In each of the following questions you will be asked to find the integral (or integrals) that compute something (lengths, areas, volumes or centroids). You just need to write down the definite integral that would give the solution: there is NO NEED TO COMPUTE ANY INTEGRAL in this test. The value of every question is indicated at the beginning of it. You may only use scratch paper and a small note card. No cell phones, calculators, notes, books or music players are allowed during the test.

Name: ______________________ UID: __________

1. (15 points)

(i) (5 points) Let R be the region bounded by the graphs of $x = 3 - y^2$, $y = x - 1$ and $y = 0$. Write down the integral that would compute the area of R using vertical slices.

Solution: Using vertical slices, the area is given by

$$A = \int_{-1}^{2} \left(x - 1 + \sqrt{3 - x}\right)\, dx + \int_{2}^{3} 2\sqrt{3 - x}\, dx$$

(ii) (5 points) Write down the integral that would compute the area of R using horizontal slices.

Solution: Using horizontal slices, the area is given by

$$A = \int_{-2}^{1} \left(3 - y^2 - (y + 1)\right)\, dy$$

(iii) (5 points) Now consider the region bounded by the graphs of $x = 3 - y^2$, $y = x - 1$ and $x = 0$. Write down the integral that would compute the area of R.

Solution: Using vertical slices, the area is given by

$$A = \int_{0}^{2} \left(\sqrt{3 - x} - (x - 1)\right)\, dx$$

2. (15 points) Let R be the region in the upper half-plane bounded by the graphs of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $y = 0$.

   (i) Write down the integral that would compute the volume of the solid obtained by revolving R about the x-axis.

      (a) (5 points) Using shells:

      Solution: Using shells, the volume is given by

$$V = \int_0^b 2\pi y \cdot 2a\sqrt{1 - \frac{y^2}{b^2}}\, dy$$

      (b) (5 points) Using disks:

      Solution: Using disks, the volume is given by

$$V = \int_{-a}^{a} \pi b^2 \left(\sqrt{1 - \frac{x^2}{a^2}}\right)^2 dx$$

   (ii) (5 points) Use any method to write down the integral that would compute the volume of the solid obtained by revolving R about the line $y = -1$.

   Solution: Using disks, the volume is given by

$$V = \int_{-a}^{a} \pi \left[\left(b\sqrt{1 - \frac{x^2}{a^2}} + 1\right)^2 - 1^2\right] dx$$

   and using shells, the volume is given by

$$V = \int_0^b 2\pi(y+1)2a\sqrt{1 - \frac{y^2}{b^2}}\, dy$$

3. (10 points) Let R be the region bounded by the graphs of $y = 4x$ and $y = 4x^2$. Write down the integral that would compute the volume of the solid obtained by revolving R about the y-axis.

   (i) (5 points) Using shells:

   Solution: Using shells, the volume is given by

$$V = \int_0^1 2\pi x(4x - 4x^2)\, dx$$

   (ii) (5 points) Using disks:

   Solution: Using disks, the volume is given by

$$\int_0^4 \pi \left[\left(\frac{\sqrt{y}}{2} \right)^2 - \left(\frac{y}{4} \right)^2 \right] dy$$

4. (15 points) For each of the following curves, write down the integral that would compute its arc length.

(i) (5 points) $y = x^{3/2}, \quad 1 \le x \le 4.$

Solution:

$$L = \int_1^4 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\, dx$$

(ii) (5 points) $30xy^3 - y^8 = 15, \quad y = 1, \quad y = 3.$

Solution: Solving for x, he curve is given by

$$x = \frac{15 + y^8}{30y^3} = \frac{1}{2y^3} + \frac{y^5}{30}$$

so

$$x'(y) = -\frac{3}{2y^4} + \frac{y^4}{6}$$

so the length of the curve is given by

$$L = \int_1^3 \sqrt{1 + [x'(y)]^2}\, dy = \int_1^3 \sqrt{1 + \left[-\frac{3}{2y^4} + \frac{y^4}{6}\right]^2}\, dy$$

(iii) (5 points) $x = 3t^2 + 2, \quad y = 2t^3 - \frac{1}{2}, \quad 1 \le t \le 4.$

Solution:

$$L = \int_1^4 \sqrt{(6t)^2 + (6t^2)^2}\, dt$$

5. (10 points) Write down the integral that would compute the area of the surface generated by revolving the following curves about the x-axis.

(i) (5 points) $y = \frac{x^6+2}{8x^2}, \quad 1 \le x \le 3.$

Solution:

$$\begin{aligned} S &= \int_1^3 2\pi \left(\frac{x^6+2}{8x^2}\right) \sqrt{1+[y'(x)]^2}\, dx \\ &= \int_1^3 2\pi \left(\frac{x^6+2}{8x^2}\right) \sqrt{1+\left[\frac{6x^5 \cdot 8x^2 - (x^6+2)16x}{64x^4}\right]^2}\, dx \end{aligned}$$

(ii) (5 points) $x = t, \quad y = t^3, \quad 0 \le t \le 1$ (sketch a graph of this curve).

Solution:

$$S = \int_0^1 2\pi t^3 \sqrt{1^2 + [3t^2]^2}\, dt$$

6. EXTRA CREDIT (10 points) The integral

$$\pi \int_a^b [f(x)]^2 \, dx$$

computes the volume of the solid obtained by revolving the region under the curve $y = f(x), \quad a \leq x \leq b$ about the x-axis.

Use a picture and a couple of sentences to justify why this is the case.

Solution: