This exam consists of 2 sections, A and B. Section A is conceptual, whereas section B is more computational. The value of every question is indicated at the beginning of it. You may only use scratch paper and a small note card. No cell phones, calculators, notes, books or music players are allowed during the exam.

Name: ______________________________ UID: ______________

Section A: Conceptual questions.

1. (i) (4 points) What does it mean for a function $f(x)$ to be continuous at $x = c$?

   Solution: A function is continuous at $x = c$ if $\lim_{x\to x} f(x) = f(c)$, namely:

   (i) f must be defined at $x = c$.
   (ii) The limit $\lim_{x\to c} f(x)$ must exist.
   (iii) Both $f(c)$ and $\lim_{x\to c} f(x)$ must coincide.

   (ii) (4 points) What is the derivative of a function $f(x)$ at $x = c$ (write it at as a limit)?

   Solution: We define $f'(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ or equivalently, $f'(c) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

   (iii) (4 points) Give a geometric interpretation of the derivative of $f(x)$ in terms of the curve $y = f(x)$.

   Solution: For every x-value c, the derivative $f'(c)$ computes the slope of the tangent to the curve given by the equation $y = f(x)$.

2. (8 points) In Figure 1 below you can see the graph of a function $f(x)$ and the graph of its derivative $f'(x)$. Indicate **which one is which** and support your answer by giving **at least two reasons**.

   Solution: The dotted curve is the graph of a some function f and the continuous curve is the graph of its derivative $f'(x)$.

   (i) Over those intervals where the dotted curve is increasing (resp. decreasing), the continuous curve lies above (resp. below) the x-axis; namely, f is increasing (resp. decreasing) whenever f' is positive (resp. negative).
   (ii) Whenever the dotted curve presents a local maximum or minimum value, the continuous curve crosses the x-axis (namely if $f(x)$ is a local extreme point, then $f'(x) = 0$).

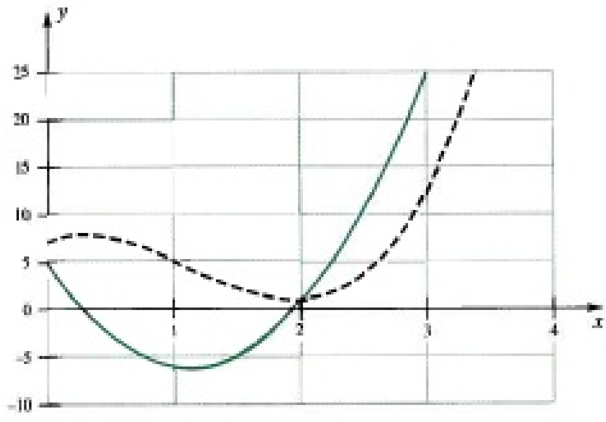

Figure 1: Question 2

3. (4 points) Use the definition (limit) to compute the derivative of the function

$$f(x) = 2x^3 + 5$$

Recall that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(2(x+h)^3 + 5) - (2x^3 + 5)}{h} \\ &= \lim_{h\to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) + 5 - 2x^3 - 5}{h} = \lim_{h\to 0} \frac{6x^2h + 6xh^2 + 2h^3}{h} \\ &= \lim_{h\to 0} 6x^2 + 6xh + 2h^2 \\ &= 6x^2 \end{aligned}$$

4. (8 points) Find the equation of the tangent line to the curve $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$

Solution: Recall that the slope of the tangent line to the curve given by the equation $y = f(x)$ at the point $(c, f(c))$ is $f'(c)$, so that the equation os the tangent line is given by

$$y - f(c) = f'(c)(x - c)$$

In our case $f(x) = \tan x$ and $c = \frac{\pi}{4}$; the derivative is given by

$$f'(x) = \frac{1}{\cos^2 x}, \qquad \text{so} \qquad f'\left(\frac{\pi}{4}\right) = \frac{1}{\cos^2 \frac{\pi}{4}} = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{1}{\frac{2}{4}} = 2$$

so the equation of the tangent line is given by

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

Section B: Practical questions.

5. (16 points) Study the continuity of the function

$$f(x) = \begin{cases} \frac{3}{x-2}, & x < -1 \\ x, & -1 \leq x \leq 1 \\ \frac{-x^2+4x-3}{x-3}, & x > 1 \end{cases}$$

by answering the following questions.

(i) (2 points) What is the domain of the function f (namely, at what points is f defined)?

Solution: The function is defined over $(-\infty, +\infty)$, except for the x-value $x = 3$. Note that when $x > 1$ (and clearly $3 > 1$), the function is given by $\frac{-x^2+4x-3}{x-3}$ and the denominator becomes zero at $x = 3$. Also observe that when $x < -1$, the function is given by $f(x) = \frac{3}{x-2}$, and even though the denominator becomes zero at $x = 2$, there is no discontinuity at $x = 2$ since $2 > -1$.

(ii) (6 points) Are there any removable discontinuities? If so, what is the limit at those points? (namely, if there is a removable discontinuity at $x = c$, compute $\lim_{x \to c} f(x)$).

Solution: The discontinuity at $x = 3$ is removable; indeed, the limit is given by

$$\begin{aligned} \lim_{x \to 3} f(x) &= \lim_{x \to 3} \frac{-x^2 + 4x - 3}{x - 3} = \lim_{x \to 3} \frac{-(x-1)(x-3)}{x-3} = \lim_{x \to 3} -(x-1) \\ &= -(3-1) = -2 \end{aligned}$$

(iii) (6 points) Are there any jump discontinuities? If so, what are the right-hand and left-hand limits?

Solution: Note that as we approach the x-values $x = 1$ and $x = -1$ from the left and from the right, the function f has different expressions, so it is likely that f presents jump discontinuities at those x-values if the one-sided limits differ.

At $x = -1$ we have

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1} \frac{3}{x-2} = -1, \qquad \lim_{x \to -1^+} f(x) = \lim_{x \to -1} x = -1$$

Since both limits coincide, we conclude that f is continuous at $x = -1$.

On the other hand, at $x = 1$ we have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} x = 1, \qquad \lim_{x \to 1^+} f(x) = \lim_{x \to 1} \frac{-x^2 + 4x - 3}{x - 3} = \frac{-1 + 4 - 3}{1 - 3} = 0$$

Since both limits differ, f has a jump discontinuity at $x = 1$.

(iv) (2 points) Are there any vertical asymptotes? If so, what are the right-hand and left-hand limits ($-\infty$ or $+\infty$)?

Solution: The function f has no vertical asymptotes.

6. (4 points) Use the intermediate value theorem to show that the equation

$$x^3 - 7x^2 + 14x - 8 = 0$$

has a solution in the interval [0,5] (Recall that a solution of the equation $f(x) = 0$ is a number c such that $f(c) = 0$.)

Solution: The intermediate value theorem states that if a function f is continuous over a closed interval $[a, b]$ and $f(a)f(b) < 0$ (namely, the values of f at $x = a$ and $x = b$ have opposite signs) then there exists some c inside $[a, b]$ such that $f(c) = 0$.

In plain words, if you draw a curve without lifting your pen (i.e. a continuous curve), starting from below the x-axis and ending above the x-axis (or viceversa), then you must have crossed the x-axis at least once.

In our case, the function $f(x) = x^3 - 7x^2 + 14x - 8$ is a polynomial, so it is clearly continuous and besides,

$$f(0) = -8 < 0 \qquad \text{and} \qquad f(5) = 5^3 - 7 \cdot 5^2 + 14 \cdot 5 - 8 = 12 > 0$$

Since the values of the function at the endpoints of the interval have opposite signs, the Intermediate Value Theorem guarantees that the equation

$$x^3 - 7x^2 + 14x - 8 = 0$$

must have a solution inside $[0, 5]$.

7. (4 points) Compute the limit

$$\lim_{x\to\infty} \sqrt[3]{\frac{1+8x^2}{x^2+4}}$$

Solution:

$$\lim_{x\to\infty} \sqrt[3]{\lim_{x\to\infty} \frac{1+8x^2}{x^2+4}} = \sqrt[3]{\frac{1+8x^2}{x^2+4}} = \sqrt[3]{\lim_{x\to\infty} \frac{\frac{1}{x^2}+8}{1+\frac{4}{x^2}}} = \sqrt[3]{\frac{0+8}{1+0}} = \sqrt[3]{8} = 2$$

8. (4 points) Compute the limit

$$\lim_{x\to 0} \frac{3x \tan x}{\sin x}$$

Solution:

$$\lim_{x\to 0} \frac{3x \tan x}{\sin x} = \lim_{x\to 0} \frac{3x \frac{\sin x}{\cos x}}{\sin x} = \lim_{x\to 0} \frac{3x}{\cos x} = \frac{0}{1} = 0$$

9. (4 points) Compute the limit

$$\lim_{x\to\sqrt{2}} \frac{\sqrt[3]{2x^2+3}}{7x^2+13}$$

Solution:

$$\lim_{x\to\sqrt{2}} \frac{\sqrt[3]{2x^2+3}}{7x^2+13} = \frac{\sqrt[3]{2(\sqrt{2})^2+3}}{7(\sqrt{2})^2+13} = \frac{\sqrt[3]{2\cdot 2+3}}{7\cdot 2+13} = \frac{\sqrt[3]{7}}{27}$$

10. (4 points) Compute the limit

$$\lim_{x\to -2} \frac{(x+2)(x^2-x-6)}{x^2+4x+4}$$

Solution: Evaluating at $x=-2$ we obtain an indeterminate expression of the form $\frac{0}{0}$. This means that at least one monomial $x+2$ can be factored out from both the numerator and the denominator. Indeed,

$$\lim_{x\to -2} \frac{(x+2)(x^2-x-6)}{x^2+4x+4} = \lim_{x\to -2} \frac{(x+2)(x+2)(x-3)}{(x+2)^2} = \lim_{x\to -2} x-3 = -5$$

11. (4 points) Compute the limits

$$\lim_{x\to 2^+} \frac{x+1}{x^2-5x+6}, \qquad \lim_{x\to 2^-} \frac{x+1}{x^2-5x+6}$$

Solution: Evaluating at $x=2$ we obtain $\frac{3}{0}$, so the function $f(x)=\frac{x+1}{x^2-5x+6}$ has a vertical asymptote at $x=2$. In order to compute the left-hand and right-hand limits, we factor the denominator as $x^2-5x+6=(x-2)(x-3)$ so that

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x+1}{(x-2)(x-3)} = -\infty \qquad \frac{+}{+-}$$

and

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} \frac{x+1}{(x-2)(x-3)} = +\infty \qquad \frac{+}{--}$$

12. (20 points) Compute the derivatives of the following functions (you don't need to simplify your solution).

(i) (4 points) $f(x) = (x^4 + 1)(x^2 + 1)$

Solution: Using the product rule we get

$$f'(x) = (x^4 + 1)'(x^2 + 1) + (x^4 + 1)(x^2 + 1)' = 4x^3(x^2 + 1) + (x^4 + 1) \cdot 2x$$

(ii) (4 points) $f(x) = \frac{5x^2+2x-6}{3x-1}$

Solution: Using the quotient rule we get

$$\begin{aligned} f'(x) &= \frac{(5x^2 + 2x - 6)'(3x - 1) - (5x^2 + 2x - 6)(3x - 1)'}{(3x - 1)^2} \\ &= \frac{(10x + 2)(3x - 1) - (5x^2 + 2x - 6) \cdot 3}{(3x - 1)^2} \end{aligned}$$

(iii) (4 points) $f(x) = \sin x \tan x$

Solution:

$$f'(x) = (\sin x)' \tan x + \sin x(\tan x)' = \cos x \tan x + \sin x \frac{1}{\cos^2 x}$$

(iv) (8 points) $f(x) = \frac{x \cos x + \sin x}{x^2+1}$

Solution: Note that

$$(x \cos x + \sin x)' = \cos x - x \sin x + \cos x = 2 \cos x - x \sin x$$

so that

$$\begin{aligned} f'(x) &= \frac{(x \cos x + \sin x)'(x^2 + 1) - (x \cos x + \sin x)(x^2 + 1)'}{(x^2 + 1)^2} \\ &= \frac{(2 \cos x - x \sin x)(x^2 + 1) - (x \cos x + \sin x) \cdot 2x}{(x^2 + 1)^2} \end{aligned}$$