This exam consists of 2 sections, A and B. Section A is conceptual, whereas section B is more computational. The value of every question is indicated at the beginning of it. You may only use scratch paper and a small note card. No cell phones, calculators, notes, books or music players are allowed during the test.

Name: ______________________ UID: __________

Section A: Conceptual questions.

1. (5 points) If f is a function which is differentiable over the open interval (a, b) and continuous over the closed interval $[a, b]$, the mean value theorem for derivatives states that there exists some c inside of $[a, b]$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

   Consider the function $f(x) = x^{2/3}$ over the interval $[-8, 27]$. Show that the mean value theorem for derivatives fails and explain why.

   Solution: One the one hand $f'(x) = \frac{2}{3}\frac{1}{x^{1/3}}$, and on the other hand

$$\frac{f(27) - f(-8)}{27 - (-8)} = \frac{9-4}{35} = \frac{1}{7}$$

   Imposing $f'(c) = \frac{2}{3}\frac{1}{x^{1/3}} = \frac{1}{7}$ we find that $c = \left(\frac{14}{3}\right)^3 > 4^3 = 64$ so clearly c doesn't lie inside the interval $[-8, 27]$ and the mean value theorem fails. The reason is that the function f is not differentiable at $x = 0$.

2. (5 points) Recall that a function f is odd if $f(-x) = -f(x)$ for all x. Show that if f is an odd function, then for any real number a we have

$$\int_{-a}^{a} f(x)\,dx = 0$$

   Hint: use the change of variables $u = -x$.

   Solution: Write

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

   We apply the suggested change of variable $u = -x$ to the first one and we get

$$\int_{-a}^{0} f(x)\,dx \overset{[1]}{=} \int_{a}^{0} f(-u)(-\,du) \overset{[2]}{=} -\int_{a}^{0} f(-u)\,du \overset{[3]}{=} \int_{0}^{a} f(-u)\,du \overset{[4]}{=} -\int_{0}^{a} f(u)\,du$$
$$\overset{[5]}{=} -\int_{0}^{a} f(x)\,dx$$

   where in [1] we applied the change of variable $u = -x$ (note the change in the limits of integration), in [2] we took the minus sign out if the integral, in [3] we reversed the limits of integration by changing the sign, in [4] we used the fact the f is an odd function, so $f(-u) = -f(u)$, and in [4] we wrote x instead of u (these are just dummy variables). Overall, going back to our original integral we get

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

as claimed.

Section B: Practical questions.

3. (15 points) Compute the following indefinite integrals:

   (i) (5 points) $\int \frac{(x^2+1)^2}{\sqrt{x}}\,dx$ **Solution**: We start by expanding the numerator:

$$\int \frac{(x^2+1)^2}{\sqrt{x}}\,dx = \int x^{7/2} + x^{-1/2} + 2x^{3/2}\,dx = \frac{2}{9}x^{9/2} + 2x^{1/2} + \frac{4}{5}x^{5/2} + C$$

   (ii) (5 points) $\int \frac{3x}{\sqrt{2x^2+5}}\,dx$ **Solution**:

$$\int \frac{3x}{\sqrt{2x^2+5}}\,dx = \frac{3}{4}\int 4x(2x^2+5)^{-1/2}\,dx = \frac{3}{2}\sqrt{2x^2+5} + C$$

   (iii) (5 points) $\int x^2(x^3+5)^8 \cos\left[(x^3+5)^9\right]\,dx$
   Hint: Use the change of variables $u = (x^3+5)^9$.

   Solution:

$$\begin{aligned}
\int x^2(x^3+5)^8 \cos\left[(x^3+5)^9\right]\,dx &= \int x^2(x^3+5)^8 \cos(u) \left(\frac{du}{27x^2(x^3+5)^8}\right) \\
&= \frac{1}{27}\int \cos(u)\,du = \frac{1}{27}\sin(u) + C \\
&= \frac{1}{27}\sin\left[(x^3+5)^9\right] + C
\end{aligned}$$

4. (15 points) Find the area under the curve $y = 2x + 2$ over the interval $[-1, 1]$ as follows.

(i) (2 points) Subdivide the interval $[-1, 1]$ into n equal subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$$

What is the length Δx of every subinterval? For every k, write an expression for x_k.

Solution:

$$\begin{aligned} \Delta x &= \frac{2}{n} \\ x_k &= -1 + \frac{2k}{n} \end{aligned}$$

(ii) (4 points) Write down an expression for the area of the rectangle over $[x_k, x_{k+1}]$ which depends ONLY on k and n. **Solution**: The area of the rectangle over $[x_k, x_{k+1}]$ is given by

$$f(x_k)\Delta x = (2x_k + 2)\Delta x = \left(2\left(-1 + \frac{2k}{n}\right) + 2\right)\frac{2}{n} = \frac{8k}{n^2}$$

(iii) (7 points) Find the sum $A(R_n)$ of the areas of the n rectangles.

Hint: Remember that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Solution: The sum of the areas of the n rectangles is given by

$$A(R_n) = \sum_{k=0}^{n-1} \frac{8k}{n^2} = \frac{8}{n^2} \sum_{k=0}^{n-1} k = \frac{8}{n^2} \frac{(n-1)n}{2} = 4\left(1 - \frac{1}{n}\right)$$

(iv) (2 points) Find the limit $A = \lim_{n\to\infty} A(R_n)$. **Solution**: From (iii) we clearly have

$$\lim_{n\to\infty} A(R_n) = \lim_{n\to\infty} 4\left(1 - \frac{1}{n}\right) = 4$$

5. (15 points) Consider the function $F(x) = \int_{\sin x}^{\cos x} t^5\, dt$

(i) (2 points) Compute $F\left(\frac{\pi}{4}\right)$ and $F(0)$.

Solution:

$$\begin{aligned} f\left(\frac{\pi}{4}\right) &= \int_{\sin\frac{\pi}{4}}^{\cos\frac{\pi}{4}} t^3\, dt = \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} t^3\, dt = 0 \\ f(0) &= \int_{\sin 0}^{\cos 0} t^3\, dt = \int_0^1 t^3\, dt = \frac{1}{4}t^4\Big|_0^1 = \frac{1}{4} \end{aligned}$$

(ii) (10 points) Compute $F'(x)$
Hint: Write $F(x) = \int_{\sin x}^{a} t^3\, dt + \int_a^{\cos x} t^3\, dt$.

Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_{\sin x}^{a} t^3\, dt + \frac{d}{dx}\int_a^{\cos x} t^3\, dt = -\frac{d}{dx}\int_a^{\sin x} t^3\, dt + \frac{d}{dx}\int_a^{\cos x} t^3\, dt \\ &= -\sin^3 x\cos x - \cos^3 x(-\sin x) = -\sin^3 x\cos x - \cos^3 x\sin x \end{aligned}$$

6. (10 points) Consider the function $f(x) = \int_0^x \frac{u}{\sqrt{1+u^2}}\, du$

(i) (5 points) Find the intervals where f is increasing or decreasing.

Solution: This boils down to determining over which intervals is $f'(x)$ positive or negative. By the first fundamental theorem of calculus,

$$f'(x) = \frac{d}{dx}\int_0^x \frac{u}{\sqrt{1+u^2}}\, du = \frac{x}{\sqrt{1+x^2}}$$

so f is decreasing over $(-\infty, 0)$ and increasing over $(0, \infty)$.

(ii) (5 points) Find the intervals where f is concave up or down.

This boils down to studying the sign of $f''(x)$. Note that

$$f''(x) = \left(\frac{x}{\sqrt{1+x^2}}\right)' = \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2} = \frac{1}{(1+x^2)^{3/2}}$$

which is always positive, so f is concave-up over the whole reals.

7. (10 points) Compute the definite integral

$$\int_0^{\pi/2} \sin x \sin(\cos x)\, dx$$

Solution: We use the change of variable $u = \cos x$, so that $du = -\sin x dx$ and hence

$$\begin{aligned}\int_0^{\pi/2} \sin x \sin(\cos x)\, dx &= \int_{u(0)}^{u(\pi/2)} \sin x \sin(u) \frac{-du}{\sin x} = -\int_1^0 \sin u\, du = \cos u\big|_1^0 \\ &= 1 - \cos 1\end{aligned}$$

8. (10 points) Use symmetry to compute the following integral

$$\int_{-\pi/2}^{\pi/2} \frac{\sin x}{1+\cos x} + |x| \sin^5 x + x^2\, dx$$

Be explicit in your justification.

Solution: We split the integral into 3

$$\int_{-\pi/2}^{\pi/2} \frac{\sin x}{1+\cos x} + |x| \sin^5 x + x^2\, dx = \int_{-\pi/2}^{\pi/2} \frac{\sin x}{1+\cos x}\, dx + \int_{-\pi/2}^{\pi/2} |x| \sin^5 x\, dx + \int_{-\pi/2}^{\pi/2} x^2\, dx$$

Note that the functions $f(x) = \frac{\sin x}{1+\cos x}$ and $g(x) = |x| \sin^5 x$ are odd. Indeed:

$$\begin{aligned}f(-x) &= \frac{\sin(-x)}{1+\cos(-x)} = \frac{-\sin x}{1+\cos x} = -f(x) \\ g(-x) &= |-x|\,(\sin(-x))^5 = |x|\,(-\sin x)^5 = -|x| \sin^5 x = -g(x)\end{aligned}$$

The integrals of f and g over symmetric intervals are hence 0 and therefore

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \frac{\sin x}{1+\cos x} + |x| \sin^5 x + x^2\, dx &= \int_{-\pi/2}^{\pi/2} x^2\, dx = 2\int_0^{\pi/2} x^2\, dx = 2\,\frac{1}{3} x^3\Big|_0^{\pi/2} \\ &= \frac{2}{3}\left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{12}\end{aligned}$$

9. (15 points) YOU DON'T HAVE TO COMPUTE ANY INTEGRAL IN THIS EXERCISE.

(i) (5 points) Let R be the region bounded by the graphs of $y = \sqrt{x}$, $y = -x + 6$ and $y = 0$. Write down the integral that would compute the area of R using vertical slices.

Solution: We first need to determine the points of intersection of the 2 curves, namely the points satisfying both equations. The x-value of such a point must satisfy

$$\sqrt{x} = -x + 6$$

Taking squares yields

$$x = x^2 + 36 - 12x \Longrightarrow x^2 - 13x + 36 = 0$$

and this equations has the following solutions

$$x = \frac{13 \pm \sqrt{169 - 144}}{2} = \frac{13 \pm 5}{2} = \begin{cases} 9 \Longrightarrow \text{out of bound} \\ 4 \Longrightarrow y = 2 \end{cases}$$

so the curves intersect at the point $(4, 2)$.

Using vertical slices, the area will be given by

$$A = \int_0^4 \sqrt{x}\, dx + \int_4^6 -x + 6\, dx$$

(ii) (5 points) Write down the integral that would compute the area of R using horizontal slices.

Using horizontal slices, we need to integrate from $y = 0$ to $y = 2$ in order to cover the region. The height of a thin rectangular slice will be dy and the base will be given by $6 - y - y^2$.

You need to invert the equations defining the curves to do this: given any x-value $x = a$, the corresponding y-value of a point lying on the graph of $y = \sqrt{x}$ must be $y = \sqrt{a}$. Conversely, given a y-value $y = b$, the corresponding x-value of a point lying on the graph of $y = \sqrt{x}$ must be $x = b^2$.

The area will then be given by

$$A = \int_0^2 6 - y - y^2\, dy$$

(iii) (5 points) Now consider the region bounded by the graphs of $y = \sqrt{x}$, $y = -x + 6$ and $x = 0$. Write down the integral that would compute the area of R.

Solution: In this case we may use vertical slices and the area will be given by

$$A = \int_0^4 -x + 6 - \sqrt{x}\, dx$$