Practice Exam
Math 2250-4

1. See the problems from the previous practice exams.

2. Find the general solutions of the two equations below.

$$\text{(a)} \quad y'' + y' + y = 0$$

$$\text{(b)} \quad y^{iv} + 4y'' = 0$$

   Answer:
   a) $y(x) = e^{-x/2}[C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x]$,
   b) $y(x) = C_1 \cos 2x + C_2 \sin 2x + C_3 x + C_4$

3. Given $4x''(t) + 4x(t) + x(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 4$, $k = 1$, solve the differential equation and classify the answer as over-damped, critically damped or under-damped.
   Answer: $x(t) = e^{-t/2}(C_1 + C_2 t)$, critically damped.

4. Determine (from the table on page 341 of the textbook) the final form of a trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

$$y^{iv} - 9y'' = xe^{3x} + x^3 + e^{-3x}$$

   Answer: $y_p = C_1 xe^{-3x} + e^{3x}(C_2 x + C_3 x^2) + C_4 x^5 + C_5 x^4 + C_6 x^3 + C_7 x^2$

5. Find the steady-state periodic solution for the equation

$$x'' + 2x' + 6x = 5cos(3t).$$

   Answer: $x_p(t) = -\frac{1}{3}\cos(3t) + \frac{2}{3}\sin(3t)$

6. Find the eigenvalues of the matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

   Answer: $\lambda = 1, 4$

7. Given a 3×3 matrix $\mathbf{A}$ has eigenpairs (eigenvalue and eigenvector)

$$3, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}; \quad 1, \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix}; \quad 0, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix},$$

find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{AP} = \mathbf{PD}$. ***Answer***:

$$\mathbf{P} = \begin{bmatrix} a & 0 & 0 \\ 0 & 2b & c \\ 2a & -5b & -3c \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where a, b, c are arbitrary constants.

8. Give an example of a 3×3 matrix $\mathbf{C}$ which has exactly one eigenpair

$$2, \ [1, 0, 0]^T$$

Answer:

$$\mathbf{C} = \begin{bmatrix} 2 & a & b \\ 0 & 2 & c \\ 0 & 0 & 2 \end{bmatrix}$$

where a, c are arbitrary nonzero constants and b can be zero.

9. Solve for $x(t)$, $y(t)$ in the system below. The answers depend upon two arbitrary constants, because $x(0)$ and $y(0)$ are not supplied.

$$x' = x - y,$$

$$y' = 10x + y.$$

Answer:

$$\begin{cases} x(t) = e^t \left[C_1 \cos(\sqrt{10}t) - \dfrac{C_2}{\sqrt{10}} \sin(\sqrt{10}t) \right] \\ y(t) = e^t \left[C_2 \cos(\sqrt{10}t) + C_1 \sqrt{10} \sin(\sqrt{10}t) \right] \end{cases}$$

10. Let the real 2×2 matrix $\mathbf{A}$ have a complex eigenpair

$$7i, \ \begin{bmatrix} 1+i \\ -1 \end{bmatrix}.$$

Find all real solutions $x(t)$ of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
Answer: see Chapter 7.3 (pp. 421-422)

$$\begin{cases} x(t) = (C_2 - C_1)\cos(7t) + (C_1 + C_2)\sin(7t) \\ y(t) = C_1 \cos(7t) - C_2 \sin(7t) \end{cases}$$

11. Given $x'' + 10x' + 650x = 100\cos(\omega t)$, find: (a) The steady-state solution $x = A\cos(\omega t) + B\sin(\omega t)$. (b) The practical resonant frequency ω_0.
Answer:

$$A = \frac{100(650 - \omega^2)}{(650 - \omega^2)^2 + 100\omega^2}, \qquad B = \frac{1000\omega}{(650 - \omega^2)^2 + 100\omega^2}, \quad \omega_0 = 10\sqrt{6}$$

12. Solve for a particular solution $y_p(x)$:

$$y''' - y' = 2e^{1+\pi} + e^{x-\pi}$$

Answer:

$$y_p(x) = -2e^{1+\pi}x + \frac{1}{2}xe^{x-\pi}$$

13. Show the steps in the solution of the differential equation to obtain the general solution y.

$$y'' - 4y = 1 - xe^{-2x},$$

Answer: $y(x) = -\frac{1}{4} + \frac{e^{-2x}}{64}(4x + 8x^2) + C_1e^{2x} + +C_2e^{-2x}$

$$y'' - 4y' = 1 - xe^{4x},$$

Answer: $y(x) = -\frac{1}{4} + \frac{1}{36}e^{4x}\,(2 - 3x) + C_1e^{2x} + C_2e^{-2x}$

$$y'' - 16y = x - xe^{-4x},$$

Answer: $y(x) = -\frac{x}{16} + \frac{e^{-4x}}{64}(x + 4x^2) + C_1e^{4x} + C_2e^{-4x}$

$$y'' + 4y = x - xe^{-4x},$$

Answer: $y(x) = \frac{x}{4} - \frac{e^{-4x}}{100}(2 + 5x) + C_1\cos(2x) + C_2\sin(2x)$

$$y''' - y'' = x^3 + e^x - \cos(3x),$$
$$y''' - y'' = 1 + x^3 + xe^x - \sin x,$$
$$y''' - 4y'' = x + x^3 + e^{4x} - \cos(2x),$$
$$y''' + 4y'' = x^3 + x^2 + xe^{4x|} - \sin x.$$

14. Problems from Chapter 7.4: 1-16.