

Example

Straight line. Differentiation.

1. Find the equation of the line which goes through the point $(2, -1)$ and is parallel to the line given by the equation $2x - y = 1$.
2. Find the equation of the line which goes through the point $(2, -1)$ and is perpendicular to the line given by the equation $2x - y = 1$.
3. Find the equation of the line which goes through the point $(1, 2)$ and is parallel to the line through the points $(0, 1)$ and $(-2, 7)$.

4. The relation

$$2x^2 + y^2 = 9$$

determines a curve in the $x - y$ plane (an ellipse).

- a) Find the equation of the line which goes through the point $(2, -1)$ on the curve and is *perpendicular* to the tangent line to the curve at this point.
 - b) At what points (x_0, y_0) does this curve have a horizontal tangent line.
5. A man standing at the edge of the roof of a building 120 feet high throws a ball directly upwards at a velocity of 48 ft/sec. a) How high does the ball go? b) Assuming that it proceeds to fall along the side of the building, how long does it take to hit ground level?
 6. Another man standing on ground level throws the ball back to his friend on the roof. At what initial velocity must he throw it in order to reach the roof?
 7. Let $y = \frac{x}{x^2 + 1}$ Find the equation of the tangent line to the graph at the point $(2, 0.4)$.
 8. Let C_1 and C_2 be curves given by the equation $C_1: y = x^3 + x^2$, $C_2: y = x^2 + x$. For what values of x do these curves have parallel tangent lines?
 9. A point moves around the unit circle so that the angle it makes with the x -axis at time t is $\theta(t) = (t^2 + t)\pi$. Let $(x(t), y(t))$ be the cartesian coordinates of the point at time t . What is dy/dt when $t = 3$?
 10. Find the derivative: $f(x) = x^3 - x^2 + 1$.
 11. Find the derivative: $f(x) = x^5 + 3x^4 - 2x^2 + 4x - 7$.
 12. Find the derivative: $f(x) = x^4 - 2x^3 + 5x^2 - x +$.
 13. Find the derivative: $f(x) = 3x^{-1} + x^3$.

14. Find the equation of the tangent line to the graph of $y = x^3 - 3x^2 + x$ at the point $(2, -2)$.
15. Let $y = 16x^{-1} + x^2$. At what point(s) is the tangent line horizontal?
16. Let $y = 4x^4 + x$. At what point is the tangent line to the graph perpendicular to the line tangent to the graph at $(0, 0)$?
17. Find the derivative:

$$y = \left(x^2 + \frac{1}{x^3}\right)(x^3 - x^2 + 1)$$

18. Find f' and f'' : $f(x) = \left(x + \frac{1}{x}\right)(x^2 + 1)$.
19. Find the derivative of $f(x) = \frac{x^2 + 1}{x + 1}$
20. Evaluate the derivative of the function

$$y(x) = \sin(\pi \sin(\pi \sin(\pi x)))$$

at $x = 0$.

21. Find the derivative: $f(x) = \sin x \cos x$.
22. Find the derivative: $g(x) = \frac{\sin x}{\cos x}$.
23. Differentiate $\sqrt{\frac{x-1}{x+1}}$
24. Consider the curves $C_1: y = \sin x$ and $C_2: y = \cos x$.
 a) At which points x between $-\pi/2$ and $\pi/2$ do the curves have parallel tangent lines? b) At which such points do they have perpendicular tangent lines?
25. Differentiate: $f = \frac{1 + \tan x}{1 - \tan x}$
26. Let $y = x + 25x^{-1}$. Find an approximate value of y when $x = 3.2$.
27. Find an approximate value of $\tan(0.26\pi)$.
28. Find the equation of the tangent line to $y = x^2(x^3 - 1)$ at $(2, 28)$.
29. Find the equation of the tangent line to the curve $y = x \cos x$ at $(\pi/4, \pi\sqrt{2}/8)$.
30. Find the equation of the tangent line to the curve $y = x - x^2$ at $(2, 7/4)$.
31. Differentiate: $y = (x^2 - 1) \sin(x^2 + 1)$.

32. Find $f'(x)$: $f(x) = \frac{(x+1)^2}{(x-1)^2}$
33. Find $g'(x)$, $g''(x)$: $g(x) = (x^3 + 1)^4$.
34. Find the derivative: $h(x) = (\cos(2x) + 1) \sin(3x)$.
35. Find the derivatives of the following functions:
 a) $f(x) = \cos^2 x$;
 b) $g(x) = \frac{\sin^2 x}{\cos x}$.
36. Find the first and second derivatives of $f(x) = x\sqrt{1-x^2}$.
37. Differentiate: $g(x) = (\sin(3x) + 1)^3$.
38. Differentiate: $h(t) = \frac{1-t^2}{1+t^3}$.
39. Differentiate: $f(x) = \sqrt{2x^2 - 3x + 1}$.
40. Find the derivatives of the given function: $h(x) = (\cos(2x) + 1) \sin(3x)$
41. Find the equation of the line tangent to the curve $y = \cos(x/2)$ at $(3\pi, 0)$
 Let $f(x) = (x - \sqrt{x})^2$. Find $f'(x)$ and $f''(x)$.
42. Find the points on the curve $y = 3x^2 - 3x + 1$ whose tangent line is perpendicular to the line $x + 2y = 7$.
43. Consider the curves $C_1: x^2 + y^2 = 2$, $C_2: 2x^2 + y^2 = 2$ for y positive. For each x , the vertical line through $(x, 0)$ intersects the curves C_1 , C_2 at the points (x, y_1) , (x, y_2) . Let $L(x)$ be the length of the line segment joining these two points. Find $L'(x)$.
44. Suppose that a point moves along the x -axis according to the formula $x(t) = 1/(t^2 + 1)$. Let $A(t)$ be the area of the circle with diameter joining the origin to the point $x(t)$. Find $A'(t)$ when $t = 3$.
45. a) Let $f(x) = x^2 + 3x - 1$. Find the slope of the line joining the points $(2, 9)$ and $(x, f(x))$.
 b) Find the slope of the tangent line to the curve $y = f(x)$ at the point $(2, 9)$.
 c) What is the equation of this tangent line?

Implicit differentiation

1. A curve is given by the equation $x^2 - xy + y^2 = 7$. Find the equation of the line tangent to this curve at the point $(2, -1)$.

- Find the slope of the curve defined by the relation $4(x^2 + xy) = 2y^3 - y^2$ at the point $(1, 2)$.
- Variables x and y are related by the formula $x \sin y + y \sin x = \pi$. If $dy/dt = 3$ when $x = 3\pi/2$ and $y = \pi/2$, what is dx/dt ?
- The relation $\cos y + x = \sin y$ determines a curve in the $x - y$ plane. Find the slope of the line tangent to the curve at the point $(1, \pi/2)$.
- Consider the curve given by the equation: $y^2 + xy + x^2 + 1$. At what points does this curve have a horizontal tangent line?
- Consider the curve given by the equation: $x^2y - y^3 = 1$. At what points does this curve have a vertical tangent line?

Minima-Maxima. Graph of a function.

- Let $y = (x^2 - 1)(x^2 - 5)$. For what value of x in the interval $[-2, 2]$ is y a maximum? a minimum? Find the points of inflection of the graph.

- Let

$$y = 10 + 7x + 2x^2 - x^3.$$

For what value of x is y a local maximum? a local minimum?

- For the following function, find a) all critical values, b) intervals in which the function is increasing and where it is decreasing, c) intervals in which the function is concave up or concave down:

$$y = x^4 - 4x^3 + 4x^2 + 2$$

Sketch the graph.

- Consider the function $y = \sin x + (\cos x)^2$ as defined on the interval $[-\pi, \pi]$. Find a) all critical values, b) all points of inflection, c) the value at which the function takes its maximum. Sketch the graph.

- Let

$$y = \frac{1}{x^2 - 1}.$$

Find the x coordinate of the points of inflection of the curve so defined. Sketch the graph.

- Graph

$$y = \frac{x(x-2)}{x^2-1} = \frac{x^2-2x}{x^2-1}$$

You must show enough work to explain how you found the various features of the graph.

7. In each of the following, derive enough information about the curve to sketch the graph, showing clearly all asymptotes.

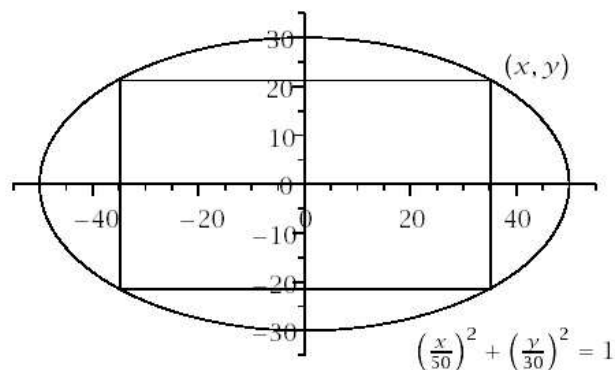
$$y = \frac{x}{x^2 + 1}$$

$$y = \frac{x}{(x - 1)(x - 2)}$$

$$y = \frac{x^2}{(x - 1)(x - 2)}$$

$$y = \sqrt{1 - \cos x}$$

8. I have to make a closed cylindrical can to hold 12 cu. ft. The material to make the top and bottom costs \$6 a sq. ft., and the material to make the side costs \$10 per sq. ft. What are the dimensions which minimize the cost? (It will suffice to give either the radius of the base or the height).
9. I want to build a rectangular building on a square base which has 60000 ft³ of space. My painter will charge \$ $h/2$ per ft² to paint a building h feet high, and will paint the roof at a cost of \$30 per ft². How high should the building be to minimize the cost of painting?
10. I have an elliptical meadow whose major axis is 100 meters and whose minor axis is 60 meters. I want to put a rectangular soccer field in the meadow, two of whose sides are parallel to the major axis, What are the dimensions of the field which maximize its area?



Integration

1. Find the indefinite integral of:

a) $f(x) = (x^2 + 1)^2 x$

b) $g(x) = (x^2 + 1)(x^3 + 3x)^3$

c) $h(t) = 4t(t^2 - 1)^{-3}$

2. Find the indefinite integral of:

a) $h(x) = \tan x \sec^2 x$

b) $g(x) = \sin^3 x$

c) $f(x) = \sin(2x)(\cos(2x))^2$

3. Find the indefinite integrals:

a) $\int \sqrt{x}(x + 1) dx;$

b)¹ $\int x\sqrt{x + 1} dx$

4. Find the function whose value at 0 is 0 and whose derivative is given:

a) $\frac{x}{(2x^2 + 1)^2}$

b) $\frac{\sin x}{\cos^4 x}$

b) $\frac{\sin^2 x}{\cos^4 x}$

5. Find the solution to the differential equation:

$$\frac{dy}{dx} = y^2 x^2 + y^2$$

such that $y(1) = 2$.

6. Given

$$\frac{dy}{dx} = x^2 \sqrt{y}, \quad y = 4 \text{ when } x = 1$$

find y as a function of x .

7. Find y as a function of x , given

$$\frac{dy}{dx} = \frac{\sin x}{y}, \quad y = 5 \text{ when } x = 0.$$

8. Find $f(x)$ given that $f(2) = 1$, $f'(1) = 1$ and $f''(x) = x - x^3$

¹Hint: use $u = x + 1$

9. Find y as a function of x , given that $y = 4$ when $x = 0$ and

$$\frac{dy}{dx} = x + \sin x$$

10. An automobile is travelling down the road a speed of 100 ft/sec. a) At what constant rate must the automobile decelerate in order to stop in 300 ft.? b) How long does that take?
11. A ball is thrown from ground level so as to just reach the top of a building 150 ft. high. At what initial velocity must the ball be thrown?
12. Calculate the definite integrals:

a) $\int_{-4}^4 (x^3 + 3x + \sin(2x)) dx$

b) $\int_0^{\pi/2} (\sin x \cos x) dx$

c) $\int_1^3 x(x+1)^2 dx$

d) $\int_0^{\pi} (\sin x + \cos x) dx$

13. Evaluate

a) $\frac{d}{dx} \int_{-3}^{2x+1} (\cos t)^{2t} dt$

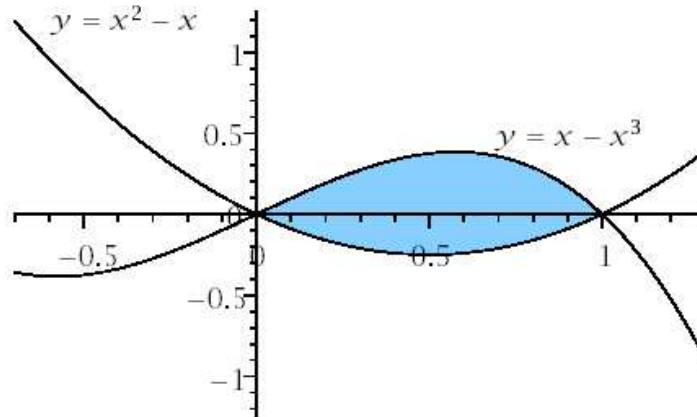
b) $\frac{d}{dt} \int_1^{2t^3+1} x^3 dx$

Area, Volume, Arc Length, Surface, . . .

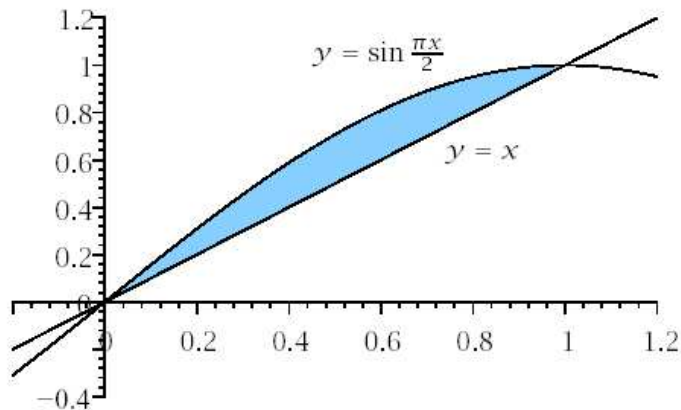
1. Find the area of the region bounded by the curves $y_1 = x^3 - x^2 + x$ and $y_2 = x^3 + 2x^2 - 10$. (Ans.: $1331/54 \simeq 24.65$)

²Be careful with all the negative signs!

2. Find the area of the region bounded by the curves $y_1 = x^3$ and $y_2 = x^2 + 2x$. (Ans.: $37/12$)
3. Find the area of the region in the right half plane ($x > 0$) bounded by the curves $y_1 = x - x^3$ and $y_2 = x^2 - x$. (Ans.: $5/12$)

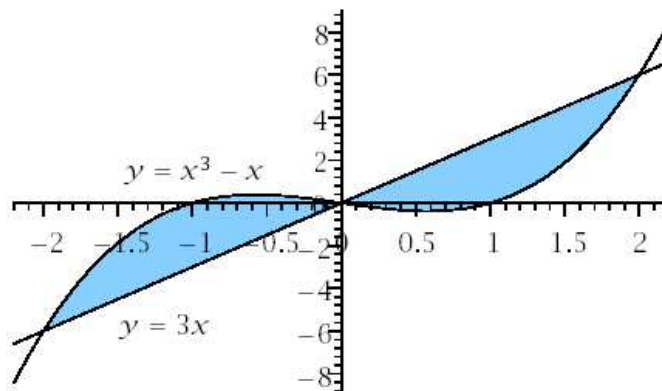


4. Find the area of the region in the first quadrant bounded by the curves $y_1 = \sin(\frac{\pi x}{2})$ and $y_2 = x$. (Ans. $2/\pi - 1/2$)



5. Find the area of the region under the curve $y = x\sqrt{x^2 + 1}$, above the x -axis and bounded by the lines $x = 1$ and $x = 2$. (Ans.: $[10\sqrt{10} - 2\sqrt{2}]/3$)

6. Find the area under the curve $y = x^2 + x^{-2}$, above the x -axis and between the lines $x = 1$ and $x = 2$. (Ans.: $17/5$)
7. What is the area of the region bounded by the curves $y = x^3 - x$ and $y = 3x$. (Ans. 8)



8. A solid is formed over the region in the first quadrant bounded by the curve $y = \sqrt{10 - x}$ so that the section by any plane perpendicular to the x -axis is a semicircle. What is the volume of this solid? (Ans.: $25\pi/4$)
9. A solid is formed over the region in the first quadrant bounded by the curve $y = \sqrt{4 - x}$ so that the section by any plane perpendicular to the x -axis is a square. What is the volume of this solid? (Ans.: 8)
10. A solid is formed over the region in the first quadrant bounded by the curve $y = 2x - x^2$ so that the section by any plane perpendicular to the x -axis is a semicircle. What is the volume of this solid? (Ans.: $2\pi/15$)
11. The region in the first quadrant bounded by $y = \sqrt{x^2 - 1}$; $y = 0$; $x = 1$; $x = 4$ is revolved around the x -axis. Find the volume of the resulting solid. (Ans.: 18π)
12. Find the volume of the solid obtained by rotating about the y -axis the region bounded by $y = x^2$; $x = 2$ and the x -axis. (Ans.: 8π)
13. The region in the first quadrant under the curve $y = 2x - x^2$ is rotated about the y -axis. Find the volume of the resulting solid. (Ans.: $8\pi/3$)
14. The region in the first quadrant bounded by $y = x^4$ and $x = 1$ is revolved around the y -axis. Find the volume of the resulting solid. (Ans.: $\pi/3$)
15. The region in the first quadrant bounded by $y = x - x^2$ and $y = x - x^3$ is revolved around the x -axis. Find the volume of the resulting solid. (Ans.: $3\pi/70$)

16. A pencil sharpener is made by drilling a cone out of a sphere; the cone has as its axis a diameter of the sphere, and its vertex is on the surface of the sphere. If the ratio of the height to base radius in the cone is 4 to 1, and the sphere has a 1 inch radius, what is the volume of the pencil sharpener? (Ans.: $1024\pi/867 \simeq 3.71$)
17. An ellipsoid is formed by rotating the curve $4x^2 + y^2 = 1$ around the x -axis. What is its volume? What is the volume of the ellipsoid obtained by rotating this curve about the y -axis? (Ans. $2\pi/3, \pi/3$)
18. Consider the region in the first quadrant bounded by $y = \sin x, x = 0, x = \pi$. Find the volume of the solid obtained by rotating this region about the x -axis. Hint: $\cos 2x = 1 - 2\sin^2 x$. (Ans.: $\pi^2/2$)
19. Consider the region in the first quadrant bounded by $y = \sin x, x = 0, x = \pi$. Find the volume of the solid obtained by rotating this region about the y -axis. Hint: The derivative of $\sin x - x \cos x$ is $x \sin x$. (Ans.: $2\pi^2$)
20. Find the length of the curve $y = t^3, x = t^2, 0 \leq t \leq 1$. (Ans.: $(13\sqrt{13} - 8)/27$)
21. Find the integral giving the length of the curve $y = \sqrt{1 + x^2}$ for $0 \leq x \leq 1$.
(Ans.: $\int_0^1 \sqrt{\frac{1 + 2x^2}{1 + x^2}} dx$)
22. Find the area of the surface obtained by rotating the line segment $y = 3x - 3, 3 \leq x \leq 5$ about the x -axis. (Ans.: $36\sqrt{10}\pi$)
23. Find the area of the bowl obtained by rotating the parabola $y = x^2, 0 \leq x \leq a$ about the y -axis. (Ans.: $\pi[(1 + 4a)^{3/2} - 1]/6$)
24. A 2 lb. weight will extend a certain spring 5 inches. How much work is done in extending the spring 14 inches? (Ans.: 39.2)
25. A 10 kg mass extends a spring 45 cm, to a new equilibrium position. The spring is then extended another meter and released. With what velocity does it pass the equilibrium position? (Ans.: $k = 2180\text{N/m}, v = 14.7\text{m/sec.}$)
26. Find the center of mass of the homogeneous region in the first quadrant bounded by the curve $x^4 + y = 1$. (Ans.: $(5/12, 4/9)$)
27. Find the center of mass of the region bounded by the curves $y = x - x^3$ and $y = x - x^2$. (Ans.: $(3/5, 9/35)$)