Exotic t-structures for two-block Springer fibres

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Outline

Two-block Springer fibres: the definitions

Fix $m \in \mathbb{Z}_{\geq 0}$, and let $n \in \mathbb{Z}_{\geq 0}$. Recall the following definitions:

Definition

Let $G = SL_{m+2n}(\mathbb{C}), \mathfrak{g} = \mathfrak{sl}_{m+2n}(\mathbb{C})$, $B \subset G$ be the Borel subgroup of upper triangular matrices; and G/B the flag variety:

$$\mathcal{B}_n = G/B = \{0 \subset V_1 \subset \cdots \subset V_{m+2n-1} \subset V_{m+2n} = \mathbb{C}^{m+2n} \mid \dim V_i = i\}$$

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Definition

Let $\mathcal{N}_n = \{x \in \mathfrak{g} \mid x \text{ nilpotent}\}$ be the nilpotent cone, $\widetilde{\mathcal{N}_n}$ it's Springer resolution (with the natural map $\pi_n : \widetilde{\mathcal{N}_n} \to \mathcal{N}_n$):

$$\widetilde{\mathcal{N}_n} := T^*\mathcal{B}_n = \{(0 \subset V_1 \subset \cdots \subset V_{m+2n-1} \subset V_{m+2n}), x \mid x(V_i) \subseteq V_{i-1}\}$$

Let $z_n \in \mathcal{N}_n$ be the standard nilpotent with Jordan type $(m+n, n)$, and let $\mathcal{B}_{m+n,n} = \pi_n^{-1}(z_n)$ be the corresponding Springer fiber.

Two-block Springer fibres: transverse slices

Definition

Define the Mirkovic-Vybornov transverse slice as follows:

$$S_n = \{z_n + \sum_{1 \leq i \leq m+n} a_i e_{m+n,i} + \sum_{1 \leq j \leq n} b_j e_{m+n,m+n+j} + \sum_{1 \leq j \leq n} c_j e_{m+2n,j} + \sum_{1 \leq j \leq m+2n} d_j e_{m+2n,m+n+j}\}$$

Let $U_n = \pi_n^{-1}(S_n \cap \mathcal{N}_n)$ denote the resolution of the variety $S_n \cap \mathcal{N}_n$.

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Note that:

$$\mathcal{B}_{m+n,n} = \{(0 \subset V_1 \subset \cdots \subset V_{m+2n-1} \subset V_{m+2n}) \mid z_n V_i \subseteq V_{i-1}\}$$

$$U_n = \{(0 \subset V_1 \subset \cdots \subset V_{m+2n-1} \subset V_{m+2n}), x \mid x \in S_n, x V_i \subseteq V_{i-1}\}$$

$\mathcal{D}_n := D^b(\mathrm{Coh}_{\mathcal{B}_{m+n,n}}(U_n))$, the bounded derived category of coherent sheaves on U_n supported on $\mathcal{B}_{m+n,n}$, will be our primary object of interest.

Two-block Springer fibres: some identities

Define the partial flag variety $\mathcal{P}_{k,n}$ (and $T^*\mathcal{P}_{k,n}$) as follows:

$$\mathcal{P}_{k,n} = \{(0 \subset V_1 \subset \cdots \subset \widehat{V_k} \subset \cdots \subset V_{m+2n} = \mathbb{C}^{m+2n})\}$$

$$T^*\mathcal{P}_{k,n} =\{(0 \subset V_1 \subset \cdots \subset \widehat{V_k} \subset \cdots \subset V_{m+2n} = \mathbb{C}^{m+2n}), x \mid x \in \mathfrak{gl}_{m+2n}, xV_{k+1} \subset V_{k-1}, xV_i \subset V_{i-1} \text{ for } i \neq k, k+1\}$$

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Proposition

We have: $S_{n+1} \times_{\mathfrak{gl}_{m+2n+2}} T^*\mathcal{P}_{k,n+1} \simeq U_n$.

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Proof (Sketch): Below, $xV_{k+1} = V_{k-1}$; $\exists$ a canonical isomorphism ϕ_x : $xV_{m+2n+2} \simeq V_{m+2n}$ inducing $\phi(x) \in \mathrm{End}(\mathbb{C}^{m+2n})$.

$$S_{n+1} \times_{\mathfrak{gl}_{m+2n+2}} T^*\mathcal{P}_{k,n+1} = \{(0 \subset V_1 \subset \cdots \subset \widehat{V_k} \subset \cdots \subset V_{m+2n+2}) \mid x \in S_{n+1}, xV_{k+1} \subseteq V_{k-1}, xV_i \subset V_{i-1}\}$$

Map this to $\{(0 \subset V_1 \subset \cdots V_{k-1} \subset xV_{k+2} \subset \cdots xV_{m+2n+2}), \phi(x)\} \in U_n$.

Two-block Springer fibres: some identities

Let $V_{m,n} = \mathbb{C}\{e_i, f_i\}_{1 \leq i \leq m+2n}$; define z by $ze_i = e_{i-1}, zf_i = f_{i-1}$. Let $W_{m,n} = \mathbb{C}\{e_k, f_l\}_{1 \leq k \leq m+n, 1 \leq l \leq n}$, and $P : V_{m,n} \to W_{m,n}$ be the natural projection.

$$Y_{m+2n} = \{(L_1 \subset \cdots \subset L_{m+2n} \subset V_{m,n})| \operatorname{dim} L_i = i, zL_i \subset L_{i-1}\}$$
$$\widetilde{U}_{m+2n} = \{(L_1 \subset \cdots \subset L_{m+2n}) \in Y_{m+2n} | P(L_{m+2n}) = W_{m,n}\}$$

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The categories $\widetilde{\mathcal{D}_n} := D^b(\mathrm{Coh}(Y_{m+2n}))$ have been studied by Cautis and Kamnitzer. The below fact allows us to apply their results to study $\mathcal{D}_n$.

Proposition

There is a closed embedding $U_n \to \widetilde{U}_{m+2n}$ (hence U_n is locally closed in Y_{m+2n}).

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Proof (Sketch): Let $S'_n = \{z_n + \sum_{1 \leq i \leq m+2n} (u_i e_{m+n,i} + v_i e_{m+2n,i})\}$. It suffices to show that $\widetilde{U}_{m+2n} \simeq S'_n \times_{\mathfrak{gl}_{m+2n}} T^*\mathcal{B}_n$, since $U_n = S_n \times_{\mathfrak{gl}_{m+2n}} T^*\mathcal{B}_n$ is closed in $S'_n \times_{\mathfrak{gl}_{m+2n}} T^*\mathcal{B}_n$.

Affine tangles: definitions

Definition

If $p \equiv q \pmod 2$, a (p, q) affine tangle is an embedding of $\frac{p+q}{2}$ arcs and a finite number of circles into the region $\{(x, y) \in \mathbb{C} \times \mathbb{R} | 1 \leq |x| \leq 2\}$, such that the end-points of the arcs are given $\{(1, 0), (\zeta_p, 0), \cdots, (\zeta_p^{p-1}, 0), (2, 0), (2\zeta_q, 0), \cdots, (2\zeta_q^{q-1}, 0)\}$; where $\zeta_k = e^{\frac{2\pi i}{k}}$.

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Definition

- Let g_n^i denote the $(n-2, n)$ tangle with an arc connecting $(2\zeta_n^i, 0)$ to $(2\zeta_n^{i+1}, 0)$. Let f_n^i denote the $(n, n-2)$ tangle with an arc connecting $(\zeta_n^i, 0)$ and $(\zeta_n^{i+1}, 0)$.
- Let $t_n^i(1)$ denote the (n, n) tangle with a strand connecting $(\zeta_n^i, 0)$ to $(2\zeta_n^{i+1}, 0)$ passes beneath a strand connecting $(\zeta_n^{i+1}, 0)$ to $(2\zeta_n^i, 0)$.
- Let r_n denote the (n, n) tangle connecting $(\zeta_n^j, 0)$ to $(2\zeta_n^{j+1}, 0)$ for each $1 \leq j \leq n$. Also let $r_n' := r_n^{-1}, t_n^i(2) := t_n^i(1)^{-1}$.

Affine tangles: generators and relations

Proposition

Any affine tangle is a composition of the above tangles, and any relation between those generators is a composition of the following relations:

- $f_n^i \circ g_n^{i+1} = f_n^{i+1} \circ g_n^i = \mathrm{id}$
- $t_n^i(1) \circ t_n^{i+1}(1) \circ t_n^i(1) = t_n^{i+1}(1) \circ t_n^i(1) \circ t_n^{i+1}(1)$
- $f_n^{i+k-2} \circ f_{n+2}^i = f_n^i \circ f_{n+2}^{i+k}$
- $f_n^i \circ t_n^{i+k}(q) = t_{n-2}^{i+k-2}(q) \circ f_n^i, f_n^{i+k} \circ t_n^i(q) = t_{n-2}^i(q) \circ f_n^{i+k}$
- $t_n^i(1) \circ g_n^{i+1} = t_n^{i+1}(2) \circ g_n^i, t_n^i(2) \circ g_n^{i+1} = t_n^{i+1}(1) \circ g_n^i$
- $r'_{n-2} \circ f_n^i \circ r_n = f_n^{i+1}, f_n^{n-1} \circ r_n^2 = f_n^1, r'_n \circ t_n^i(q) \circ r_n = t_n^{i+1}(q)$
- $f_n^i \circ t_n^{i\pm1}(2) \circ g_n^i = f_n^i \circ t_n'^{i\pm1}(1) \circ g_n^i = \mathrm{id}$
- $g_{n+2}^{i+k} \circ g_n^i = g_{n+2}^i \circ g_n^{i+k-2}$
- $g_n^{i+k-2} \circ f_n^i = f_{n+2}^i \circ g_{n+2}^{i+k}, g_n^i \circ f_n^{i+k-2} = f_{n+2}^{i+k} \circ g_{n+2}^i$
- $g_n^i \circ t_{n-2}^{i+k-2}(q) = t_n^{i+k}(q) \circ g_n^i, g_n^{i+k} \circ t_{n-2}^i(q) = t_n^i(q) \circ g_n^{i+k}$
- $t_n^i(p) \circ t_n^{i+k}(q) = t_n^{i+k}(q) \circ t_n^i(p)$
- $r'_n \circ g_n^i \circ r_{n-2} = g_n^{i+1}, r_n'^2 \circ g_n^{n-1} = g_n^1$

Affine tangles: generators and relations

It will be more convenient to use a slightly different set of relations.

Definition

Let s_n^i denote the (n, n)-tangle with a strand connecting $(\zeta_j, 0)$ to $(2\zeta_j, 0)$ for each j, and a strand connecting $(\zeta_i, 0)$ to $(2\zeta_i, 0)$ passing clockwise around the circle, beneath all the other strands.

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Proposition

The following relations are equivalent to the two relations mentioned involving the generators r_n.

- $s_n^n \circ g_n^i = g_n^i \circ s_{n-2}^{n-2}, s_{n-2}^{n-2} \circ f_n^i = f_n^i \circ s_n^n, s_n^n \circ t_n^i(p) = t_n^i(p) \circ s_n^n$
- $f_n^{n-1} \circ s_n^n \circ t_n^{n-1}(2) \circ s_n^n \circ t_n^{n-1}(2) = f_n^{n-1}$
- $s_n^n \circ t_n^{n-1}(2) \circ s_n^n \circ t_n^{n-1}(2) \circ g_n^{n-1} = g_n^{n-1}$

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- $f_n^{n-1} \circ s_n^n \circ t_n^{n-1}(2) \circ s_n^n \circ t_n^{n-1}(2) = f_n^{n-1}$
- $s_n^n \circ t_n^{n-1}(2) \circ s_n^n \circ t_n^{n-1}(2) \circ g_n^{n-1} = g_n^{n-1}$

Proof (Sketch): Use the identity $r_n = s_n^n \circ t_n^{n-1}(2) \circ \cdots \circ t_n^1(2)$ to interchange the generators r_n and s_n^n.

Affine tangles: functors

We closely follow the method used by Anno in the $m = 0$ case below.

Goal:

For each $(m+2p, m+2q)$ affine tangle α, we would like to construct a functor $\Psi(\alpha) : \mathcal{D}_p \to \mathcal{D}_q$ that is compatible with composition (i.e. given an $(m+2q, m+2r)$ tangle β, $\Psi(\beta \circ \alpha) = \Psi(\beta) \circ \Psi(\alpha)$.)

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Proposition (Cautis-Kamnitzer)

Given a linear $(m + 2p, m + 2q)$ tangle α, one can construct a functor $\widetilde{\Psi}(\alpha) : \widetilde{D}_p \to \widetilde{D}_q$ that is compatible with composition.

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Lemma

Given a linear $(m + 2p, m + 2q)$ tangle α, one can construct a functor $\Psi(\alpha) : \widetilde{D}_p \to \widetilde{D}_q$ by "restricting" the functors $\widetilde{\Psi}(\alpha)$, that is compatible with composition.

Affine tangles: functors

Now we describe the functors $\Psi(f_n^i), \Psi(g_n^i)$ and $\Psi(t_n^i(p))$ mentioned above; since $f_n^i, g_n^i, t_n^i(p)$ generate all linear tangles, this is sufficient to describe $\Psi(\alpha)$ for arbitrary α. Below all functors are derived.

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$$X_{n,k} := S_n \times_{\mathfrak{gl}_{m+2n}} T^*\mathcal{P}_{k,n} \times_{\mathcal{P}_{k,n}} \mathcal{B}_n =$$
$$\{(0 \subset V_1 \subset \cdots \subset V_{m+2n}), x \mid x \in S_n, xV_{k+1} \subset V_{k-1}, xV_i \subset V_{i-1} \forall\ i\}$$

Consider the $\mathbb{P}^1$-bundle $\pi_{n,k} : X_{n,k} \to S_n \times_{\mathfrak{gl}_{m+2n}} T^*\mathcal{P}_{k,n} \simeq U_{n-1}$, and the divisor $i_{n,k} : X_{n,k} \to U_n$. Let $\mathcal{V}_k$ be the vector bundle on U_n corresponding to V_k; define $\mathcal{E}_k = \mathcal{V}_k/\mathcal{V}_{k-1}$. Let $G^k_{m+2n}(\mathcal{F}) = i_{n,k*}(\pi^*_{n,k}\mathcal{F} \otimes \mathcal{E}_k)$ for $\mathcal{F} \in \mathcal{D}_{n-1}$, and $F^k_{m+2n}(\mathcal{G}) = \pi_{n,k*}(i^*_{n,k}\mathcal{G} \otimes \mathcal{E}^{-1}_{k+1})$ for $\mathcal{G} \in \mathcal{D}_n$. Then $\Psi(f_n^i) = F_n^i, \Psi(g_n^i) = G_n^i$.

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Lemma

G^i_{m+2n} has right adjoint $F_{m+2n}[-1]$, and left adjoint $F^i_{m+2n}[1]$.

Affine tangles: functors

For $\mathcal{G} \in \mathcal{D}_n$, define $T_n^i(1)\mathcal{G}$ to be the cone of $G_{m+2n}^i F_{m+2n}^i[-1]\mathcal{G} \to \mathcal{G}$, and $T_n^i(2)\mathcal{G}$ to be the cone of $\mathcal{G} \to G_{m+2n}^i F_{m+2n}^i[1]\mathcal{G}$. Let $\Psi(t_n^i(1)) = T_n^i(1)$ and $\Psi(t_n^i(2)) = T_n^i(2)$. Now let $\mathcal{S}_n(\mathcal{F}) = \mathcal{F} \otimes \mathcal{E}_{m+2n}^{-1}$, and let $\Psi(s_{m+2n}^{m+2n}) := \mathcal{S}_n$. To check that this assignment satisfies the compatibility under compositions, we check the relations involving the generator s_{m+2n}^{m+2n} below.

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Proposition

The following identities hold, where $1 \leq i \leq m+2n-2, 1 \leq p \leq 2$:

- $\mathcal{S}_{n-1} \circ F_{m+2n}^i \simeq F_{m+2n}^i \circ \mathcal{S}_n$, $\mathcal{S}_n \circ G_{m+2n}^i \simeq G_{m+2n}^i \circ \mathcal{S}_{n-1}$
- $\mathcal{S}_n \circ T_{m+2n}^i(p) \simeq T_{m+2n}^i(p) \circ \mathcal{S}_n$
- $F_{m+2n}^{m+2n-1} \circ \mathcal{S}_n \circ T_{m+2n}^{m+2n-1}(2) \circ \mathcal{S}_n \circ T_{m+2n}^{m+2n-1}(2) \simeq F_{m+2n}^{m+2n-1}$
- $\mathcal{S}_n \circ T_{m+2n}^{m+2n-1}(2) \circ \mathcal{S}_n \circ T_{m+2n}^{m+2n-1}(2) \circ G_{m+2n}^{m+2n-1} \simeq G_{m+2n}^{m+2n-1}$

Affine tangles: functors

For $\mathcal{G} \in \mathcal{D}_n$, define $T_n^i(1)\mathcal{G}$ to be the cone of $G_{m+2n}^i F_{m+2n}^i[-1]\mathcal{G} \to \mathcal{G}$, and $T_n^i(2)\mathcal{G}$ to be the cone of $\mathcal{G} \to G_{m+2n}^i F_{m+2n}^i[1]\mathcal{G}$. Let $\Psi(t_n^i(1)) = T_n^i(1)$ and $\Psi(t_n^i(2)) = T_n^i(2)$. Now let $\mathcal{S}_n(\mathcal{F}) = \mathcal{F} \otimes \mathcal{E}_{m+2n}^{-1}$, and let $\Psi(s_{m+2n}^{m+2n}) := \mathcal{S}_n$. To check that this assignment satisfies the compatibility under compositions, we check the relations involving the generator s_{m+2n}^{m+2n} below.

Proposition

The following identities hold, where $1 \leq i \leq m+2n-2, 1 \leq p \leq 2$:

- $\mathcal{S}_{n-1} \circ F_{m+2n}^i \simeq F_{m+2n}^i \circ \mathcal{S}_n$, $\mathcal{S}_n \circ G_{m+2n}^i \simeq G_{m+2n}^i \circ \mathcal{S}_{n-1}$
- $\mathcal{S}_n \circ T_{m+2n}^i(p) \simeq T_{m+2n}^i(p) \circ \mathcal{S}_n$
- $F_{m+2n}^{m+2n-1} \circ \mathcal{S}_n \circ T_{m+2n}^{m+2n-1}(2) \circ \mathcal{S}_n \circ T_{m+2n}^{m+2n-1}(2) \simeq F_{m+2n}^{m+2n-1}$
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This gives us functors $\Psi(\alpha) : \mathcal{D}_p \to \mathcal{D}_q$ for each $(m+2p, m+2q)$ tangle α, compatible under composition.

Exotic t-structure: Braid group action and definition

Let $\mathbb{B}_{aff}$ denotes the braid group attached to the affine Weyl group $W_{aff} = W \ltimes \Lambda$, where W is the Weyl group for G, Λ is the weight lattice and Q the root lattice. Let $\mathbb{B}^{+}_{aff}$ be the semigroup generated by the lifts of the simple reflections $\widetilde{s}_{\alpha}$ in the Coxeter group $W \ltimes Q \subset W_{aff}$.

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Lemma

$\mathbb{B}_{aff}$ can be identified with the group of all bijective $(m+2n, m+2n)$ affine tangles (i.e. where each strand must connect a point in the inner circle with a point in the outer circle), quotiented out by the generator r_n.

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Thus for each $b \in \mathbb{B}_{aff}$, we have a functor $\Psi(b) : \mathcal{D}_n \to \mathcal{D}_n$.

Definition

The exotic t-structure on $\mathcal{D}_n$ is given by:

$$\mathcal{D}_n^{\geq 0} = \{\mathcal{F} \mid R\Gamma(\Psi(b^{-1})\mathcal{F}) \in D^{\geq 0}(\mathrm{Vect}) \ \forall \ b \in \mathbb{B}^+_{aff}\}$$
$$\mathcal{D}_n^{\leq 0} = \{\mathcal{F} \mid R\Gamma(\Psi(b)\mathcal{F}) \in D^{\leq 0}(\mathrm{Vect}) \ \forall \ b \in \mathbb{B}^+_{aff}\}$$

Exotic t-structure: Exactness of G^i_{m+2n}

Proposition

The functor $G^i_{m+2n} : \mathcal{D}_{n-1} \to \mathcal{D}_n$ is exact with respect to the exotic t-structures on the two categories (i.e. it maps the heart to the heart).

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Proof (Sketch): To show that G^i_{m+2n} is right t-exact (i.e. it maps $\mathcal{D}^{\leq 0}_{n-1}$ to $\mathcal{D}^{\leq 0}_n$) is equivalent to showing $(G^i_{m+2n})^L = F^i_{m+2n}[-1]$ is left t-exact. We must show that for each $b' \in \mathbb{B}'^{\,+}_{aff}, \mathcal{G} \in \mathcal{D}^{\geq 0}_n$, $R\Gamma(\Psi(b^{-1})F^i_{m+2n}[-1]\mathcal{G}) \in \mathcal{D}^{\geq 0}(Vect)$. One can construct $\eta_i(b) \in \mathbb{B}^+_{aff}$ such that $b^{-1} \circ f^i_{m+2n} = f^i_{m+2n} \circ \eta_i(b)^{-1}$; now we need to show that $R\Gamma(F^i_{m+2n}[-1]\Psi(\eta_i(b)^{-1})\mathcal{G}) \in \mathcal{D}^{\geq 0}(Vect)$. Since $\Psi(\eta_i(b)^{-1})\mathcal{G} \in \mathcal{D}^{\geq 0}_n$, it remains to show that we have $R\Gamma(F^i_{m+2n}\mathcal{G}[-1]) \in \mathcal{D}^{\geq 0}(Vect)$.

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Proposition

The functor $G^i_{m+2n} : \mathcal{D}_{n-1} \to \mathcal{D}_n$ maps irreducible objects in the heart of the exotic t-structure on $\mathcal{D}_{n-1}$ to irreducible objects in the heart of the exotic t-structure on $\mathcal{D}_n$.

Exotic t-structure: Irreducible objects in the heart

Let a $(m, m+2n)$ crossingless matching to be affine $(m, m+2n)$-tangle whose vertical projection has no crossings, and where the m inner points are not labelled. Given a crossingless $(m, m+2n)$ matching α, we obtain a functor $\Psi(\alpha) : \mathcal{D}_0 \to \mathcal{D}_n$. Let $\Psi_\alpha = \Psi(\alpha)\underline{\mathbb{C}}$, where $\underline{\mathbb{C}} \in \mathcal{D}_0 \simeq D^b(\text{Vect})$. Any crossingless matching is a product $g_{m+2n}^{i_n} \circ \cdots \circ g_{m+4}^{i_2} \circ g_{m+2}^{i_1}$; thus since G_n^i maps irreducibles to irreducibles, Ψ_α is an irreducible object in the heart of the exotic t-structure on $\mathcal{D}_n$. The following result generalizes work of Anno, in the $m = 0$ case.

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The irreducible objects in the heart of the exotic t-structure on $\mathcal{D}_n$ are precisely given by Ψ_α, as α ranges across $(m, m+2n)$ crossingless matchings.

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Theorem

The irreducible objects in the heart of the exotic t-structure on $\mathcal{D}_n$ are precisely given by Ψ_α, as α ranges across $(m, m+2n)$ crossingless matchings.

Proof (Sketch): One shows that there are $\binom{m+2n}{n}$ crossingless matchings, and $\binom{m+2n}{n}$ irreducible objects in the heart of the exotic t-structure, D^0_n (using the fact that $K^0(\mathcal{D}^0_n) = K^0(\mathcal{B}_{m+n,n})$).