This exam consists of 2 sections, A and B. Section A is conceptual, whereas section B is more computational. The value of every question is indicated at the beginning of it. You may only use scratch paper and a small note card. No cell phones, calculators, notes, books or music players are allowed during the test.

Name: ______________________________ UID: ______________

Section A: Conceptual questions.

1. 1. (3 points) Let f be a function defined over an interval $[a, b]$ which contains c. If $f(c)$ is an extreme value of f, what can you say about c?

   Solution: c must be a critical point: it must be either an endpoint, or a singular point ($f'(c)$ does not exist) or a stationary point (i.e. $f'(c) = 0$).

2. (3 points) Sketch the graph of the functions $f(x) = \cos x$ and $g(x) = |\cos x|$ over the interval $[0, \pi]$.

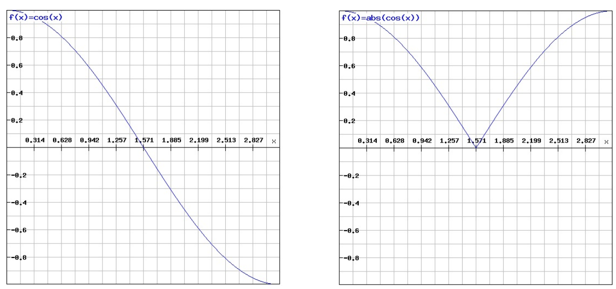

Figure 1: Graphs of $f(x) = \cos(x)$ and $g(x) = |\cos(x)|$

3. (3 points) What are the absolute extreme values of $\cos x$ over $[0, \pi]$? (even though this question can be answered by looking at the graphs you drew in part 2, you should justify your answer using calculus).

   Solution: Note that $f'(x) = -\sin(x) = 0$ if and only if $x = 0$ or $x = \pi$ so we have

	Critical points c	Critical values $f(c)$
Endpoints	$c = 0$	$f(0) = 1$
	$c = \pi$	$f(\pi) = -1$
Stationary points	$c = 0$	$f(0) = 1$
	$c = \pi$	$f(\pi) = -1$
Singular points	–	–

   and we see that the absolute maximum value is $f(0) = 1$ and the absolute minimum value is $f(\pi) = -1$.

4. (3 points) What are the absolute extreme values of $|\cos x|$ over $[0, \pi]$? (even though this question can be answered by looking at the graphs you drew in part 2, you should justify your answer using calculus).

Solution: We proceed analogously for the function g: the endpoints are the same, but the value of the function might switch sign; the stationary points also stay the same, but the value of the function might switch sign; whenever the function f crosses the x-axis, the function g has a singularity (and has 0 value). Bearing this in mind we have

	Critical points c	Critical values $f(c)$
Endpoints	$c = 0$	$g(0) = 1$
	$c = \pi$	$g(\pi) = 1$
Stationary points	$c = 0$	$g(0) = 1$
	$c = \pi$	$g(\pi) = 1$
Singular points	$c = \frac{\pi}{2}$	0

and we see that the absolute maximum value is $f(0) = f(\pi) = 1$ and the absolute minimum value is $f(\frac{\pi}{2}) = 0$.

2. (9 points) In Figure 1 below you can see the graph of the derivative f' of a function f. Assume that $f(-3) = 2$, $f(-2) = 0$, $f(-1) = -2$, $f(0) = 5$ and $f(2) = 0$.

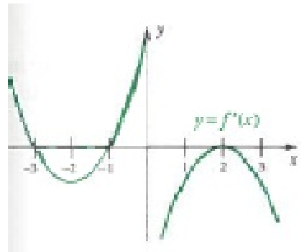

Figure 2: Question 2

(i) (3 points) Over which intervals is f increasing/decreasing? What are (if any) its local extreme values? **Solution**: f is increasing (resp. decreasing) whenever f' is positive (resp. negative). By looking at the graph of f' we conclude that:

   (i) f is increasing over $(-\infty, -3)$ and $(-1, 0)$.

   (ii) f is decreasing over $(-3, -1)$ and $(0, +\infty)$.

(ii) (3 points) Over which intervals is f concave up/down? What are (if any) the inflection points of f?

   Solution: f is concave up (resp. down) whenever f'' is positive (resp. negative), namely when f' is increasing (resp. decreasing). By looking at the graph of f' we conclude that:

   (i) f is concave up over $(-2, 0)$ and $(0, 2)$.

   (ii) f is concave down over $(-\infty, -2)$ and $(2, +\infty)$.

(iii) (3 points) Sketch the graph of f.

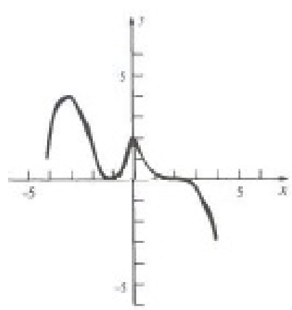

Figure 3: Sketch of the graph of f. This is how the function looks like, but as you can see the prescribed values are wrong.

Section B: Practical questions.

3. (12 points) Use the chain rule to compute the following derivatives (there is no need to simplify your answer).

   (i) (3 points) $f(x) = (3x^7 - 2)^3(3 - x^2)^3$

   Solution: $f'(x) = 3(3x^7 - 2)^2 \cdot 21x^6 \cdot (3 - x^2)^3 + (3x^7 - 2) \cdot 3 \cdot (3 - x^2)^2(-2x)$.

   (ii) (3 points) $f(x) = \frac{1}{(3x^2+x-3)^9}$

   Solution: $f'(x) = -9(3x^2 + x - 3)^{-10}(6x + 1)$.

   (iii) (3 points) $f(x) = \cos\left(\frac{x+1}{x-1}\right)$

   Solution: $f'(x) = -\sin\left(\frac{x+1}{x-1}\right) \frac{x-1-(x+1)}{(x-1)^2}$.

   (iv) (3 points) $f(x) = \cos^4\left(\frac{x+1}{x-1}\right)$

   Solution: $f'(x) = 4\cos^3\left(\frac{x+1}{x-1}\right) \cdot \left[\cos\left(\frac{x+1}{x-1}\right)\right]'$.

4. (10 points) Suppose that the equation

$$y + \cos(xy^2) + 3x^2 = 4$$

defines y as an implicit function of x

   (i) (6 points) Find $y' = D_x(y)$

   Solution: Thinking of $y = f(x)$ as a function of x and differentiating with respect to x we obtain

$$y' - \sin(xy)(y + xy') + 6x = 0, \qquad [*]$$

   , so $y'\left[1 - x\sin(xy)\right] - y\sin(xy) = 0$ and hence

$$y' = \frac{-6x + y\sin(xy)}{1 - x\sin(xy)}$$

   (ii) (4 points) Find $y'' = D_x^2(y)$

   Solution: We differentiate the equation [∗] with respect to x (bearing in mind that $y = f(x)$ is a function of x) and we obtain:

$$y'' - \cos(xy)(y + xy')^2 - \sin(xy)(2y' + xy'') + 6 = 0$$

   so $y''\left[1 - x\sin(xy)\right] - \cos(xy)(y + xy')^2 - 2\sin(xy)y' + 6 = 0$ and hence

$$y'' = \frac{\cos(xy)(y + xy')^2 + 2\sin(xy)y' - 6}{1 - x\sin(xy)}$$

5. (8 points) A particle is moving along the graph of the curve $y = \sqrt{x^2 - 4}$ (for $x \geq 2$) in such a way that its x-coordinate increases at a rate of 5 units per second. How fast is the y-coordinate increasing when $x = 3$?

Solution: Denote by $x(t)$ and $y(t)$ the x-coordinate and the y-coordinate of the particle at time t. We are being told that $x'(t) = \frac{d}{dt}x(t) = 5$ and we are being asked about the value of $y' = \frac{d}{dt}y(t)$ when $x = 3$.

Since the particle is always over the curve $y = \sqrt{x^2 - 4}$ we know that for every t we have

$$y(t) = \sqrt{[x(t)]^2 - 4}$$

Differentiating with respect to t we see that

$$y'(t) = \frac{2x(t)x'(t)}{2\sqrt{(x(t))^2 - 4}} = \frac{x(t)x'(t)}{\sqrt{(x(t))^2 - 4}}$$

so knowing that $x'(t) = 5$ for all t, when $x = 3$ we have

$$y' = \frac{3 \cdot 5}{\sqrt{3^2 - 4}} = \frac{15}{\sqrt{5}}$$

6. (8 points) Approximate the value of $\sqrt{35.7}$ using differentials (you just have to write down the final expression that you would plug into your calculator).

Hint: Recall that $f(x + dx) \simeq f(x) + f'(x)dx$ if dx is a small perturbation of x.

Solution: We simply apply the expression in the hint to the function $f(x) = \sqrt{x}$ for $x = 36$ and $dx = -0.3$:

$$\sqrt{35.7} = \sqrt{36 - 0.3} \simeq \sqrt{36} + \frac{1}{2\sqrt{36}} \cdot (-0.3)$$

7. (8 points) A flower bed will be in the shape of a sector of circle (a pie slice) of radius r and vertex angle θ. Find the values of r and θ such that the perimeter is minimum, assuming that the area is constant and equal to A.

Hint: The perimeter of a sector of circle of radius r and vertex angle θ is $r\theta$ and its area is $\frac{1}{2}\theta r^2$.

Solution: The perimeter of the flower bed is given in terms of r and θ by

$$P(r, \theta) = 2r + r\theta$$

.

The area A is constant and provides a relation between r and θ, namely $A = \frac{1}{2}\theta r^2$, so that $\theta = \frac{2A}{r^2}$ and we may thus rewrite the perimeter as a function of r only:

$$P(r) = 2r + \frac{2A}{r}, \qquad \text{over } (0, \infty)$$

We are being asked to determine the absolute minimum of this function: the interval of definition is open so there are no endpoints, and the only singular point is $r = 0$ which is not inside the interval of definition. Our only candidates to absolute minimum are thus the stationary points of $P(r)$.

The derivative of $P(r)$ is given by $P'(r) = 2 - \frac{2A}{r^2}$ so that $P'(r) = 0$ if and only if $r = \sqrt{A}$.

We finally need to verify that this is indeed a minimum. Using the first derivative criterion, it suffices to check the sign of P' as we approach $r = \sqrt{A}$ from above and from below. Writing

$$P'(r) = \frac{2}{r^2}(r - \sqrt{A})(r + \sqrt{A})$$

we see that $P'(r) < 0$ when $r < \sqrt{A}$ and that $P'(r) < 0$ when $r > \sqrt{A}$, so that $r = \sqrt{A}$ yields a minimum value.

8. (20 points) Consider the function $f(x) = \frac{(x-1)(x-3)}{(x+1)(x-2)}$ over $(-\infty, +\infty)$.

1. (2 points) What is the domain of f? Describe the discontinuities at the x-values for which f is not defined.

   Solution: The only x-values for which the function f is not defined are $x = -1$ and $x = 2$, at which the denominator vanishes. Since the numerator is non-zero at these x-values, the lines $x = -1$ and $x = 2$ are vertical asymptotes. The one-sided limits are given by

   $$\lim_{x \to -1^-} f(x) = +\infty, \quad \lim_{x \to -1^+} f(x) = -\infty, \quad \lim_{x \to 2^-} f(x) = +\infty, \quad \lim_{x \to 2^+} f(x) = -\infty$$

2. (2 point) Compute the limits $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

   Solution: $\lim_{x \to +\infty} f(x) = 1$ and $\lim_{x \to -\infty} f(x) = 1$.

3. (4 points) Over which intervals is f increasing (resp. decreasing)? Namely, for which x is $f'(x)$ positive (resp. negative)?

   Solution: The derivative of f is given by

   $$\begin{aligned} f'(x) &= \frac{(2x-4)(x^2-x-2) - (x^2-4x+3)(2x-1)}{(x+1)^2(x-2)^2} \\ &= \frac{2x^3 - 2x^2 - 4x - 4x^2 + 4x + 8 - 2x^3 + x^2 + 8x^2 - 4x - 6x + 3}{(x+1)^2(x-2)^2} \\ &= \frac{3x^2 - 10x + 11}{(x+1)^2(x-2)^2} \end{aligned}$$

   Clearly $f'(x) = 0$ if and only if $3x^2 - 10x + 11 = 0$ and using the quadratic formula we see that this happens only when $x = \frac{-10 \pm \sqrt{100-132}}{6}$. We thus conclude that $f'(x) \neq 0$ for all x.

   The function $f'(x)$ is continuous over the intervals $(-\infty, -1)$, $(-1, 2)$ and $(2, \infty)$. Since $f'(x) \neq 0$, we conclude that the sign of $f'(x)$ must remain constant in each of the three intervals (otherwise, f' should vanish somewhere by the intermediate value theorem). Checking the sign of f' at one x-value in each interval we conclude that $f'(x) > 0$ for all x, so that f is always increasing in all its domain.

4. (2 points) What are, if any, the local extreme values of f?

   Solution: We have seen that f is increasing in all its domain, so it cannot have local extreme values (for instance, by the first derivative criterion).

5. (4 points) Over which intervals is f concave up (resp. down)? Namely, for which x is $f''(x)$ positive (resp. negative)?

Solution: The second derivative of f is given by

$$\begin{aligned} f''(x) &= \frac{(6x-10)(x+1)^2(x-2)^2 - (3x^2-10x+11)\left[2(x+1)(x-2)^2 + 2(x+1)^2(x-2)\right]}{(x+1)^4(x-2)^4} \\ &\overset{[*]}{=} \frac{(6x-10)\overbrace{(x+1)(x-2)}^{x^2-x-2} - (3x^2-10x+11)\overbrace{[2(x-2)+2(x+1)]}^{2(2x-1)}}{(x+1)^3(x-2)^3} \end{aligned}$$

where in [$*$] we divided the numerator and the denominator by $(x+1)(x-2)$.

Clearly $f''(x) = 0$ if and only if the numerator is zero, namely

$$\begin{aligned} 0 &= (6x-10)(x^2-x-1) - 2(3x^2-10x+11)(2x-1) \\ &= 2\left[(3x-5)(x^2-x-2) - (3x^2-10x+11)(2x-1)\right] \\ &= 2\left[3x^3 - 3x^2 - 6x - 5x^2 + 5x + 10 - 6x^3 + 3x^2 + 20x^2 - 10x - 22x + 11\right] \\ &= 2(-3x^3 + 15x^2 - 33x + 21) \\ &= -6(x^3 - 5x^2 + 11x - 7) \\ &\overset{[*]}{=} -6(x-1)(x^2-4x+7) \end{aligned}$$

where [$*$] follows from Ruffini's theorem or by long division. From this we see that $f''(x) = 0$ if and only if either $x = 1$, or $x^2 - 4x‘7 = 0$, but the latter is not possible, again, by the quadratic formula $x = \frac{4 \pm \sqrt{16-28}}{2}$ and hence $x = 1$ is the only x-value at which f'' vanishes.

We finally study the sign of $f''(x) = 6\frac{(x-1)(-x^2+4x-7)}{(x+1)^3(x-2)^3}$:

(i) If $x < -1$, then $f''(x) = \frac{--}{--} > 0$, so f is concave up.
(ii) If $-1 < x < 1$, then $f''(x) = \frac{--}{+-} < 0$, so f is concave down.
(iii) If $1 < x < 2$, then $f''(x) = \frac{+-}{+-} > 0$, so f is concave up.
(iv) If $2 < x$, then $f''(x) = \frac{+-}{++} < 0$, so f is concave down.

6. (2 points) What are, if any, the inflection points of f?

   Solution: The only inflection point has x-value $x = 1$ (f goes from being concave down to being concave up).

7. (4 points) Sketch the graph of f.

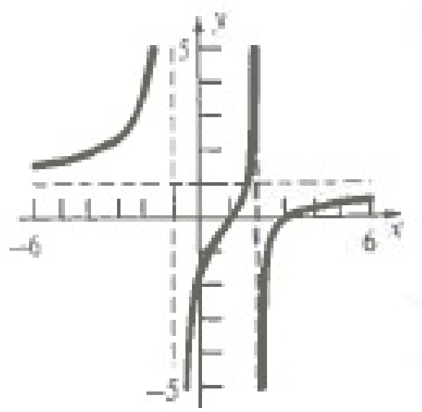

Figure 4: Graph of $f(x) = \frac{(x-1)(x-3)}{(x+1)(x-2)}$