Section A: Conceptual questions.

1. (3 points) Let \( f \) be a function defined over an interval \([a, b]\) which contains \( c \). If \( f(c) \) is an extreme value of \( f \), what can you say about \( c \)?

Solution: \( c \) must be a critical point: it must be either an endpoint, or a singular point (\( f'(c) \) does not exist) or a stationary point (i.e. \( f'(c) = 0 \)).

2. (3 points) Sketch the graph of the functions \( f(x) = \cos x \) and \( g(x) = |\cos x| \) over the interval \([0, \pi]\).

![Figure 1: Graphs of \( f(x) = \cos(x) \) and \( g(x) = |\cos(x)| \)](image)

3. (3 points) What are the absolute extreme values of \( \cos x \) over \([0, \pi]\)? (even though this question can be answered by looking at the graphs you drew in part 2, you should justify your answer using calculus).

Solution: Note that \( f'(x) = -\sin(x) = 0 \) if and only if \( x = 0 \) or \( x = \pi \) so we have

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<th>Critical points</th>
<th>Critical values ( f(c) )</th>
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<tr>
<td>( c = 0 )</td>
<td>( f(0) = 1 )</td>
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and we see that the absolute maximum value is \( f(0) = 1 \) and the absolute minimum value is \( f(\pi) = -1 \).
4. (3 points) What are the absolute extreme values of $| \cos x |$ over $[0, \pi]$? (even though this question can be answered by looking at the graphs you drew in part 2, you should justify your answer using calculus).

**Solution:** We proceed analogously for the function $g$: the endpoints are the same, but the value of the function might switch sign; the stationary points also stay the same, but the value of the function might switch sign; whenever the function $f$ crosses the x-axis, the function $g$ has a singularity (and has 0 value). Bearing this in mind we have

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<td>$g(\pi) = 1$</td>
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<tr>
<td><strong>Singular points</strong></td>
<td>$c = \frac{\pi}{2}$</td>
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and we see that the absolute maximum value is $f(0) = f(\pi) = 1$ and the absolute minimum value is $f(\frac{\pi}{2}) = 0$. 
2. (9 points) In Figure 1 below you can see the graph of the derivative $f'$ of a function $f$. Assume that $f(-3) = 2$, $f(-2) = 0$, $f(-1) = -2$, $f(0) = 5$ and $f(2) = 0$.

![Figure 1: Graph of the derivative $f'$](image)

Figure 2: Question 2

(i) (3 points) Over which intervals is $f$ increasing/decreasing? What are (if any) its local extreme values? **Solution:** $f$ is increasing (resp. decreasing) whenever $f'$ is positive (resp. negative). By looking at the graph of $f'$ we conclude that:

(i) $f$ is increasing over $(-\infty, -3)$ and $(-1, 0)$.
(ii) $f$ is decreasing over $(-3, -1)$ and $(0, +\infty)$.

(ii) (3 points) Over which intervals is $f$ concave up/down? What are (if any) the inflection points of $f$?

**Solution:** $f$ is concave up (resp. down) whenever $f''$ is positive (resp. negative), namely when $f'$ is increasing (resp. decreasing). By looking at the graph of $f'$ we conclude that:

(i) $f$ is concave up over $(-2, 0)$ and $(0, 2)$.
(ii) $f$ is concave down over $(-\infty, -2)$ and $(2, +\infty)$.

(iii) (3 points) Sketch the graph of $f$.

![Figure 3: Sketch of the graph of $f$.](image)

Figure 3: Sketch of the graph of $f$. This is how the function looks like, but as you can see the prescribed values are wrong.
Section B: Practical questions.

3. (12 points) Use the chain rule to compute the following derivatives (there is no need to simplify your answer).

   (i) (3 points) \( f(x) = (3x^7 - 2)^3(3 - x^2)^3 \)

   Solution: \( f'(x) = 3(3x^7 - 2)^2 \cdot 21x^6 \cdot (3 - x^2)^3 + (3x^7 - 2) \cdot 3 \cdot (3 - x^2)^2(-2x) \).

   (ii) (3 points) \( f(x) = \frac{1}{(3x^2 + x - 3)^9} \)

   Solution: \( f'(x) = -9(3x^2 + x - 3)^{-10}(6x + 1) \).

   (iii) (3 points) \( f(x) = \cos(\frac{x+1}{x-1}) \)

   Solution: \( f'(x) = -\sin(\frac{x+1}{x-1}) \cdot \frac{x-1-(x+1)}{(x-1)^2} \).

   (iv) (3 points) \( f(x) = \cos^4(\frac{x+1}{x-1}) \)

   Solution: \( f'(x) = 4 \cos^3(\frac{x+1}{x-1}) \cdot \left[ \cos\left(\frac{x+1}{x-1}\right) \right]' \).

4. (10 points) Suppose that the equation

   \[ y + \cos(xy^2) + 3x^2 = 4 \]

   defines \( y \) as an implicit function of \( x \)

   (i) (6 points) Find \( y' = D_x(y) \)

   Solution: Thinking of \( y = f(x) \) as a function of \( x \) and differentiating with respect to \( x \) we obtain

   \[ y' - \sin(xy)(y + xy') + 6x = 0, \quad [*] \]

   so \( y'[1 - x \sin(xy)] - y \sin(xy) = 0 \) and hence

   \[ y' = \frac{-6x + y \sin(xy)}{1 - x \sin(xy)} \]

   (ii) (4 points) Find \( y'' = D_x^2(y) \)

   Solution: We differentiate the equation \([*]\) with respect to \( x \) (bearing in mind that \( y = f(x) \) is a function of \( x \)) and we obtain:

   \[ y'' - \cos(xy)(y + xy')^2 - \sin(xy)(2y' + xy'') + 6 = 0 \]

   so \( y''[1 - x \sin(xy)] - \cos(xy)(y + xy')^2 - 2 \sin(xy)y' + 6 = 0 \) and hence

   \[ y'' = \frac{\cos(xy)(y + xy')^2 + 2 \sin(xy)y' - 6}{1 - x \sin(xy)} \]
5. (8 points) A particle is moving along the graph of the curve \( y = \sqrt{x^2 - 4} \) (for \( x \geq 2 \)) in such a way that its \( x \)-coordinate increases at a rate of 5 units per second. How fast is the \( y \)-coordinate increasing when \( x = 3 \)?

**Solution:** Denote by \( x(t) \) and \( y(t) \) the \( x \)-coordinate and the \( y \)-coordinate of the particle at time \( t \). We are being told that \( x'(t) = \frac{dx}{dt}x(t) = 5 \) and we are being asked about the value of \( y' = \frac{dy}{dt}y(t) \) when \( x = 3 \).

Since the particle is always over the curve \( y = \sqrt{x^2 - 4} \) we know that for every \( t \) we have

\[
y(t) = \sqrt{[x(t)]^2 - 4}
\]

Differentiating with respect to \( t \) we see that

\[
y'(t) = \frac{2x(t)x'(t)}{2\sqrt{(x(t))^2 - 4}} = \frac{x(t)x'(t)}{\sqrt{(x(t))^2 - 4}}
\]

so knowing that \( x'(t) = 5 \) for all \( t \), when \( x = 3 \) we have

\[
y' = \frac{3 \cdot 5}{\sqrt{3^2 - 4}} = \frac{15}{\sqrt{5}}
\]

6. (8 points) Approximate the value of \( \sqrt{35.7} \) using differentials (you just have to write down the final expression that you would plug into your calculator).

**Hint:** Recall that \( f(x + dx) \approx f(x) + f'(x)dx \) if \( dx \) is a small perturbation of \( x \).

**Solution:** We simply apply the expression in the hint to the function \( f(x) = \sqrt{x} \) for \( x = 36 \) and \( dx = -0.3 \):

\[
\sqrt{35.7} = \sqrt{36 - 0.3} \approx \sqrt{36} + \frac{1}{2\sqrt{36}} \cdot (-0.3)
\]
7. (8 points) A flower bed will be in the shape of a sector of circle (a pie slice) of radius \( r \) and vertex angle \( \theta \). Find the values of \( r \) and \( \theta \) such that the perimeter is minimum, assuming that the area is constant and equal to \( A \).

Hint: The perimeter of a sector of circle of radius \( r \) and vertex angle \( \theta \) is \( r\theta \) and its area is \( \frac{1}{2}r^2\theta \).

Solution: The perimeter of the flower bed is given in terms of \( r \) and \( \theta \) by

\[
P(r, \theta) = 2r + r\theta
\]

The area \( A \) is constant and provides a relation between \( r \) and \( \theta \), namely \( A = \frac{1}{2}r^2\theta \), so that \( \theta = \frac{2A}{r^2} \) and we may thus rewrite the perimeter as a function of \( r \) only:

\[
P(r) = 2r + 2A/r, \quad \text{over } (0, \infty)
\]

We are being asked to determine the absolute minimum of this function: the interval of definition is open so there are no endpoints, and the only singular point is \( r = 0 \) which is not inside the interval of definition. Our only candidates to absolute minimum are thus the stationary points of \( P(r) \).

The derivative of \( P(r) \) is given by \( P'(r) = 2 - \frac{2A}{r^2} \) so that \( P'(r) = 0 \) if and only if \( r = \sqrt{A} \).

We finally need to verify that this is indeed a minimum. Using the first derivative criterion, it suffices to check the sign of \( P' \) as we approach \( r = \sqrt{A} \) from above and from below. Writing

\[
P'(r) = \frac{2}{r^2}(r - \sqrt{A})(r + \sqrt{A})
\]

we see that \( P'(r) < 0 \) when \( r < \sqrt{A} \) and that \( P'(r) < 0 \) when \( r > \sqrt{A} \), so that \( r = \sqrt{A} \) yields a minimum value.
8. (20 points) Consider the function \( f(x) = \frac{(x-1)(x-3)}{(x+1)(x-2)} \) over \((-\infty, +\infty)\).

1. (2 points) What is the domain of \( f \)? Describe the discontinuities at the \( x \)-values for which \( f \) is not defined.

Solution: The only \( x \)-values for which the function \( f \) is not defined are \( x = -1 \) and \( x = 2 \), at which the denominator vanishes. Since the numerator is non-zero at these \( x \)-values, the lines \( x = -1 \) and \( x = 2 \) are vertical asymptotes. The one-sided limits are given by

\[
\lim_{x \to -1^-} f(x) = +\infty, \quad \lim_{x \to -1^+} f(x) = -\infty, \quad \lim_{x \to 2^-} f(x) = +\infty, \quad \lim_{x \to 2^+} f(x) = -\infty.
\]

2. (2 point) Compute the limits \( \lim_{x \to +\infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \).

Solution: \( \lim_{x \to +\infty} f(x) = 1 \) and \( \lim_{x \to -\infty} f(x) = 1 \).

3. (4 points) Over which intervals is \( f \) increasing (resp. decreasing)? Namely, for which \( x \) is \( f'(x) \) positive (resp. negative)?

Solution: The derivative of \( f \) is given by

\[
f'(x) = \frac{(2x-4)(x^2-x-2)-(x^2-4x+3)(2x-1)}{(x+1)^2(x-2)^2}
\]

\[
= \frac{2x^3-2x^2-4x-4x^2+4x+8-2x^3+x^2+8x^2-4x-6x+3}{(x+1)^2(x-2)^2}
\]

\[
= \frac{3x^2-10x+11}{(x+1)^2(x-2)^2}
\]

Clearly \( f'(x) = 0 \) if and only if \( 3x^2-10x+11 = 0 \) and using the quadratic formula we see that this happens only when \( x = \frac{-10 \pm \sqrt{100-132}}{6} \). We thus conclude that \( f'(x) \neq 0 \) for all \( x \).

The function \( f'(x) \) is continuous over the intervals \((-\infty, -1), (1, 2) \) and \((2, +\infty)\). Since \( f'(x) \neq 0 \), we conclude that the sign of \( f'(x) \) must remain constant in each of the three intervals (otherwise, \( f' \) should vanish somewhere by the intermediate value theorem). Checking the sign of \( f' \) at one \( x \)-value in each interval we conclude that \( f'(x) > 0 \) for all \( x \), so that \( f \) is always increasing in all its domain.

4. (2 points) What are, if any, the local extreme values of \( f \)?

Solution: We have seen that \( f \) is increasing in all its domain, so it cannot have local extreme values (for instance, by the first derivative criterion).

5. (4 points) Over which intervals is \( f \) concave up (resp. down)? Namely, for which \( x \) is \( f''(x) \) positive (resp. negative)?
Solution: The second derivative of $f$ is given by

$$f''(x) = \frac{(6x - 10)(x + 1)^2(x - 2)^2 - (3x^2 - 10x + 11)(2(x + 1)(x - 2)^2 + 2(x + 1)^2(x - 2))}{(x + 1)^4(x - 2)^4}$$

where in [*] we divided the numerator and the denominator by $(x + 1)(x - 2)$.

Clearly $f''(x) = 0$ if and only if the numerator is zero, namely

$$0 = (6x - 10)(x^2 - x - 1) - 2(3x^2 - 10x + 11)(2x - 1)$$

$$= 2[(3x - 5)(x^2 - x - 2) - (3x^2 - 10x + 11)(2x - 1)]$$

$$= 2[3x^3 - 3x^2 - 6x - 5x^2 + 5x + 10 - 6x^3 + 3x^2 + 20x^2 - 10x - 22x + 11]$$

$$= 2(-3x^3 + 15x^2 - 33x + 21)$$

$$= -6(x^3 - 5x^2 + 11x - 7)$$

$$\Rightarrow -6(x - 1)(x^2 - 4x + 7)$$

where [*] follows from Ruffini’s theorem or by long division. From this we see that $f''(x) = 0$ if and only if either $x = 1$, or $x^2 - 4x + 7 = 0$, but the latter is not possible, again, by the quadratic formula $x = \frac{4 \pm \sqrt{16 - 28}}{2}$ and hence $x = 1$ is the only x-value at which $f''$ vanishes.

We finally study the sign of $f''(x) = \frac{6(x - 1)(-x^2 + 4x - 7)}{(x + 1)^3(x - 2)^3}$:

(i) If $x < -1$, then $f''(x) = \frac{6x - 10}{(x + 1)^3(x - 2)^3} > 0$, so $f$ is concave up.

(ii) If $-1 < x < 1$, then $f''(x) = \frac{6x - 10}{(x + 1)^3(x - 2)^3} < 0$, so $f$ is concave down.

(iii) If $1 < x < 2$, then $f''(x) = \frac{6x - 10}{(x + 1)^3(x - 2)^3} > 0$, so $f$ is concave up.

(iv) If $2 < x$, then $f''(x) = \frac{6x - 10}{(x + 1)^3(x - 2)^3} < 0$, so $f$ is concave down.

6. (2 points) What are, if any, the inflection points of $f$?

Solution: The only inflection point has x-value $x = 1$ ($f$ goes from being concave down to being concave up).

7. (4 points) Sketch the graph of $f$.  

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Figure 4: Graph of $f(x) = \frac{(x-1)(x-3)}{(x+1)(x-2)}$