CRYSTALS FROM PREPROJECTIVE ALGEBRAS OF MODULATED GRAPHS

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ABSTRACT. Kashiwara and Saito give a geometric construction of the crystal $B(-\infty)$ for a symmetric Kac-Moody algebra by using irreducible components of Lusztig's quiver varieties, which are varieties of nilpotent representations of a pre-projective algebra. Here we generalize their construction to symmetrizable Kac-Moody algebras by replacing Lusztig's preprojective algebra with a more general one due to Dlab and Ringel. This requires non-trivial field extensions (in non-symmetric cases), so we must work over fields which are not algebraically closed. Our work also gives a non-trivial deformation of Kashiwara and Saito's realization even in symmetric type and working over $\mathbb{C}$.

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1. Introduction

1.1. **Overview.** Fix a symmetrizable Kac-Moody algebra $\mathfrak{g}$. Kashiwara's crystal $B(-\infty)$ is a combinatorial object (a discrete set along with some partial-permutations $\tilde{f}_i$) that encodes a lot of information about $\mathfrak{g}$ and its integrable representations. It is usually defined using the corresponding quantized universal enveloping algebra, but it can also be realized by other methods. In symmetric type, Kashiwara and Saito [KS97] developed a very useful geometric realization, where the underlying set consists of the collection of irreducible components of some algebraic varieties.

The varieties used in Kashiwara and Saito's construction are Lustzig's nilpotent varieties from [Lus91, §12]. They are the varieties of nilpotent representations of Lusztig's preprojective algebra Λ acting on a fixed (graded) vector space. Lustzig's preprojective algebra is only defined in symmetric cases, which is why Kashiwara and Saito's construction is restricted to that generality.

However, even before Lusztig's work, Dlab and Ringel [DR80] define the preprojective algebra of a "modulated graph". There is a natural way to associate a symmetrizable (but not necessarily symmetric) Cartan datum to any modulated graph, and all symmetriable Cartan matrices arise this way. If that Cartan datum happens to be symmetric then, by making the appropriate choices in Dlab and Ringel's construction, one recovers Lusztig's preprojective algebra (see [Rin98]).

Our main result is to generalize Kashiwara and Saito's realization of $B(-\infty)$ by replacing Lusztig's preprojective algebra with Ringel and Dlab's version. This gives a realization of $B(-\infty)$ for any symmetrizable Kac-Moody algebra.

1.2. **Modulated graphs and preprojective algebras.** Modulated graphs (also sometimes called species) date back to work of Gabriel [Gab73]. The preprojective algebra construction here is due to Dlab and Ringel [DR80].

Fix an undirected graph Γ, and denote the set of vertices by I and the set of edges by E. Let A be the set of directed edges, which we will call arrows; so there are two arrows in A for each edge in E. We do not allow edges connecting a vertex to itself, or multiple edges. Denote the arrow from i to j by ${}_ja_i$; since we do not allow multiple edges this does not cause confusion.

A modulated graph M is a graph Γ along with a choice of a field $\mathbb{F}$ and:

- A choice of field $\mathbb{F}_i$ for each vertex i of the graph, such that $\mathbb{F}_i$ is a finite extension of $\mathbb{F}$, and $\cap_i \mathbb{F}_i = \mathbb{F}$.
- For each arrow ${}_ja_i$, an $(\mathbb{F}_j, \mathbb{F}_i)$ bimodule ${}_jM_i$ such that the two actions of $\mathbb{F} \subset \mathbb{F}_i, \mathbb{F}_j$ agree.
- For each arrow ${}_ja_i$, a non-degenerate $\mathbb{F}_i$-bilinear form $\epsilon_i^j : {}_iM_j \otimes_{\mathbb{F}_j} {}_jM_i \to \mathbb{F}_i$.

The tensor algebra T_M is

$$T_M = \bigoplus_{i_1 i_2 \cdots i_k \text{ a path in } \Gamma} {}_kM_{k-1} \otimes_{\mathbb{F}_{k-1}} \cdots \otimes_{\mathbb{F}_{i_3}} {}_{i_2}M_{i_2} \otimes_{F_{i_2}} {}_{i_2}M_{i_1}, \tag{1}$$

with multiplication being tensor product if the end of one path agrees with the beginning of the next, and 0 otherwise.

For each arrow ${}_ja_i$, the bilinear form ϵ_i^j defines a canonical element r_j^i in ${}_jM_i \otimes_{\mathbb{F}_i} {}_iM_j$, which can be taken to be

$$r_j^i := \sum_k v_k \otimes v^k \tag{2}$$

for any pair of dual $\mathbb{F}_i$ bases $\{v_k\} \subset {}_jM_i, \{v^k\} \subset {}_iM_j$ with respect to ϵ_i^j. It is well known that this does not depend on the choice of dual bases. Furthermore, although the two $\mathbb{F}_j$ actions on ${}_jM_i \otimes_{\mathbb{F}_i} {}_iM_j$ need not agree, it is true that $zr_i = r_iz$ for all $z \in \mathbb{F}_j$.

For each $i \in I$, define

$$r_i := \sum_{j: {}_ja_i \in A} r_i^j \tag{3}$$

Definition 1.1. (see [DR80]) The preprojective algebra Λ_M is the quotient of T_M by $\{r_i\}_{i\in I}$.

One can associate a symmetrizable (but not necessarily symmetric) Cartan matrix $C = (c_{ij})$ to such a modulated graph as follows:

$$c_{i,j} = \begin{cases} 2 & \text{if } i = j \\ -\dim_{\mathbb{F}_i} {}_iM_j & \text{if there is an arrow from } i \text{ to } j \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

As in [DR80], Λ_M is finite dimensional over $\mathbb{F}$ if and only if C is of finite type. If C is symmetric then, letting $\mathbb{F}_i = \mathbb{C}$ for all i and taking an obvious choice of bimodules and bilinear forms, one recovers Lusztig's preprojective algebra from [Lus91] (see [Rin98]). However, even in this case, different choices of bilinear form give non-isomorphic algebras (see [Rin98] or §4.2).

Example 1.2. Consider the modulated graph with $\mathbb{F}_1 = \mathbb{R}$, $F_2 = \mathbb{C}$, ${}_1M_2 = {}_2M_1 = \mathbb{C}$, with the standard actions of $\mathbb{R}$ and $\mathbb{C}$ by multiplication. Use the bilinear forms:

$$\begin{aligned} \epsilon_1^2 &: \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \to \mathbb{R} & \qquad \epsilon_2^1 &: \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \\ & z \otimes w \to \operatorname{Re}(zw) & & z \otimes w \to zw \end{aligned} \tag{5}$$

The corresponding Cartan matrix is of type C_2. Consider the elements of the tensor algebra:

- $e_1 = 1 \in \mathbb{F}_1$ and $e_2 = 1 \in \mathbb{F}_2$ in degree 0.
- $\tau = 1 \in {}_2M_1$ and $\bar{\tau} = 1 \in {}_1M_2$ in degree 1.

then the relations defining the preprojective algebra Λ are

$$\overline{\tau}\tau = 0 \quad \text{and} \quad \tau\overline{\tau} - i\tau\overline{\tau}i = 0. \tag{6}$$

As a vector space, the preprojective algebra decomposes as

$$\mathbb{R}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}\tau \oplus \mathbb{R}\overline{\tau} \oplus \mathbb{R}\overline{\tau}i \oplus \mathbb{C}\tau\overline{\tau} \oplus \mathbb{R}\overline{\tau}i\tau, \tag{7}$$

where the field shown in each case is the one that naturally acts on the left.

1.3. **Nilpotent Representation varieties.** There is a natural partition of the identity $e \in \Lambda_M$ as $e = \sum_i e_i$, where e_i is the lazy path at node i. Given a representation of Λ_M on a vector space V, notice that $V = \bigoplus_i e_i V$, and in fact each $e_i V$ is naturally a left $\mathbb{F}_i$ module. Given a dimension vector $\mathbf{v} = (v_i)_{i \in I}$, fix a v_i dimensional left $\mathbb{F}_i$ module V_i for each i. Define the nilpotent representation variety $\Lambda(\nu)$ to be the variety of representations of Λ_M on $V = \oplus_i V_i$ such that $e_i V = V_i$, the induced left $\mathbb{F}_i$ module structure on V_i agrees with the original $\mathbb{F}_i$ module structure, and all sufficiently long paths act as 0. Note that, up to isomorphism, $\Lambda(\nu)$ does not depend on any choice.

The representation variety $\Lambda(\nu)$ is contained in

$$\bigoplus_{(i,j) \in A} \mathrm{Hom}_{\mathbb{F}_j}({}_jM_i \otimes_{\mathbb{F}_i} V_i, V_j), \tag{8}$$

and is cut out by the polynomial equations stating that

- each r_i acts as 0, and
- for all paths of length at least $\sum_i \dim_{\mathbb{F}_i} V_i$, every element in the right side of (1) acts as 0.

This is the set of $\mathbb{F}$ points of an algebraic variety, so inherits the Zariski topology.

1.4. **Summary of main result.** We will show that the union over all $\mathbf{v}$ of the set of irreducible components of $\Lambda(\mathbf{v})$ realizes $B(-\infty)$, where the crystal operators are natural analogues of those used by Kashiwara and Saito [KS97]. We also establish some related results, such as the fact that $\Lambda(\mathbf{v})$ is always pure dimensional, with each irreducible component being quasi-affine. To a large extent, Kashiwara and Saito's original proof goes through, although we have made a few modifications.

We then give a few examples of the new behavior our construction exhibits, and discuss some potential applications. Interestingly, our construction is more general than Lusztig's even when the Cartan matrix is symmetric and all the $\mathbb{F}_i$ are chosen to be $\mathbb{C}$. See §4.2.

1.5. **Connections with the literature.** There is already a well known way to study $B(-\infty)$ in symmetrizable types by "folding" the quiver variety for a larger symmetric Kac-Moody algebra (see [Sav05]). There the crystal for the symmetrizable Kac-Moody algebra is the set of irreducible components of the quiver variety for the symmetric Kac-Moody algebra that are fixed set-wise by an automorphism related to an automorphism of the Dynkin diagram. Due to that construction, most results that can be proven about $B(\infty)$ using quiver varieties have been extended to symmetrizable type. However, with our symmetrizable quiver varieties, the proofs of those results should be simplified, as many of the symmetric type proofs should simply carry over without modification. We also feel it aesthetically important to have a quiver variety in symmetrizable types that is *actually a representation variety* for some algebra.

One could hope to see a direct relationship between our construction and the folding construction, and in fact we believe this is possible: It seems that, by composing the Diagram automorphism with a Galois automorphism of the field that has the same order, the irreducible components that are fixed set-wise are in bijection with the irreducible components of the set of actual fixed points, which is in turn isomorphic to our quiver variety. Perhaps this will be the subject of a future work.

This paper does not consider the natural question of where $U^-(\mathfrak{g})$ itself can be realized using our quiver varieties, in a way analogous to the symmetric case.

We also note that the current work can perhaps be generalized: Dlab and Ringel actually allow division rings everywhere where we have used fields. We believe that one should be able to construct representation varieties for these more general modulated graphs and still realize $B(-\infty)$, but this involves some technicalities which we prefer to avoid for the moment.

Finally, we note two recent papers of Geiss, Leclerc and Schroer [GLSa, GLSb] which also address preprojective algebras in symmetrizable type. They take a different approach, via quivers with relations, and so their constructions is quite different from ours. They do not consider crystals, but do have a realization of $U^-(\mathfrak{g})$.

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2. Background

2.1. **Symmetrizable Kac-Moody algebras.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra with Cartan matrix $C = (c_{ij})_{i\in I}$, and let $D = \operatorname{diag}\{d_i\}_{i\in I}$ be such that DC is symmetric, with the d_i relatively prime positive integers. Let P be the weight lattice of $\mathfrak{g}$ and Q the root lattice, and denote by $\{\alpha_i\}$ be the simple roots. Let $Q_+ = \operatorname{span}_{\mathbb{Z}_{\geq 0}}\{\alpha_i\}_{i\in I}$. Recall the usual (symmetric) bilinear form on Q defined by $(\alpha_i, \alpha_j) = d_i c_{ij}$. We also have the form $\langle \cdot, \cdot \rangle$ between the root lattice, and the co-root lattice, defined by $\langle \alpha_i, \check{\alpha_j} \rangle = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)} = c_{j,i}$.

2.2. **Crystals.** The following definitions and results are by now standard. Here we roughly follow [Kas95] and [KS97, §3].

Definition 2.1. (see [Kas95, §7.2]) A **combinatorial crystal** is a set B along with functions $\operatorname{wt}\colon B \to P$ (where P is the weight lattice), and, for each $i \in I$, functions $\varepsilon_i, \varphi_i \colon B \to \mathbb{Z} \cup \{-\infty\}$ and $e_i, f_i : B \to B \sqcup \{\emptyset\}$, such that

(i) $\varphi_i(b) = \varepsilon_i(b) + \langle \operatorname{wt}(b), \alpha_i^\vee \rangle$.
(ii) e_i increases φ_i by 1, decreases ε_i by 1 and increases wt by α_i.
(iii) $f_i b = b'$ if and only if $e_i b' = b$.
(iv) If $\varphi_i(b) = -\infty$, then $e_i b = f_i b = \emptyset$.

We often denote a combinatorial crystal simply by B, suppressing the other data.

Definition 2.2. A **lowest weight** combinatorial crystal is a combinatorial crystal with a distinguished element b_- (the lowest weight element) such that

(i) The element b_- can be reached from any $b \in B$ by applying a finite sequence of f_i for various $i \in I$.
(ii) For all $b \in B$ and all $i \in I$, $\varphi_i(b) = \max\{n : f_i^n(b) \neq \emptyset\}$.

For lowest weight combinatorial crystals, the functions φ_i, ε_i and wt are determined from the operators e_i, f_i and the weight of b_-.

Here we are concerned with the "infinity crystal" $B(-\infty)$, which can be thought of as the crystal for $U_q^-(\mathfrak{g})$. We will use the following result essentially as our definition. It is a rewording of [KS97, Proposition 3.2.3] designed to make the roles of the usual crystal operators and the $*$-crystal operators more symmetric; see [TW] for this exact statement.

Proposition 2.3. *Fix a set B with operators e_i, f_i, e_i^*, f_i^*. Assume (B, e_i, f_i) and (B, e_i^*, f_i^*) are both lowest weight combinatorial crystals with the same lowest weight element b_-, where the other data is determined by setting $wt(b_-) = 0$. Assume further that, for all $i \neq j \in I$ and all $b \in B$,*

(i) $e_i(b), e_i^*(b) \neq 0$.
(ii) $e_i^* e_j(b) = e_j e_i^*(b)$,
(iii) *For all* $b \in B$, $\varphi_i(b) + \varphi_i^*(b) - \langle wt(b), \alpha_i^\vee \rangle \geq 0$
(iv) *If* $\varphi_i(b) + \varphi_i^*(b) - \langle wt(b), \alpha_i^\vee \rangle = 0$ *then* $e_i(b) = e_i^*(b)$,
(v) *If* $\varphi_i(b) + \varphi_i^*(b) - \langle wt(b), \alpha_i^\vee \rangle \geq 1$ *then* $\varphi_i^*(e_i(b)) = \varphi_i^*(b)$ *and* $\varphi_i(e_i^*(b)) = \varphi_i(b)$.
(vi) *If* $\varphi_i(b) + \varphi_i^*(b) - \langle wt(b), \alpha_i^\vee \rangle \geq 2$ *then* $e_i e_i^*(b) = e_i^* e_i(b)$.

then $(B, e_i, f_i) \simeq (B, e_i^*, f_i^*) \simeq B(-\infty)$.

2.3. **Some topology.** All our topological spaces are the $\mathbb{F}$ points of algebraic varieties for some infinite field $\mathbb{F}$, which is typically not algebraically closed. The topology is the Zariski topology; that is, closed sets are locally defined as the zero sets of some polynomials.

Since we work over non-algebraically closed fields, All the varieties we study decompose In fact, our varieties are all birationally equivalent to $\mathbb{F}^k$ for various k.

As usual, we say a space X is irreducible if it cannot be written as the union of two proper closed subsets. In that case, we say the dimension of X is the maximal d such that there is a sequence of irreducible subsets

$$\emptyset \subset X_0 \subset \cdots \subset X_d = X \tag{9}$$

with all containments proper. When X is birational to $\mathbb{F}^k$, the dimension is k.

If X is reducible, its irreducible components are the irreducible subsets which are not properly contained in larger irreducible subsets.

The following is certainly well known.

Lemma 2.4. *If $\pi : X \to Y$ is a locally trivial fiber bundle with irreducible fiber F, then there is a bijection between the irreducible components of X and Y. If Y (or equivalently X) is irreducible, then* $\dim X = \dim Y + \dim F$.

Proof. First assume that Y is irreducible; we will show that X is irreducible. Suppose that $X = X_1 \cup X_2$, where X_1 and X_2 are closed in X. For each $y \in Y$, $\pi^{-1}(y) = (\pi^{-1}(y) \cap X_1) \cup (\pi^{-1}(y) \cap X_2)$; since $\pi^{-1}(y) \simeq F$ is irreducible, $\pi^{-1}(y) \subseteq X_1$ or $\pi^{-1}(y) \subseteq X_2$. Let $Y_1 = \{y | \pi^{-1}(y) \subseteq X_1\}, Y_2 = \{y | \pi^{-1}(y) \subseteq X_2\}$. We will show that Y_1 and Y_2 are closed sets; then since $Y = Y_1 \cup Y_2$ is irreducible it will follow that $Y = Y_1$ or $Y = Y_2$; and hence $X = X_1$ or $X = X_2$ as required.

Trivialize π over an open cover $\{Y^i\}$ of Y. We will prove that $Y_1 \cap Y^i$ is closed in Y^i for each i; this implies that Y_1 is closed (since then $\tilde{Y_1} = \cup(\tilde{Y_1} \cap Y^i)$ is open, where $\tilde{Y_1}$ is the complement of Y_1). Now identifying $\pi^{-1}(Y^i) \simeq Y^i \times F$, we have

$$Y_1 \cap Y^i = \{y \in Y^i | y \times F \subseteq X_1\} = \cap_{f \in F}\{y \in Y^i | y \times f \in X_1\}. \tag{10}$$

This is closed, since $\{y \in Y^i | y \times f \in X_1\}$ is closed (it is the pre-image of X under the map $Y^i \to Y^i \times F, y \to (y, f)$). Similarly Y_2 is closed in Y.

Now if Y is not irreducible, suppose $Y = \cup_i Y_i$ is it's decomposition into irreducible components; then using the above argument, $\pi^{-1}(Y_i)$ is irreducible, so $X = \cup_i \pi^{-1}(Y_i)$ is its decomposition into irreducible components. □

3. Representation varieties and crystals

3.1. **Definition of the representation variety.** Fix a modulated graph M with Cartan matrix C, as defined in §1.2. Denote the corresponding tensor algebra and preprojective algebra by T and Λ respectively (see Definition 1.1). Fix an I-graded vector space $V = \sum_{i, \in I} V_i$, where, as in §1.3, V_i is a vector space over $\mathbb{F}_i$, and let $T(\mathbf{v})$ and $\Lambda(\mathbf{v})$ be the corresponding varieties of nilpotent representation as in §1.3. We associate to $\mathbf{v} = \dim V$ a point in Q_+ by

$$\dim V = \sum_{i \in I} v_i \alpha_i, \tag{11}$$

and sometimes abuse notation by using, for example, $(\mathbf{v}, \mathbf{v})$ to mean $(\sum_{i \in I} v_i \alpha_i, \sum_{i \in I} v_i \alpha_i)$.

Example 3.1. Continue considering the modulated quiver from Example 1.2. By definition, an element of $T(\alpha_1 + \alpha_2)$ is a choice of

- ${}_2x_1 \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C} \otimes_R \mathbb{R}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$. Call this $z = z_1 + z_i i$.
- ${}_1x_2 \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C} \otimes_C \mathbb{C}, \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{R})$. Denote this by $(w_1, w_i) \in \mathbb{R}^2$, where the homomorphism sends 1 to w_1 and i to w_i.

So, $T(\alpha_1 + \alpha_2)$ is isomorphic to $\mathbb{R}^4$.

The relations cutting out Λ are

$$ {}_1x_2 \circ {}_2x_1 = 0 \Leftrightarrow {}_1x_2 \circ {}_2x_1(1) = 0 \Leftrightarrow w_1 z_1 + w_i z_i = 0, \qquad \text{and} \tag{12}$$

$$\begin{aligned} {}_2x_1 \circ {}_1x_2 - i{}_2x_1 \circ {}_1x_2 i = 0 \quad &\Leftrightarrow \begin{pmatrix} z_1 & z_i \end{pmatrix} \begin{pmatrix} w_1 \\ w_i \end{pmatrix} - \\ &\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 & z_i \end{pmatrix} \begin{pmatrix} w_1 \\ w_i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbf{0} \\ &\Leftrightarrow z_1 w_1 + z_2 w_2 = 0, z_2 w_1 - z_1 w_2 = 0. \end{aligned} \tag{13}$$

The real solutions to these equations form two components, defined by $\{w_1 = w_i = 0\}$ and $\{z_1 = z_i = 0\}$ (equivalently, by ${}_1x_2 = 0$ or ${}_2x_1 = 0$). However, as a real algebraic variety, there would be a third component, defined by the equations $\{x_i^2 = -x_1^2, y_1^1 = -y_2^2, x_1 y_1 = -x_2 y_2\}$. This last component contains no new real points, and if we base change to $\mathbb{C}$ would decompose further into two components: $\{x_1 = x_2 i, y_1 = y_2 i\}$ and $\{x_1 = -x_2 i, y_1 = -y_2 i\}$.

This demonstrates both why we need to work with the space of $\mathbb{F}$-points as opposed to the abstract algebraic variety.

3.2. **Some important spaces.**

Definition 3.2. Fix $i \in I$. Set $V^i = \bigoplus_{j : {}_j a_i \in A} {}_i M_j \otimes_{\mathbb{F}_j} V_j$.

Definition 3.3. For ${}_j a_i \in A$, define

$$\begin{aligned} \iota_a : V_i &\to {}_i M_j \otimes_{\mathbb{F}_j} {}_j M_i \otimes_{\mathbb{F}_i} V_i \\ v &\to r_i^j \otimes v, \end{aligned}$$

where r_i^j is the canonical element as from (2). Define ${}_j \tilde{x}_i : V_i \to {}_i M_j \otimes_{\mathbb{F}_j} V_j$ to be the composition of ι_a with the map

$$1 \otimes {}_j x_i : {}_i M_j \otimes {}_j M_i \otimes V_i \to {}_i M_j \otimes V_j.$$

Definition 3.4. $\tilde{x}_i = \bigoplus_{j \in I} {}_j \tilde{x}_i : V_i \to V^i, \quad$ and $\quad {}_i \tilde{x} = \bigoplus_{j \in I} {}_i x_j : V^i \to V_i.$

Proposition 3.5. *The maps $\tilde{x}_i$ and ${}_i \tilde{x}$ are both $\mathbb{F}_i$ linear.*

Proof. The map ι_a intertwines the left $\mathbb{F}_i$ module structure on V_i with the left $\mathbb{F}_i$ module structure on $M_{\overline{a}}$, which immediately implies that each ${}_j \tilde{x}_i$ is $\mathbb{F}_i$ linear, so by summing $\tilde{x}_i$ is as well. That ${}_i \tilde{x}$ is $\mathbb{F}_i$ linear is immediate from the definition. □

Definition 3.6. For each i, let S_i be the simple Λ module such that $e_i S_i = S_i$ and $\dim_{\mathbb{F}_i} e_i S_i = 1$ (so S_i is a copy of $\mathbb{F}_i$ lying over vertex i, and all ${}_j x_i$ are 0).

Lemma 3.7. *Fix a representation V of Λ. Then* $\mathrm{Hom}(S_i, V)$, $\mathrm{Hom}(V, S_i)$ *and* $\mathrm{Ext}^1(S_i, V)$ *are all naturally $\mathbb{F}_i$ vector spaces, with*

- $\mathrm{Hom}(S_i, V) \simeq \ker \tilde{x}_i$
- $\mathrm{Hom}(V, S_i) \simeq (V_i / \operatorname{im} {}_i \tilde{x})^*$
- $\dim_{\mathbb{F}_i} \mathrm{Ext}^1(S_i, V) = \dim_{\mathbb{F}_i} V^i - \dim_{\mathbb{F}_i} \operatorname{im} \tilde{x}_i - \dim_{\mathbb{F}_i} \operatorname{im} {}_i \tilde{x}$.

Proof. The first two statements are obvious. For the third, we seek to classify extensions

$$0 \to S_i \xrightarrow{\iota} V' \xrightarrow{f} V \to 0$$

up to equivalence. Clearly $V'_j = V_j$ if $j \neq i$. Choose a vector space splitting $V'_i \simeq V_i \oplus \mathbb{F}_i$, where $\mathbb{F}_i = \operatorname{im} \iota$. The extension is uniquely determined by a map $\phi : V^i \to \mathbb{F}_i$ subject to the condition that the composition

$$V_i \oplus \mathbb{F}_i \xrightarrow{(\tilde{x}_i, 0)} V^i \xrightarrow{({}_i \tilde{x}, \phi)} V_i \oplus \mathbb{F}_i$$

is 0. This precisely says that that $\ker \phi \supset \operatorname{im} \tilde{x}_i$, so $\phi \in \mathrm{Hom}(V^i / \mathrm{im}\ \tilde{x}_i, \mathbb{F}_i)$.

Two maps ϕ, ϕ' give rise to the same class in $\mathrm{Ext}^1(S_i, V)$ if there exists a map

$$\theta : V_i \oplus \mathbb{F}_i \to V_i \oplus \mathbb{F}_i$$

which is the identity on $\mathbb{F}_i$ and on $(V_i \oplus \mathbb{F}_i)/\mathbb{F}_i$, and such that

$$({}_i \tilde{x}, \phi') = \theta \circ ({}_i \tilde{x}, \phi)$$

Such maps are exactly $(v, x) \to (v, x + \kappa(v))$ for linear $\kappa : V_i \to \mathbb{F}_i$, and such a map stabilizes the short exact sequence iff $\ker \kappa \supset \operatorname{im} {}_i \tilde{x}$. Thus the orbit of a short exact sequence is parameterized by $\kappa|_{\operatorname{im} {}_i \tilde{x}}$, and the result follows. □

Lemma 3.8. $\dim_{\mathbb{F}_i} \operatorname{Ext}^1(S_i, V) = \dim_{\mathbb{F}_i} \operatorname{Hom}(S_i, V) + \dim_{\mathbb{F}_i} \operatorname{Hom}(V, S_i) - \langle \dim V, \alpha_i \rangle$.

Remark 3.9. For Lusztig's preprojective algebra, Lemma 3.8 still holds if S_i is replaced by an arbitrary finite dimensional module W (see [C-B00, Lemma 1]). However, for Dlab and Ringel's preprojective algebras, this more general statement is false (see §4.2).

Proof. Using Lemma 3.7, we compute as follows:

$$\begin{aligned}
&\dim \operatorname{Ext}^1(S_i, V) + \langle \dim V, \alpha_i \rangle - \dim \operatorname{Hom}(S_i, V) - \dim \operatorname{Hom}(V, S_i) \\
&= \dim(V^i/\operatorname{im}(\tilde{x}_i)) - \dim(\operatorname{im}({}_i\tilde{x})) + \langle \dim V, \alpha_i \rangle - \dim(\ker(\tilde{x}_i)) - \dim(V_i/\operatorname{im}({}_i\tilde{x})) \\
&= \dim(V^i) - \dim(\ker(\tilde{x}_i)) - \dim(\operatorname{im}(\tilde{x}_i)) + \langle \dim V, \alpha_i \rangle - \dim(V_i) \\
&= \dim(V^i) - 2\dim(V_i) + \langle \dim V, \check{\alpha}_i \rangle = 0
\end{aligned}$$

To see the last equality, note that $\langle \alpha_j, \check{\alpha}_i \rangle = c_{i,j} = \dim_{\mathbb{F}_i}({}_iM_j)$. □

3.3. **Relations between components.** Define a function on $\Lambda(\mathbf{v})$ by

$$\varphi_i(x) = \dim_{\mathbb{F}_i} \ker x_i. \tag{14}$$

Certainly φ_i is constructible. Let $\Lambda(\mathbf{v})_{i;k}$ be the subset of $\Lambda(\mathbf{v})$ where φ_i takes the value k.

Fix $\mathbf{v}$ and let $\bar{\mathbf{v}} = \mathbf{v} - k\alpha_i$. Fix vector spaces $V, \bar{V}$ of graded dimensions $\mathbf{v}, \bar{\mathbf{v}}$, such that $\bar{V}_j = V_j$ for all $j \neq i$. Let $\Lambda(\mathbf{v}; i; k)$ be the variety whose points consist of an element of $\Lambda(\mathbf{v})$ along with a short exact sequence

$$0 \to \mathbb{F}_i^k \to V \to \bar{V} \to 0, \tag{15}$$

which is trivial on V_j for all $j \neq i$. More explicitly, $\Lambda(\mathbf{v}; i; k)$ is the set of triples (x, P, Q) where $x \in \Lambda(V)_{i;k}$, $P : V_i \mapsto \bar{V}_i$, $Q : F_i^k \mapsto V$, and

- P is the identity on V_j for all $j \neq i$.
- $\ker P = \ker x_i$
- $\operatorname{im} Q = \ker x_i$.

Consider the obvious projections

$$\Lambda(\bar{v})_{i;0} \xleftarrow{\pi_3} \Lambda(\bar{v})_{i;0} \times \operatorname{Surj}(V_i, \bar{V}_i) \xleftarrow{\pi_2} \Lambda(\mathbf{v}; i; k) \xrightarrow{\pi_1} \Lambda(v)_{i;k} \tag{16}$$

Proposition 3.10. *π_1 is a locally trivial fiber bundle whose fibers are isomorphic to*

$$\operatorname{GL}(\bar{V}_i) \times \operatorname{GL}(\mathbb{F}_i^k).$$

π_2 is a locally trivial fiber bundle whose fibers are isomorphic to

$$\operatorname{GL}(\mathbb{F}_i^k) \times \mathit{Hom}(\mathbb{F}_i^{\dim V^i - \dim \overline{V}_i}, \mathbb{F}_i^k).$$

In particular, the fibers of π_1 have dimension $v_i^2 - 2v_ik + 2k^2$ and the fibers of $\pi_3 \circ \pi_2$ have dimension $v_i^2 - 2v_ik + 2k^2 + k \dim_{\mathbb{F}_i} V^i$ over $\mathbb{F}_i$.

Proof. First consider π_1. Choose an $\mathbb{F}_i$ subspace W of V_i of co-dimension k and isomorphisms $M : \mathbb{F}_i^k \to V_i/W$ and $N : \bar{V}_i \to W$. Then on the locus where $W \cap \ker x_i = 0$ (which is a condition on the base) the map

$$(x, P, Q) \to (x, (P \circ N, M \circ \pi_{\ker x_i} \circ Q)) \tag{17}$$

is the required local isomorphism, where $\pi_{\ker x_i}$ is the projection that kills W. The local inverse is

$$(x, (A, B)) \to (x, A \circ N^{-1}, \pi_{\ker x_i} \circ M^{-1} \circ B). \tag{18}$$

Now consider π_2. Fix $W \in V_i$ of codimension k and $L \subset V^i$ of codimension $r = \dim V^i - \dim \bar{V}_i$, and choose $\mathbb{F}_i$ vector space isomorphisms $M : V_i/W \to \mathbb{F}_i^k$, $\iota : \mathbb{F}_i^r \to L$. Then, on the locus where $M \cap \ker P = 0$ and $L \cap \ker {}_i\tilde{x} = 0$ (a condition on the base),

$$(x, P, Q) \to (\bar{x}, P)\,, \left(M \circ \pi_{\ker x_i} \circ Q, Q^{-1} \circ \pi_{\ker x_i} \circ {}_i x \circ \iota\right) \tag{19}$$

is the necessary local isomorphism, where both $\pi_{\ker x_i}$ and $\pi_{\ker x_i'}$ are the projections onto these spaces which kill the chosen subspace W. The inverse sends $((\bar{x}, P), (R, \gamma))$ to the short exact sequence

$$0 \longrightarrow \mathbb{F}_i^k \xrightarrow{\pi_{\ker x_i} \circ M^{-1}} V \xrightarrow{P} \overline{V} \longrightarrow 0 \tag{20}$$

along with the extension of $(\bar{V}, \bar{x})$ to $V = W + \ker P$ defined by the map γ. □

Definition 3.11. $D(\mathbf{v}) = \sum_{i \in I} \dim_{\mathbb{F}} (\mathrm{Hom}_{\mathbb{F}_i}(V_i, V_i)) = \sum_{i \in I} d_i v_i^2.$

Lemma 3.12. *$\Lambda(\mathbf{v})$ has pure dimension $D(\mathbf{v}) - \frac{1}{2}(\mathbf{v}, \mathbf{v})$ (over $\mathbb{F}$). Furthermore, each $\Lambda(\mathbf{v})_{i;k}$ is also pure of this dimension, and for each i there is a bijection between Irr$\Lambda(\mathbf{v})$ and Irr $\coprod_k \Lambda(\mathbf{v})_{i;k}$ which takes X to $X \cap \Lambda(\mathbf{v})_{i;k}$ for the unique k for which this is dense in X.*

Proof. Proceed by induction on $\dim(\mathbf{v})$, the case $\mathbf{v} = 0$ being trivial. Fix $\mathbf{v}$, and assume the statement for all smaller $\bar{\mathbf{v}}$.

Fix $k > 1$. By Proposition 3.10, $\Lambda(\mathbf{v}; i; k)$ is a fiber bundle over each of $\Lambda(\mathbf{v})_{i;k}$ and $\Lambda(\mathbf{v} - k\mathbf{1}_i)_{i;0}$. By lemma 2.4 gives the desired bijection of components, and furthermore by considering the dimensions of the fibers we see that

$$\begin{aligned}
&\dim_{\mathbb{F}} \Lambda(\mathbf{v})_{i;k} \\
&= \dim_{\mathbb{F}} \Lambda(\mathbf{v} - k\alpha_i)_{i;0} + d_i k \dim_{\mathbb{F}_i} V^i \\
&= D(\mathbf{v} - k\alpha_i) - \frac{1}{2}(\mathbf{v} - k\alpha_i, \mathbf{v} - k\alpha_i) + d_i k(2\nu_i - \langle \mathbf{v}, \check{\alpha}_i \rangle) \\
&= \{D(\mathbf{v}) - 2d_i\nu_i k + d_i k^2\} - \frac{1}{2}\{(\mathbf{v}, \mathbf{v}) - 2k(\mathbf{v}, \alpha_i) + 2d_i k^2\} + d_i k(2\nu_i - \frac{(\mathbf{v}, \check{\alpha}_i)}{d_i}) \\
&= D(\mathbf{v}) - \frac{1}{2}(\mathbf{v}, \mathbf{v})
\end{aligned}$$

Now, fix an irreducible component of $\Lambda(\mathbf{v})$. Every point in $\Lambda(\mathbf{v})$ is nilpotent, so is in $\Lambda(\mathbf{v})_{i;k}$ for some i and some $k > 1$. In particular every irreducible component of $\Lambda(\mathbf{v})$ has an open dense subset contained in $\Lambda(\mathbf{v})_{i;k}$ for some i and some $k \geq 1$. Then the result follows as above.

Finally we must handle the case of $\Lambda(\mathbf{v})_{i;0}$. But this is open in $\Lambda(\mathbf{v})$, so every irreducible component is open and dense in some irreducible component of $\Lambda(\mathbf{v})$. So the result follows by the previous paragraph.

The required bijections of components are then clear. □

3.4. **Crystal operators.** Let

$$B = \coprod_{\mathbf{v}} \mathrm{Irr}\Lambda(\mathbf{v}). \tag{21}$$

Lemma 3.12 shows that, for each i, there is a bijection between $\mathrm{Irr}\Lambda(\mathbf{v})$ and $\coprod_k \mathrm{Irr}\Lambda(\mathbf{v})_{i;k}$, where X is sent to the component of $\coprod_k \mathrm{Irr}\Lambda(\mathbf{v})_{i;k}$ that is dense in X. Denote by X_o the corresponding component. Furthermore, we have bijections

$$\tilde{f}_{i;k} : \coprod_{\mathbf{v}} \mathrm{Irr}\Lambda(\mathbf{v})_{i;k} \to \coprod_{\mathbf{v}} \mathrm{Irr}\Lambda(\mathbf{v})_{i;0}. \tag{22}$$

Define

$$\tilde{f}_i := \bigsqcup_k \tilde{f}^{-1}_{i;k-1}\tilde{f}_{i;k}, \qquad \tilde{e}_i := \bigsqcup_k \tilde{f}^{-1}_{i;k+1}\tilde{f}_{i;k}. \tag{23}$$

using the bijection of components, this gives operators on B.

We also need the $*$ operators, which are constructed in a completely analogues way. Define

$$\varphi^*_i(x) = \dim_{\mathbb{F}_i} V_i/\mathrm{Im}({}_ix) \quad \text{and} \quad \Lambda(V)^k_i = \{x \in \Lambda(v) : \varphi^*_i(x) = k\}. \tag{24}$$

Let $\Lambda^*(\mathbf{v}; i; k) = \{(x, P^*, Q^*) \text{ where } x \in \Lambda(V)^k_i,\ P : \overline{V} \mapsto V, Q : V \mapsto \mathbb{F}^k_i\}$ such that :

- P is the identity on V_j for all $j \neq i$,
- $\mathrm{im}\, P = \mathrm{im}\, {}_ix$,
- $\ker Q = \mathrm{im}\, {}_ix$.

We have projections

$$\pi^*_1 : \Lambda^*(\mathbf{v}; i; k) \to \Lambda(\mathbf{v})^k_i \quad \text{and} \quad \pi^*_2 : \Lambda(\mathbf{v}; i; k) \to \Lambda(\bar{\mathbf{v}})^0_i. \tag{25}$$

As with $\Lambda(\mathbf{v}; k)$, we find that

- There is a bijection $X \to X^o$ between $\mathrm{Irr}\Lambda(\mathbf{v})$ and $\coprod_k \mathrm{Irr}\Lambda(\mathbf{v})^k$.
- $\pi^*_2 \circ (\pi^*_1)^{-1}$ defines a bijections $\tilde{f}^*_{i;k}$ on the level of irreducible components.

Define $*$ crystal operators by

$$\tilde{f}^*_i = \bigsqcup_k (\tilde{f}^*_{i;k-1})^{-1} \circ \tilde{f}^*_{i;k} \quad \text{and} \quad \tilde{e}^*_i = \bigsqcup_k (\tilde{f}^*_{i;k+1})^{-1} \circ \tilde{f}^*_{i;k}. \tag{26}$$

3.5. **Reworded operators.** The following reworded characterization of the crystal operators will be useful to us:

Proposition 3.13. *Fix $X \in Irr\Lambda(\mathbf{v})$. Then there is an open-dense subset X_o of X such that, for all $x \in X_o$ and all sufficiently generic extensions*

$$0 \to S_i \to (V', x') \to (V, x) \to 0,$$

(V', x') is in a single irreducible component $Y \in Irr\Lambda(V')$, and $Y = \tilde{e}_i X$. Furthermore, the subset of Y which can be realized in this way is open-dense.

Similarly, there is an open-dense subset X^o of X such that, for all $x \in X^o$ and all sufficiently generic extensions

$$0 \to (V, x) \to (V', x') \to S_i \to 0,$$

(V', x') is in a single irreducible component $Y \in Irr\Lambda(V')$, and $Y = \tilde{e}_i^* X$. Furthermore, the subset of Y which can be realized in this way is open-dense.

Proof. Let $\widehat{X}_o = X \cap \Lambda(\mathbf{v})_{i;k}$ where $k = \varepsilon(X)$, and let X_o be the subset of $\widehat{X}_o$ consisting of points which are not in any other irreducible components of $\mathrm{Irr}\Lambda(\mathbf{v})$. Clearly $X \backslash \widehat{X}_o$ is closed and X_o is non-empty, so since X is irreducible X_o is open dense. Consider the maps

(27)
$$\Lambda(\mathbf{v})_{i;k} \xleftarrow{\pi_1} \Lambda(\mathbf{v};k) \xrightarrow{\pi_3\pi_2} \Lambda(\mathbf{v}-k\alpha_i)_{i;0} \xleftarrow{\pi_3'\pi_2'} \Lambda(\mathbf{v}+\alpha_i;k+1) \xrightarrow{\pi_1'} \Lambda(\mathbf{v}+\alpha_i)_{i;k+1}$$

Recall that all these maps give bijections on the level of irreducible components. Let $\bar{X}$ be the component of $\Lambda(\mathbf{v}-k\alpha_i)$ corresponding to X, and X' the component of $\Lambda(\mathbf{v}+\alpha_i)$. Let $\widehat{X}_o$ be the subset of X which is in $\Lambda(\mathbf{v})_{i;k}$ and not in any other irreducible components of $\Lambda(\mathbf{v})$. Similarly let $\widehat{X}'_o$ be the subset of X' which is in $\Lambda(\mathbf{v}+\alpha_i)_{i;k}$ and not in any other irreducible components of $\Lambda(\mathbf{v}+\alpha_i)$. Then

$$X_o = \widehat{X}_o \cap \pi_1(\pi_3\pi_2)^{-1}\pi_3'\pi_2'(\pi_1')^{-1}\widehat{X}'_o \tag{28}$$

is the desired set.

The second statement follows by a completely symmetric argument. ☐

Lemma 3.14. *Fix $X \in B$ and $T \in X$ generic. Then $\varphi_i(X) + \varphi_i^*(X) - \langle wt(X), \alpha_i^\vee \rangle =$ $\mathrm{Ext}^1(T, S_i) = \mathrm{Ext}^1(S_i, T)$*

Proof. This is immediate from Lemma 3.8. ☐

Corollary 3.15. *Fix $X \in B$ and $i, j \in I$.*

(i) *If $i = j$ and $\varphi_i(b) + \varphi_i^*(b) - \langle wt(b), \alpha_i^\vee \rangle = 0$, then $e_i(X) = e_i^*(X)$*
(ii) *If $i = j$ and $\varphi_i(b) + \varphi_i^*(b) - \langle wt(b), \alpha_i^\vee \rangle \geq 1$, then $\varphi_i(e_i^*(X)) = \varphi_i(X)$ and $\varphi_i^*(e_i(X)) = \varphi_i^*(X)$.*
(iii) *If either $i \neq j$ or $\varphi_i(b) + \varphi_i^*(b) - \langle wt(b), \alpha_i^\vee \rangle > 1$, then $e_i^* e_j(X) = e_j e_i^*(X)$.*

Proof. Fix X and let T be the representation corresponding to a generic point in X, meaning one where all φ_i, φ_i^* are minimal.

In case (i), by Lemma 3.14, $\mathrm{Ext}^1(T, S_i) = \mathrm{Ext}^1(S_i, T) = 0$, so the generic extensions in Proposition 3.13 are in fact trivial extensions, and $e_i(X) = e_i^*(X)$.

In case (ii), $\dim \mathrm{Ext}^1(T, S_i) > 0$, so, if T' is the generic extension

$$0 \to S_i \to T' \to T \to 0, \tag{29}$$

then, using Lemma 3.7, we get that $\mathrm{Hom}(T', S_i) \simeq \dim \mathrm{Hom}(T, S_i)$, so $\varphi_i(e_i^*(X)) = \varphi_i(X)$ from Proposition 3.13. The other equality is true by a similar argument.

In case (iii), consider a generic T'' in $e_i^* e_j X$. First we describe why it suffices to see that the natural homomorphism from the i-socle of T'' to the j-head is trivial. By Proposition 3.13, applying either $f_i^* f_j$ or $f_j f_i^*$ generically takes a subquotient that decreases the dimension of both the i-head and j socle by 1, and the fact that the homomorphism from

i-head to j-socle is trivial implies these operations commute. So,

$$f_i^* f_j e_i^* e_j X = f_j f_i^* e_i^* e_j X = X. \tag{30}$$

But $f_i^* f_j e_j e_i^* X = X$ as well so, since crystal operators are partial permutations, $e_i^* e_j X = e_j e_i^* X$.

Now let us explain why the natural map from the i-socle of T'' to the j-head is trivial. When $i \neq j$ this is clear; in the other case, if the map is non-zero, S_i will occur as a direct summand in T'' (simply pick a copy of S_i in the i-socle, whose image in the i-head is non-zero). Now note that T does not contain S_i as a direct summand. If it did, let $T = \overline{T} \oplus S_i$; since T was chosen to be generic, it follows that $\mathrm{Ext}^1(\overline{T}, S_i) = 0$, and hence $\mathrm{Ext}^1(T, S_i) = 0$, which we know to be false by Lemma 3.14. Consequently, a generic extension $T' \in e_i X$ also doesn't contain S_i as a direct summand. To get the desired contradiction, note that a generic $T'' \in e_i^* T'$ doesn't have S_i as a direct summand either, since:

$$\dim \mathrm{Ext}^1(S_i, T') = \dim \mathrm{Ext}^1(S_i, T) - 1 > 0$$

This equality is true using Lemma 3.14, combined with the prior observation that $\varphi^*(e_i(X)) = \varphi^*(X)$ when $\varphi_i(b) + \varphi_i^*(b) - \langle \mathrm{wt}(b), \alpha_i^\vee \rangle \geq 1$. □

3.6. **Realization of $B(-\infty)$.**

Let $B = \coprod_{\mathbf{v}} \mathrm{Irr}\, \Lambda(\mathbf{v})$. For each $X \in \Lambda(\mathbf{v})$, define:

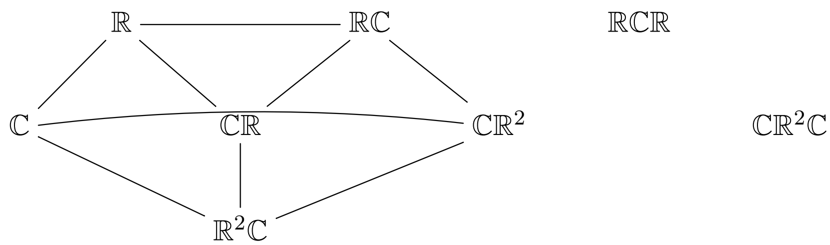
- The weight $\mathrm{wt}(X) = \mathbf{v}$ (which as Defined in §3.1 is in the root lattice)
- $\varphi_i(X) = \min_{x \in X} \varphi_i(x), \quad \varepsilon_i(X) = \varphi_i(X) - \langle \mathrm{wt}(X), \alpha_i^\vee \rangle.$
- $\varphi_i^*(X) = \min_{x \in X} \varphi_i^*(x), \quad \varepsilon_i^*(X) = \varphi_i^*(X) - \langle \mathrm{wt}(X), \alpha_i^\vee \rangle.$

Theorem 3.16. *B, along with data above and the operators $\tilde{e}_i, \tilde{f}_i$ from §3.4 is a realization of $B(-\infty)$.*

Proof. It is clear that B along with $*$ crystal operators from §3.4 is a combinatorial bicrystal. Furthermore, the condition that any $x \in \Lambda(\mathbf{v})$ is nilpotent implies that, for any $X \in B$, there is some i and some j such that $\tilde{f}_i X, \tilde{f}_j^* X \neq 0$. This, along with the definitions of $\varepsilon_i, \varepsilon_i^*$ above, implies that B is a highest weight combinatorial crystal with respect to either set of operators. It remains to check the conditions of Proposition 2.3. (i) is obvious from definitions, and (ii)-(vi) are contained in Corollary 3.15. □

4. Examples

4.1. **Continuing type C_2.** Consider again the modulated graph from Example 1.2, where the two field are $\mathbb{R}$ and $\mathbb{C}$. In this case Λ is representation-finite, and each indecomposable representation can be uniquely identified by giving its socle filtration, which we record from right to left. So, for example, $\mathbb{C}\mathbb{R}^2$ means the unique indecomposable with a copy of the simple $\mathbb{C}$ over vertex 2 in its head, and two copies of the simple $\mathbb{R}$ over vertex 1 in its socle. The following are all the isomorphism classes of indecomposable Λ modules, where we draw lines between pairs that admit a non-trivial extension.



To show that there are no other indecomposables, we argue as follows. One readily verifies that $\mathbb{R}\mathbb{C}\mathbb{R}$ and $\mathbb{C}\mathbb{R}^2\mathbb{C}$ are indecomposable projectives. Suppose we have a representation M, consisting of a real vector space V, a complex vector space W, and maps $\alpha : V \to W, \beta : W \to V$ satisfying the modulated pre-projective relations. If we can find a path of length two which acts non-trivially on some element in V or W, it is easy to see that the sub-module generated by that element is isomorphic to either $\mathbb{R}\mathbb{C}\mathbb{R}$, or $\mathbb{C}\mathbb{R}^2\mathbb{C}$, and hence occurs as a direct summand. Else, if all paths of length two act as 0, then one can break up the module as a direct sum $M_1 \oplus M_2$, where $\tau|_{M_1} = 0$, and $\overline{\tau}|_{M_2} = 0$.

In each irreducible component of $\Lambda(\mathbf{v})$, the isomorphism class of the corresponding representation is constant on an open-dense set, and the classes that show up this way are exactly the rigid representations; that is representations such that no two indecomposables in their Krull-Schmidt decomposition admit a non-trivial extension. So, the irreducible components of $\Lambda(\mathbf{v})$ correspond to collections of indecomposables none of which are connected by lines in the list above, whose total dimension in $\mathbf{v}$. For example, consider $\mathbf{v} = (3, 2)$. The number of Kostant partitions of $3\alpha_1 + 2\alpha_2$ is 5, so the crystal-theoretic result above implies there must be 5 irreducible components of $\Lambda(\mathbf{v})$. The corresponding rigid modules are:

$$(\mathbb{R})(\mathbb{C}\mathbb{R}^2\mathbb{C}), (\mathbb{C}\mathbb{R})(\mathbb{C}\mathbb{R}^2), (\mathbb{C}\mathbb{R})(\mathbb{R}\mathbb{C}\mathbb{R}), (\mathbb{R}\mathbb{C})(\mathbb{R}^2\mathbb{C}), (\mathbb{R}\mathbb{C})(\mathbb{R}\mathbb{C}\mathbb{R}).$$

Unfortunately, this method does not generalize, since usually the preprojective algebra has infinitely many isomorphism classes of indecomposables, even if the Cartan matrix is of finite type.

4.2. **Deformed construction over $\mathbb{C}$ for $\widehat{\mathfrak{sl}}_2$.** Dlab and Ringel's construction can give non-standard preprojective algebras even when all the $\mathbb{F}_i$ are chosen to be $\mathbb{C}$. An example of this is given in [Rin98, §6] for $\widehat{\mathfrak{sl}}_n$, $n \geq 3$ (where all fields are $\mathbb{C}$, and the bimodules are all rank 1). Here we will give a similar example for $\widehat{\mathfrak{sl}}_2$.

Consider $\Gamma = (I, E)$ where I consists of two vertices labeled $1, 2$ and E consists of a single edge joining these two vertices. Choose $\mathbb{F}_1, \mathbb{F}_2 = \mathbb{C}$, and ${}_1M_2 = {}_2M_1 = \mathbb{C}^2$, with the actions of both $\mathbb{F}_1$ and $\mathbb{F}_2$ given by scalar multiplication on both bimodules. The corresponding Cartan matrix, as defined in the introduction, is that of $\widehat{\mathfrak{sl}}_2$. Define

$$\begin{aligned} \epsilon_1^2 : \qquad & {}_1M_2 \otimes_{\mathbb{C}} {}_2M_1 \to \mathbb{C} \\ & (v_1, v_2) \otimes (w_1, w_2) \to v_1 w_1 + v_2 w_2 \end{aligned}$$

and

$$\begin{aligned} \epsilon_2^1 : \qquad & {}_2M_1 \otimes_{\mathbb{C}} {}_1M_2 \to \mathbb{C} \\ & (w_1, w_2) \otimes (v_1, v_2) \to w_1 v_1 - w_2 v_2 \end{aligned}$$

Fix a graded vector space $V = V_1 \oplus V_2$ and $x \in \Lambda(V)$. Define

- $m_{(0,1)} = {}_2x_1((0,1)\otimes\cdot),\ m_{(1,0)} = {}_2x_1((1,0)\otimes\cdot)$ in $\mathrm{Hom}(V_1,V_2)$,
- $\overline{m}_{(0,1)} = {}_1x_2((0,1)\otimes\cdot),\ \overline{m}_{(1,0)} = {}_1x_2((1,0)\otimes\cdot)$ in $\mathrm{Hom}(V_2,V_1)$.

These four maps determine x. The preprojective relations are

$$\overline{m}_{(1,0)}m_{(1,0)} - \overline{m}_{(0,1)}m_{(0,1)} = 0 \quad\text{and}\quad m_{(1,0)}\overline{m}_{(1,0)} + m_{(0,1)}\overline{m}_{(0,1)} = 0. \tag{31}$$

Now consider the case where V_1 and V_2 are both one dimensional, with bases $\{v_1\},\{v_2\}$ respectively. We define a representation $y \in \Lambda(v)$ by specifying the 4 maps as above:

- $m_{(1,0)} = m_{(0,1)}$ is the map which sends v_1 to v_2,
- $\overline{m}_{(1,0)} = \overline{m}_{(0,1)} = 0$.

Call this module I. Take a second copy I' of I, where the basis vectors are v_1', v_2'. Any extension of I by I' will be determined by $a,b,c,d \in \mathbb{C}$ defined by

$$m_{(1,0)}(v_1') = v_2' + av_2,\ m_{(0,1)}(v_1') = v_2' + bv_2,\ \overline{m}_{(1,0)}(v_2') = cv_1,\ \text{and}\ \overline{m}_{(0,1)}(v_2') = dv_1, \tag{32}$$

where $a,b,c,d \in \mathbb{C}$. The two preprojective relations give the equations:

$$c - d = 0 \quad\text{and}\quad c + d = 0. \tag{33}$$

The only solution is $c = d = 0$, so any such extension has a two-dimensional head.

A simple calculation shows that, for Lusztig's preprojective algebra of type $\widehat{\mathfrak{sl}}_2$, if V is any indecomposable and fits in a short exact sequence $0 \to S_2 \to V \to S_1 \to 0$, then V has a self-extension with a 1-dimensional head. Hence, for the choices made in this example, Dlab and Ringel's preprojective algebra is not isomorphic to Lusztig's.

In fact, we can consider a one parameter family of preprojective algebras which are defined as above but with

$$\epsilon_2^1 = zw_1v_1 - w_2v_2. \tag{34}$$

For $z = -1$ we get exactly Lusztig's preprojective algebra, but in all other cases the above argument shows that we do not. So in this case Dlab and Ringel's construction can be though of as non-trivially deforming Lusztig's preprojective algebras.

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