Multistage Methods II: RK Stability

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1 Introduction

We will now quickly review the stability properties of RK methods using the Butcher Tableau coefficients.

2 Stability

Let $b^T = [b_1, b_2, \ldots, b_s]$, $C = I[c_1, \ldots, c_s]^T$, and $A = a_{ij}, i, j = 1, \ldots, s$ (the Runge-Kutta matrix). Then, on the test ODE $y' = \lambda y$, we can show that any RK method yields

$$y^{n+1} = R(\Delta t \lambda)y^n,$$

$$R(\mu) = 1 + \mu b^T (I - \mu A)^{-1} 1,$$

where $1 = [1, 1, \ldots, 1]^T$ is an $s$–long vector of ones. As usual, for stability, we will require that

$$|R(\Delta t \lambda)| \leq 1.$$

2.1 A-Stability

If the region of absolute stability contains the entire left half of the complex plane, the RK method is A-stable.

2.2 L-Stability

An implicit, A-stable RK method has stiff decay if

$$\lim_{Re(\mu) \to -\infty} R(\mu) = 0.$$

Such RK methods are called L-stable methods.
2.3 B-Stability

Often, the problems we solve are nonlinear. We’d like to extend the notion of stability to these problems as well.

First, we say that a function is dissipative if \( \langle f(t, y) - f(t, z), (y - z) \rangle \leq 0 \) for all \( y \) and \( z \).

An ODE with a dissipative right-hand side \( f \) is contractive. In other words, for the ODE, \( \|y(t) - z(t)\| \leq \|y(s) - z(s)\| \) for every pair of solutions \( y \) and \( z \) when \( t \geq s \).

Armed with these definitions, we can say that a numerical method is B-Stable if every pair of numerical solutions \( u \) and \( v \) satisfy \( \|u^{n+1} - v^{n+1}\| \leq \|u^n - v^n\| \) for all \( n \), when solving a contractive ODE IVP. A B-Stable method is also A-stable.

Let \( B = Ib \), and \( M = BA + A^T B^T - bb^T \). An RK method is said to be algebraically stable if \( M \) is nonnegative semidefinite; i.e., \( x^T M x \geq 0 \) for any \( x \).

Finally, an algebraically stable RK method is B-stable.