density function. Again, by [CM14, Equation 13.122] we have
\[
\frac{S_i(T)}{S_3(t)} = \frac{S_i(t)}{S_3(t)} e^{-\frac{1}{2}||\sigma_i - \sigma_3||^2(T-t) + (\sigma_i - \sigma_3) (W(T) - W(t))} = \frac{S_i(t)}{S_3(t)} e^{-\frac{1}{2} \nu_i^2 (T-t) + \nu_i \sqrt{T} \tilde{Z}_i}
\]
for \( i = 1, 2 \), where \( \nu_i = ||\sigma_i - \sigma_3|| \) and
\[
\hat{Z}_i = \frac{1}{\nu_i \sqrt{T-t}} (\sigma_i - \sigma_3) \cdot (\hat{W}(T) - \hat{W}(t)), \quad \hat{\rho}_{12} := \text{Corr}(\hat{Z}_1, \hat{Z}_2) = \frac{(\sigma_1 - \sigma_3) \cdot (\sigma_2 - \sigma_3)}{\nu_1 \cdot \nu_2}
\]
Moreover note that
\[
\frac{S_i(T)}{S_3(t)} = \frac{S_i(t)}{S_3(t)} e^{-\frac{1}{2} \nu_i^2 (T-t) + \nu_i \sqrt{T} \tilde{Z}_i} < 1 \iff \hat{Z}_i < \frac{\ln \frac{S_i(t)}{S_3(t)} + \frac{1}{2} \nu_i^2 (T-t)}{\nu_i \sqrt{T-t}} \Rightarrow d_i, \quad i = 1, 2
\]

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and hence our expectation is

\[ S_3(t) \bar{P} \left( 1 > \frac{S_1(T)}{S_3(T)}, 1 > \frac{S_2(T)}{S_3(T)} \right) = S_3(t) \int_{-\infty}^{d_1} \int_{-\infty}^{d_2} \eta_2(x_1, x_2; \hat{\rho}_{12}) \, dx_2 \, dx_1 \]

where

\[ \eta_2(x_1, x_2; \hat{\rho}_{12}) = \frac{1}{2\pi \sqrt{1 - \hat{\rho}^2_{12}}} \exp \left( -\frac{x_1^2 - 2\hat{\rho}_{12}x_1x_2 + x_2^2}{2(1 - \hat{\rho}^2_{12})} \right) \]

\[ \square \]

**Exercise 5.10.** [CM14, Exercise 13.16] Consider an exchange option on two stocks having payoff

\[ V_T = (aS_2(T) - bS_1(T))^+, \quad a, b > 0. \]

Assume the stocks are GB; processes

\[ S_1(T) = S_1(t) e^{(\mu_1 - \frac{1}{2}\sigma_1^2)\tau + \sigma_1 \sqrt{\tau} Z_1}, \]
\[ S_2(T) = S_2(t) e^{(\mu_2 - \frac{1}{2}\sigma_2^2)\tau + \sigma_2 \sqrt{\tau}(\rho Z_1 + \sqrt{1-\rho^2} Z_2)} \]

where \( Z_1, Z_2 \) are independent standard normal random variables. Derive the time-\( t \) price \( V_t \) for this option by explicitly using one of the stocks as numraire asset and by implementing the risk-neutral pricing [CM14, Formula 13.149]. Note: the derivation is similar to that in [CM14, Example 13.3].

**Solution.** This exercise is identical to [CM14, Example 13.3]. Working with numraire \( g(t) = S_1(t) \) and denoting \( \bar{E} \equiv \bar{E}^{S_1} \), the time-\( t \) value of the option is given by

\[ V_t = S_1(t) \bar{E} \left[ \frac{V_T}{S_1(T)} \bigg| \mathcal{F}_t \right] = S_1(t) \bar{E} \left[ (a \frac{S_2(T)}{S_1(T)} - b)^+ \bigg| \mathcal{F}_t \right] = a S_1(t) \bar{E} \left[ \left( \frac{S_2(T)}{S_1(T)} - \frac{b}{a} \right)^+ \bigg| \mathcal{F}_t \right] \]

\[ \equiv a S_1(t) \bar{E} \left[ \left( Y(T) - \frac{b}{a} \right)^+ \bigg| \mathcal{F}_t \right] \]

where in the last equality we denoted \( Y(T) = \frac{S_2(T)}{S_1(T)} \).

It is easy to see\(^2\) that the process \( Y(t) = \frac{S_2(t)}{S_1(t)} \) is a \( \bar{P} \)-martingale

\[ Y(t) = \frac{S_2(t)}{S_1(t)} = Y(0) \mathcal{E}_t \left( (\sigma_2 - \sigma_1) \cdot \bar{W} \right) \]

As in [CM14, Example 13.3] we can write

\[ Y(t) = Y(0) e^{X(t)} \]

where

\[ X = -\frac{1}{2} \|\sigma_2 - \sigma_1\|^2 t + (\sigma_1 - \sigma_1) \cdot \bar{W}(t) = -\frac{1}{2} \nu^2 t + \nu \sqrt{t} \bar{Z} \]

\(^2\)Either see [CM14, Equation 13.122] or directly compute the stochastic differential \( d \left( \frac{S_2(t)}{S_1(t)} \right) \).
with \( \hat{Z} = \frac{(\sigma_2 - \sigma_1) \hat{W}(t)}{\nu \hat{Y}(t)} \sim \mathcal{N}(0, 1) \) and \( \nu^2 := \|\sigma_2 - \sigma_1\|^2 = \sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2 \). We can thus compute

\[
V_t = S_1(t) \tilde{E} \left[ \left( Y(T) - \frac{b}{a} \right)^+ | \mathcal{F}_t \right] = S_1(t) \tilde{E} \left[ \left( Y(0) e^{-\frac{1}{2} \nu^2 T + \nu \sqrt{T} \hat{Z}} - \frac{b}{a} \right)^+ | \mathcal{F}_t \right], \quad \hat{Z} \sim \mathcal{N}(0, 1)
\]

by using the Black-Scholes formula with zero interest rate and dividend, volatility \( \nu \), time to maturity \( T \), spot \( Y(t) \) and strike \( \frac{b}{a} \). In other words:

\[
V(T, S_1(0), S_2(0)) = S_1(0) \left[ Y(0) \mathcal{N}(d_+(Y(0), T)) - \frac{b}{a} \mathcal{N}(d_-(Y(0), T)) \right]
\]

\[
= S_2(0) \mathcal{N}(d_+(Y(0), T)) - \frac{b}{a} S_1(0) \mathcal{N}(d_-(Y(0), T))
\]

Exercise 5.11. [CM14, Exercise 13.17] Consider a domestic economy with constant interest rate \( r \) and two domestic stock price processes

\[
\begin{align*}
S_1(T) &= S_1(t) e^{(\mu_1 - \frac{1}{2} \sigma_1^2)T + \sigma_1 \sqrt{T} Z_1}, \\
S_2(T) &= S_2(t) e^{(\mu_2 - \frac{1}{2} \sigma_2^2)T + \sigma_2 \sqrt{T} (\rho Z_1 + \sqrt{1-\rho^2} Z_2)}
\end{align*}
\]

where \( Z_1, Z_2 \) are independent standard normal random variables. Derive the time-0 pricing function \( V_0 = V(T, S_1, S_2) \) in the spot variables \( S_1(0) = S_1, S_2(0) = S_2 \), for a European path-dependent option with payoff at maturity \( T \) given by

\[
V_T = S_1(T) \min_{0 \leq t \leq T} \frac{S_2(t)}{S_1(t)}
\]

Solution. Recall the numeraire invariant form of the risk-neutral pricing formula: for an attainable payoff \( V_T \) and numeraire \( g(t) \) we have

\[
V_t = g(t) \tilde{E}_t \left[ \frac{V_T}{g(T)} | \mathcal{F}(t) \right], \quad \tilde{E}^\text{not} = \tilde{E}(g)
\]

Picking \( g(t) = S_1(t) \) as numeraire we thus have

\[
V_t = S_1(t) \tilde{E} \left[ \min_{0 \leq t \leq T} \frac{S_2(t)}{S_1(t)} | \mathcal{F}(t) \right]
\]

It is easy to see\(^3\) that the process \( Y(t) = \frac{S_2(t)}{S_1(t)} \) is a \( \tilde{P} \)-martingale

\[
Y(t) = \frac{S_2(t)}{S_1(t)} = Y(0) \mathcal{E}_t ( (\sigma_2 - \sigma_1) \cdot \hat{W})
\]

As in [CM14, Example 13.3] we can write

\[
Y(t) = Y(0) e^{X(t)}
\]

\(^3\)Either see [CM14, Equation 13.122] or directly compute the stochastic differential \( d \left( \frac{S_2(t)}{S_1(t)} \right) \).
where
\[ X = -\frac{1}{2} \|\sigma_2 - \sigma_1\|^2 t + (\sigma_1 - \sigma_1) \cdot \hat{W}(t) = -\frac{1}{2} \nu^2 t + \nu \sqrt{t} \hat{Z} \]

with \( \hat{Z} = \frac{(\sigma_2 - \sigma_1)\hat{W}(t)}{\nu \sqrt{t}} \sim \mathcal{N}(0, 1) \) and \( \nu^2 := \|\sigma_2 - \sigma_1\|^2 = \sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2 \). We are thus reduced to computing the expectation
\[
V_t = S_1(t) \hat{E} \left[ \min_{0 \leq t \leq T} Y(t) \right] = S_1(t) \hat{E} \left[ \min_{0 \leq t \leq T} Y(0)e^{X(t)} \right]
\]
\[
= S_1(t) \hat{E} \left[ \min_{0 \leq t \leq T} Y(0)e^{-\frac{1}{2} \nu^2 t + \nu \sqrt{t} \hat{Z}} \right] \equiv S_1(t) \hat{E} \left[ \min_{0 \leq t \leq T} Y(0)e^{\nu Z(t)} \right] \not= S_1(t) \hat{E} \left[ Y(0)e^{\nu \cdot mZ(t)} \right]
\]

where \( Z(t) = \sqrt{t} \hat{Z} - \frac{1}{\nu^2} \) and \( mZ(t) = \min_{0 \leq u \leq t} Z(u) \) (see [CM14, Section 12.3.1]). The last expectation can be computed by integrating against the joint density function of the sampled minimum \( mZ(t) \) and \( Z(t) \) (c.f. [CM14, Equation 12.109]), namely
\[
\hat{E} \left[ Y(0)e^{\nu \cdot mZ(t)} \right] = Y(0) \int_{-\infty}^{0} \int_{w}^{\infty} e^{\nu w} f_{mZ(t),Z(t)}(w,x) \, dx \, dw,
\]
\[
f_{mZ(t),Z(t)}(w) = \frac{2(x - 2w)}{\tau \sqrt{2\pi \tau}} e^{-\frac{1}{2} \nu^2 + \nu x - \frac{(x - 2w)^2}{\tau}}
\]

which is a bit tedious to type. □
6 Chapter 14: American options

Exercise 6.1. [CM14, Exercise 14.1] Prove [CM14, Proposition 14.3] for an arbitrary American option with a differentiable payoff function $\Lambda$. In particular show the following.

(a) At any point $(t, S^*_t)$ of the early-exercise boundary, the American option pricing function $V$ satisfies the smooth pasting condition

$$\frac{\partial V(t, S)}{\partial S} \bigg|_{S=S^*_t} = \Lambda'(S^*_t)$$

(b) The option value $V$ satisfies the zero time-decay condition on the early-exercise domain

$$\frac{\partial V(t, S)}{\partial t} = 0, \quad \forall S \in D_t = \{(t, s) : V(t, s) = \Lambda(S(t))\}$$

Solution.

(a) We proceed as in [CM14, Section 14.2.2]. Suppose the American options has not been exercised at time $t$ and let $B$ denote the collection of all possible early-exercise boundaries defined by continuous functions $b : [t, T] \rightarrow \mathbb{R}^+$, so that for each $b \in B$ there is an exercise policy $T_b$. Then

$$V(t, S) = \sup_{b \in B} V(t, S; b), \quad V(t, S; b) = \mathcal{E}_{t, S} \left[ e^{-r(T_b-t)} \Lambda(S(T_b)) \right]$$

The total derivative of the function $V(t, S; b)$ along the boundary is given by

$$\frac{dV}{db} = \left. \frac{\partial V(t, S; b)}{\partial S} \frac{\partial S}{\partial b} \right|_{S=b} + \left. \frac{\partial V(t, S; b)}{\partial b} \right|_{S=b}$$

• Along the curve $S = b(t)$ it is clear that $\frac{\partial S}{\partial b} = 1$
• When $b = S^*$, we have $\left. \frac{\partial V(t, S; b)}{\partial b} \right|_{S=b} = 0$ (by optimality).

Therefore

$$\frac{dV}{db} = \left. \frac{\partial V(t, S; b)}{\partial S} \right|_{S=S^*}. \quad (17)$$

On the other hand, the option value is equal to the payoff function when $S = b$, that is $V(t, b; b) = \Lambda(b)$ whence

$$\left. \frac{\partial V(t, S)}{\partial S} \right|_{S=S^*} \overset{\text{def}}{=} \left. \frac{\partial V(t, S; S^*)}{\partial S} \right|_{S=S^*} \overset{\text{by (17)}}{=} \left. \frac{dV(t, S; b)}{db} \right|_{b=S^*} = \left. \frac{d\Lambda(b)}{db} \right|_{b=S^*} = \Lambda'(S^*)$$

(b) The total derivative of $V(t, S)$ with respect to time is

$$\frac{dV}{dt} = \frac{\partial V(t, S)}{\partial t} + \frac{\partial V(t, S)}{\partial S} \frac{\partial S}{\partial t}$$
On the other hand, on the early-exercise domain $D_t$ we have $V(t, S(t)) = \Lambda(S(t))$, so
\[
\frac{dV}{dt} = \frac{\partial \Lambda}{\partial S} \frac{\partial S}{\partial t} = \frac{\partial V}{\partial S} \frac{\partial S}{\partial t}
\]
and clearly
\[
\left. \frac{\partial V}{\partial t} \right|_{S \in D_t} = 0
\]

Exercise 6.2. [CM14, Exercise 14.2] Let $P(S; K, r, q)$ and $C(S; K, r, q)$, respectively, denote the price functions of the perpetual American put and call options struck at $K$. The underlying asset price follows a geometric Brownian motion
\[
S(t) = S(0)e^{(r-q-\frac{1}{2}\sigma^2)t+\sigma W(t)}
\]
Using the closed-form pricing formulas below, show that the option prices satisfy the put-call symmetry relation
\[
P(K; S, q, r) = C(S; K, r, q)
\]
where (c.f. [CM14, Equations 14.25 and 14.28])

\[
\begin{align*}
P(S; K, r, q) &= \begin{cases} 
\frac{K}{\lambda_-} \left( \frac{\lambda_--1}{\lambda_-} \right)^{\lambda_-} \left( \frac{S}{K} \right)^{\lambda_-} = -S^* \left( \frac{S}{S^*} \right)^{\lambda_-}, & S^* < S, \\
\frac{K}{\lambda_-} \left( \frac{\lambda_--1}{\lambda_-} \right)^{\lambda_-} \left( \frac{S}{K} \right)^{\lambda_-} = -S^* \left( \frac{S}{S^*} \right)^{\lambda_-}, & 0 < S \leq S^*.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
C(S; K, r, q) &= \begin{cases} 
\frac{K}{\lambda_+} \left( \frac{\lambda_+-1}{\lambda_+} \right)^{\lambda_+} \left( \frac{S}{K} \right)^{\lambda_+} = -S^* \left( \frac{S}{S^*} \right)^{\lambda_+}, & 0 < S \leq S^*, \\
\frac{K}{\lambda_+} \left( \frac{\lambda_+-1}{\lambda_+} \right)^{\lambda_+} \left( \frac{S}{K} \right)^{\lambda_+} = -S^* \left( \frac{S}{S^*} \right)^{\lambda_+}, & S^* < S.
\end{cases}
\end{align*}
\]

\[
\lambda_\pm = \frac{-(r-q-\frac{\sigma^2}{2}) \pm \sqrt{(r-q-\frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2}
\]

Solution. Recall how the pricing formulas for the perpetual American call and put are derived: ...

Exercise 6.3. [CM14, Exercise 14.3]

Solution.

Exercise 6.4. [CM14, Exercise 14.4]

Solution.

Exercise 6.5. [CM14, Exercise 14.5]
Exercise 6.6. [CM14, Exercise 14.6]

Solution.

Exercise 6.7. [CM14, Exercise 14.7]

Solution.

Exercise 6.8. [CM14, Exercise 14.8]

Solution.

Exercise 6.9. [CM14, Exercise 14.9]

Solution.
Chapter 15: Interest rate modeling and derivative pricing

Exercise 7.1. [CM14, Exercise 15.1] Suppose that the continuously compounded spot rates for the next three years are

\[
\begin{array}{c|ccc}
T & y(0, T) & 1 & 2 & 3 \\
\hline
1 & 3\% & 3.0\% & 3.25\% & 3.5\%
\end{array}
\]

Find the forward rates \( f(0; 1, 2), f(0; 1, 3) \) and \( f(0; 2, 3) \).

**Solution.** Recall the relation between yield and forward rates. From no-arbitrage, we have

\[
f(t, T, T') = \frac{1}{T' - T} \ln \frac{P(t, T)}{P(t, T')}
\]

and by definition of bond yield \( y(t, T) \) we have \( P(t, T) = e^{-y(t, T)(T - T)} \) whence

\[
f(t, T, T') = \frac{1}{T' - T} \ln \frac{P(t, T)}{P(t, T')} = \frac{1}{T' - T} \ln e^{-y(t, T)(T - T)}
\]

so

\[
f(0; 1, 2) = y(0, 2) \cdot \frac{2}{1} - y(0, 1) \cdot \frac{1}{1} = 0.0325 \cdot 2 - 0.03 \cdot 1,
\]

\[
f(0; 1, 3) = y(0, 3) \cdot \frac{3}{2} - y(0, 1) \cdot \frac{1}{2} = 0.035 \cdot 1.5 - 0.03 \cdot 0.5,
\]

\[
f(0; 2, 3) = y(0, 3) \cdot \frac{3}{1} - y(0, 2) \cdot \frac{2}{1} = 0.035 \cdot 3 - 0.0325 \cdot 2
\]

□

Exercise 7.2. [CM14, Exercise 15.3] Consider the Hull-White model

\[
dr(t) = [\alpha(t) - \beta(t)r(t)] dt + \sigma(t)dW(t)
\]

Show that the short rate is a Gaussian process. Find the mean and variance of \( r(T) \) conditional on \( r(t) \) for \( 0 \leq t \leq T \).

**Solution.** Given the market price of risk \( \gamma(t) \), the Hull-White model has risk-neutral dynamics

\[
dx(t) = [\tilde{\alpha}(t) - \beta(t)r(t)] dt + \sigma(t)d\tilde{W}(t), \quad \tilde{\alpha}(t) = \alpha(t) - \lambda(t)\sigma(t)
\]

Integrating this SDE (c.f. [CM14, Equation 11.31] or [CM14, Example 11.9]) we obtain

\[
r(T) = r(t)e^{-\int_t^T \beta(s) ds} + \int_t^T e^{-\int_s^T \beta(u) du} \alpha(s) ds + \int_t^T e^{-\int_s^T \beta(u) du} \sigma(s) dW(s)
\]

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The integrand in the It integral is non-random, so \( r(T) \) is indeed Gaussian conditional on \( r(t) \) and has mean and variance

\[
E[r(T)|r(t) = r] = re^{-\int_t^T \beta(s) \, ds} + \int_t^T e^{-\int_t^u \beta(u) \, du} \alpha(s) \, ds
\]

\[
\text{Var}[r(T)|r(t) = r] = \int_t^T e^{-2\int_t^s \beta(u) \, du} \sigma^2(s) \, ds
\]

where the second equation follows from It isometry. \( \square \)

**Exercise 7.3.** [CM14, Exercise 15.5] Find the forward rates \( f(t, T) \) in the Vasicek model

\[
 dr(t) = [\alpha - \beta r(t)] \, dt + \sigma dW(t)
\]

**Solution.** Recall that by definition of the forward rate \( f(t, T) \) we have

\[
e^{f(t, T, T') (T' - T)} = \frac{P(t, T)}{P(t, T')} \implies f(t, T, T') = \frac{1}{T' - T} \ln \frac{P(t, T)}{P(t, T')}
\]

and by definition the instantaneous forward rate is

\[
f(t, T) := \lim_{T' \to T} f(t, T, T') = -\frac{\partial}{\partial T} \ln P(t, T)
\]

so it suffices to establish the discount bond \( P(t, T) \) reconstitution formula in the Vasicek model (c.f. [CM14, Equation 15.53]) and differentiate it with respect to maturity \( T \).

We first outline the derivation of the bond reconstitution formula for \( P(t, T) \) following [CM14, Section 15.2.5]. Assuming that the market price of risk is constant \( \gamma \) the Vasicek model has risk-neutral dynamics

\[
 dr(t) = [\tilde{\alpha} - \beta r(t)] \, dt + \sigma d\tilde{W}(t), \quad \tilde{\alpha} = \alpha - \gamma \sigma
\]

Integrating this SDE we obtain

\[
 r(T) = e^{-\beta(T-t)} r(t) + \frac{\tilde{\alpha}}{\beta} \left( 1 - e^{-\beta(T-t)} \right) + \sigma \int_t^T e^{-\beta(T-s)} \, d\tilde{W}(s)
\]

\[
\ldots
\]

The bond pricing formula for the Vasicek model is hence

\[
P(t, T) = \exp \left[ \frac{1 - e^{-\beta(T-t)}}{\beta} (y_\infty - r) - y_\infty (T - t) - \frac{\sigma^2}{4\beta} \left( \frac{1 - e^{-\beta(T-t)}}{\beta} \right)^2 \right]
\]

where \( y_\infty = \frac{\tilde{\alpha}}{\beta} - \frac{\sigma^2}{2\beta^2} \). The forward rate is hence given by

\[
f(t, T) = -\frac{\partial}{\partial T} \ln P(t, T) = y_\infty + \frac{\sigma^2}{2\beta} \left( \frac{1 - e^{-\beta(T-t)}}{\beta} \right) e^{-\beta(T-t)} - (y_\infty - r) e^{-\beta(T-t)}
\]

\( \square \)
Exercise 7.4. [CM14, Exercise 15.7] A $d$-factor HJM model is given by the SDE

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)\dagger dW(t)$$

where $W(t) = (W_1(t), \ldots, W_d(t))$ is a vector of independent Brownian motions.

(a) Find the SDE for the discounted zero-coupon bond price process $\bar{P}(t,T) = D(t)P(t,T)$ and show that

$$\bar{P}(t,T) = \bar{P}(0,T) \exp\left(-\int_0^t A(s,T)ds - \int_0^t \Sigma(s,T)\dagger dW(s)\right)$$

where

$$\Sigma(t,T) = \int_t^T \sigma_f(t,u)du$$

$$A(t,T) = \int_t^T \alpha(t,u)du$$

(b) Show that the no-arbitrage condition is given by

$$\alpha(t,T) = \sigma_f(t,T)\dagger \int_t^T \sigma_f(t,s)ds$$

**Solution.** One can argue by extending the discussion in [CM14, Section 15.3.1]. We instead follow [AP10, Section 4.4].

(a) In the absence of arbitrage, the deflated bond values are $Q$-martingales, so by the martingale representation theorem there exists a $d$-dimensional stochastic process $\sigma_P(t,T)$ with $\sigma_P(T,T) = 0$ such that

$$d\bar{P}(t,T) = -\bar{P}(t,T)\sigma_P(t,T)\dagger dW(t), \quad t \leq T$$

By It’s lemma it follows that

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - \sigma_P(t,T)\dagger dW(t)$$

By It’s lemma it also follows that

$$\frac{dP(t,T, T + \tau)}{P(t,T, T + \tau)} = -[\sigma_P(t,T + \tau) - \sigma_P(t,T)]\dagger \sigma_P(t,T)dt - [\sigma_P(t,T + \tau) - \sigma_P(t,T)]\dagger dW(t)$$

In the T-forward measure, $P(t,T, T + \tau)$ is a martingale (by definition), so we must have

$$\frac{dP(t,T, T + \tau)}{P(t,T, T + \tau)} = -[\sigma_P(t,T + \tau) - \sigma_P(t,T)]\dagger dW^T(t)$$

where $W^T(t)$ is a $Q^T$-Brownian motion.
Comparison of the last two expressions yields
\[ dW^T(t) = dW(t) + \sigma_P(t, T)dt \]
which by Girsanov’s theorem identifies the Radon-Nikodym process for the measure shift
\[ \xi(t) = E_t^Q \left[ \frac{dQ^T}{dQ} \right] = \mathcal{E}_t(\sigma_P \cdot W) = \exp \left( -\int_0^t ||\sigma_P(s, T)||^2 ds - \int_0^t \sigma_P(s, T) dW(s) \right) \]

HJM models are traditionally stated in terms of instantaneous forward rates \( f(t, T) \). Once again by It’s lemma we have (omitting the drift term)
\[ d\ln P(t, T) = \sigma_P(t, T)^T dW(t) \]
Differentiating with respect to \( T \) on both sides and recalling that \( f(t, T) = -\frac{\partial P(t, T)}{\partial T} \), we have
\[ df(t, T) = \mu_f(t, T) dt + \sigma_f(t, T)^T \sigma_P(t, T)^T dW(t) \]
which shows (a).

(b) In order to establish the drift term \( \mu_f(t, T) \) note that since
\[ f(t, T) = E_t^T [r(u)] \]
is a martingale in the \( T \)-forward measure we necessarily have
\[ df(t, T) = \sigma_f(t, T)^T dW^T(t) \]
Applying the change of measure above
\[ dW^T(t) = dW(t) + \sigma_P(t, T)dt, \quad \sigma_P(t, T) = \int_t^T \sigma_f(t, u) du \]
it follows that
\[
\begin{align*}
    df(t, T) &= \sigma_f(t, T)^T \sigma_P(t, T) dt + \sigma_f(t, T)^T dW(t) \\
    &= \sigma_f(t, T)^T \int_t^T \sigma_f(t, u) du dt + \sigma_f(t, T)^T dW(t)
\end{align*}
\]
so the drift term in the risk-neutral measure is indeed
\[ \mu_f(t, T) = \sigma_f(t, T)^T \int_t^T \sigma_f(t, u) du \]

\[ \square \]

**Exercise 7.5.** [CM14, Exercise 15.8]
Exercise 7.6. [CM14, Exercise 15.9]

Solution.

Exercise 7.7. [CM14, Exercise 15.12]

Solution.

Exercise 7.8. [CM14, Exercise 15.13]

Solution.

Exercise 7.9. [CM14, Exercise 15.14]

Solution.
References


