Nilpotent Cones

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CONTENTS

Introduction

This essay deals with nilpotent cones, in the framework on complex Lie groups. There have been many established connections between the structure theory and representation theory of Lie groups and their nilpotent cones. For instance, the Springer correspondence relates nilpotent orbits to representations of the Weyl group. More generally, for any representation of a Lie group it is possible to define its null-cone; the nilpotent cone is the nullcone of the adjoint representation.

Chapter 1 begins with a description of the classical Lie algebras in types A, B, C and D. Then a definition of what the nilpotent cone is, and what nilpotent orbits are, is given. This is followed by a description of the nilpotent orbits in type A, which is a straightforward consequence of the Jordan form theorem. A description of the nilpotent orbits in type C is then given, which requires more machinery. A description of the nilpotent orbits in types B and D are given without proof, since the proof is similar in nature to that of type C. Following this, an explicit description of the centralizers of nilpotent elements in type A is given.

The second chapter of the essay deals with resolution of singularities of nilpotent orbit closures. Given a nilpotent orbit, its closure will have the structure of a singular algebraic variety. In types A and C, a resolution of singularities is given for the nilpotent orbit closures; a different technique is required to construct the resolution in each of these cases. As an application of the resolution of singularities in type A, a description of the closure ordering for nilpotent orbits in type A is given.

In the third chapter, we move on to study some variations of nilpotent cones. In type A, we study the enhanced nilpotent cone, which is the product of a vector space with the nilpotent cone of type A. Using a slightly different method to that originally used, we classify the orbits for the action of the general linear group on the enhanced nilpotent cone. We next examine Kato's exotic nilpotent cone in type C, which is the product of a vector space with a variant of the nilpotent cone in type C. We describe the orbits of the action of the symplectic group on the exotic nilpotent cone, and describe part of the proof. We then study the 2-enhanced nilpotent cone in type A, which is the product of two vector spaces with the nilpotent cone of type A. This is a problem which has not previously been studied, and we describe some partial results about the orbits in this cases.

In the final chapter, we study Springer fibres in type A. Springer fibres are fibres of the resolution of singularities for the full nilpotent cone, which is a special case of the resolution of singularities of nilpotent orbit closures. Following a paper of

Spaltenstein, we describe the irreducible components of Springer fibres.

I expect that this essay should be accessible to a fourth year student with a basic knowledge in Lie theory and algebraic geometry. Complete proofs have been given, except for the Jacobson-Morozov theorem, and the classification of orbits in the exotic nilpotent cone.

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CHAPTER 1

Classifying nilpotent orbits and centralizers of nilpotents in classical Lie algebras

1.1. Definitions

In the following, we work over the field $\mathbb{C}$ for convenience; however, almost all of the results obtained will be valid for any algebraically closed field $\mathbb{F}$ in characteristic 0.

Let $\mathfrak{g}$ denote a reductive Lie algebra, and let G be a connected Lie group having Lie algebra $\mathfrak{g}$. Recall the following definition of the classical Lie groups, and their Lie algebras. In type A, we have the general linear group; in types B and D we have the special orthogonal group, and in type C we have the symplectic group.

Type A: Here $G = GL_n(\mathbb{C}), \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. For convenience, we will work with $GL_n(\mathbb{C})$ as opposed to $SL_n(\mathbb{C})$.

Type B: Here $G = SO_{2n+1}(\mathbb{C}) = \{A \in SL_{2n+1}(\mathbb{C}) : A^tA = I_{2n+1}\}$, and $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C}) = \{A \in \mathfrak{gl}_{2n+1}(\mathbb{C}) : A + A^t = 0\}$.

Type C: Fix an invertible $2n \times 2n$ matrix J such that $J = -J^t$, so that J specifies a non-degenerate skew-symmetric form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{2n}$. Here $G = Sp_{2n}(\mathbb{C}) = \{A \in GL_{2n}(\mathbb{C}) : A^tJA = J\}$, and $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C}) = \{A \in \mathfrak{gl}_{2n}(\mathbb{C}) : JA + A^tJ = 0\}$.

Type D: Here $G = SO_{2n}(\mathbb{C}) = \{A \in SL_{2n}(\mathbb{C}) : A^tA = I_{2n}\}$, and $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C}) = \{A \in \mathfrak{gl}_{2n}(\mathbb{C}) : A + A^t = 0\}$.

Proposition 1.1. *In each of the above cases, we can define a map, known as the adjoint representation, $G \to Aut(\mathfrak{g})$, in which $g \cdot x = gxg^{-1}$ for $g \in G, x \in \mathfrak{g}$.*

Proof. We need to check that this action is well-defined, i.e. given $g \in G$ and $x \in \mathfrak{g}$, $gxg^{-1} \in \mathfrak{g}$.

In Type A, the result is clear.

In types B and D, suppose $g \in SO_k(\mathbb{C})$, so $g^t = g^{-1}$, and $x \in \mathfrak{so}_k(\mathbb{C})$.

$$\begin{aligned} gxg^{-1} + (gxg^{-1})^t &= gxg^{-1} + (g^{-1})^t x^t g^t = gxg^{-1} + (g^t)^{-1} x^t g^{-1} \\ &= gxg^{-1} + gx^t g^{-1} = g(x + x^t)g^{-1} = 0 \end{aligned}$$

Hence $gxg^{-1} \in \mathfrak{so}_k(\mathbb{C})$, as required.

In Type C, suppose $g \in Sp_{2n}(\mathbb{C})$, so $g^t J g = g$, and $x \in \mathfrak{sp}_{2n}(\mathbb{C})$, so $Jx + x^t J = 0$. Note $g^t J g = J$ implies $g^t J = J g^{-1}$, and $Jg = (g^t)^{-1} J$.

$$\begin{aligned} J(gxg^{-1}) + (gxg^{-1})^t J &= Jgxg^{-1} + (g^t)^{-1} x^t g^t J \\ &= (g^t)^{-1} J x g^{-1} + (g^t)^{-1} x^t J g^{-1} \\ &= (g^t)^{-1} (Jx + x^t J) g^{-1} = 0 \end{aligned}$$

Hence $gxg^{-1} \in \mathfrak{sp}_{2n}(\mathbb{C})$, as required. □

In the definition above of $\mathfrak{sp}_{2n}(\mathbb{C})$ as $\{A \in \mathfrak{gl}_n(\mathbb{C}) : JA + A^t J = 0\}$, an equivalent formulation is $\{A \in \mathfrak{gl}_{2n}(\mathbb{C}) : \langle Av, w \rangle = -\langle v, Aw \rangle \ \forall v, w \in \mathbb{C}^{2n}\}$.

In a Lie algebra $\mathfrak{g}$, recall that an element $X \in \mathfrak{g}$ is “nilpotent" if ad_X is a nilpotent endomorphism of the vector space $\mathfrak{g}$. In the case of $\mathfrak{g}$ being a classical Lie algebra, this is equivalent to X being nilpotent in the sense of matrices.

Definition 1.2. *The nilpotent cone of the Lie algebra $\mathfrak{g}$, denoted as $\mathcal{N}$, consists of all nilpotent elements in $\mathfrak{g}$.*

The group G has a well-defined action on $\mathcal{N}$, since in each of the above cases, G acts on $\mathfrak{g}$ by conjugation, and the conjugate of a nilpotent matrix will remain nilpotent.

Definition 1.3. *The orbits of the group action of G on $\mathcal{N}$ are nilpotent orbits.* [**3**]

Thus the nilpotent cone $\mathcal{N}$ is a union of a nilpotent orbits, and two nilpotent elements are in the same nilpotent orbit iff they are conjugate by an element of G. Alternatively, nilpotent orbits are often referred to as “conjugacy classes", or “adjoint orbits" in $\mathcal{N}$.

Next, we give an explicit description of nilpotent orbits for the classical Lie algebras. Type A ($\mathfrak{gl}_n$) serves as a straightforward example to illustrate the concepts above, while type C ($\mathfrak{sp}_{2n}$) takes more work. Types B ($\mathfrak{so}_{2n+1}$) and type D ($\mathfrak{so}_{2n}$) involve similar ideas to type C, and here we state the results without proof.

1.2. Classification of the nilpotent orbits

1.2.1. Type A. In type A, we have that $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, and the connected Lie group with Lie algebra $\mathfrak{g}$ is $G = \mathrm{GL}_n(\mathbb{C})$. The nilpotent cone $\mathcal{N}$ of $\mathfrak{gl}_n(\mathbb{C})$ is then simply the set of all nilpotent matrices in $\mathfrak{gl}_n(\mathbb{C})$.

The action of the group G on $\mathcal{N}$ is by conjugation, so two nilpotent matrices in $\mathcal{N}$ lie in the same nilpotent orbit iff they are conjugate under $GL_n(\mathbb{C})$. Now the classification of orbits follows from the Jordan form theorem. Keeping in mind that nilpotent matrices in $GL_n(\mathbb{C})$ have all eigenvalues equal to 0, the Jordan form theorem says that any matrix $x \in \mathcal{N}$ is conjugate to precisely one matrix n_λ of the following form, where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ are positive integers whose sum is n:

$$n_\lambda = \begin{pmatrix} N_{\lambda_1} & & & \\ & N_{\lambda_2} & & \\ & & ... & \\ & & & N_{\lambda_k} \end{pmatrix}$$

Here the matrix N_i denotes the following matrix (where there are i rows and columns):

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & .. & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

Now let us make the following definitions:

Definition 1.4. *Let $\mathcal{P}(n)$ denote the set of all k-tuples $(\lambda_1, ..., \lambda_k)$, with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ satisfying $\sum_{i=1}^k \lambda_i = n$.*

Definition 1.5. *For each partition $\lambda \in \mathcal{P}(n)$, let $\mathcal{O}_\lambda$ denote the nilpotent orbit containing the matrix n_λ.*

Using this, it follows that every nilpotent orbit in $\mathfrak{gl}_n(\mathbb{C})$ corresponds to a unique element of $\mathcal{P}(n)$. Thus we have proven the following classification of nilpotent orbits in type A:

Proposition 1.6. *There is a bijection between the set of nilpotent orbits for the Lie algebra $\mathfrak{gl}_n$, and the set $\mathcal{P}(n)$, given by the Jordan canonical form. Specifically, the orbit $\mathcal{O}_\lambda$ corresponding to the partition λ consists of all matrices in $\mathfrak{gl}_n$ conjugate to n_λ.*

If $x \in \mathcal{O}_\lambda$, we refer to λ as the Jordan type of the matrix x. A Jordan basis for x, is a basis $\{e_{i,j}\}$ of V, with $1 \leq i \leq k, 1 \leq j \leq \lambda_i$, such that $xe_{i,j} = e_{i,j-1}$ if $j > 1$, and 0 otherwise. The matrix of x with respect to such a basis, ordered so that $e_{i,j}$ precedes $e_{i',j'}$ if $i < i'$ or $i = i'$ and $j < j'$, will then be n_λ. It will often be convenient to represent λ as a diagram of boxes, with λ_1 boxes in the first row, λ_2 boxes in the second row, and so forth. Then the boxes in this diagram can be thought of as representing the elements of this Jordan basis (with $e_{i,j}$ being the j-th box in the i-th row), with x moving each box to the box on the left.

1.2.2. Type C. Following Collingwood and McGovern, [**3**], pg 70-73, we describe the nilpotent orbits in the symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$. While in the case of the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, the nilpotent orbits were in $1-1$ correspondence with the set of all partitions of n, here we will find that the nilpotent orbits of the symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ are in $1-1$ correspondence with the set of all partitions of $2n$ in which every odd part occurs with even multiplicity. In effect, the problem of classifying of nilpotent orbits in $\mathfrak{sp}_{2n}(\mathbb{C})$ and proving the above statement is equivalent to proving the following three propositions:

Proposition 1.7. *Given any nilpotent element $n \in \mathfrak{sp}_{2n}(\mathbb{C})$, the Jordan type of n is a partition $\lambda \in \mathcal{P}(2n)$, satisfying the condition that every odd part in λ occurs with even multiplicity.*

Proposition 1.8. *Given any partition $\lambda \in \mathcal{P}(2n)$ where every odd part in λ occurs with even multiplicity, there exists a nilpotent element $n_\lambda \in \mathfrak{sp}_{2n}(\mathbb{C})$ such that the Jordan type of n_λ is λ.*

Proposition 1.9. *Given any two nilpotent elements $n_\lambda, n'_\lambda \in \mathfrak{sp}_{2n}(\mathbb{C})$ which both have Jordan type λ, then we can find $g \in Sp_{2n}(\mathbb{C})$ such that $g^{-1}n_\lambda g = n'_\lambda$.*

In the proofs of the above propositions, we will use the Jacobson-Morozov Theorem, [**3**], pg 37:

Jacobson-Morozov Theorem: Given a semisimple Lie algebra $\mathfrak{g}$, and a nilpotent element X in $\mathfrak{g}$, then there exists a $\mathfrak{sl}_2$ triple (H, X, Y) containing X. (Equivalently, we can find $H, Y \in \mathfrak{g}$ such that $[X, Y] = H, [H, X] = 2X, [H, Y] = -2Y$).

To illustrate the Jacobson-Morozov Theorem, we will prove it for $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. (Note that here $\mathfrak{g}$ is not semisimple; nonetheless, the Jacobson-Morozov theorem is still true in this case). It suffices to find an $\mathfrak{sl}_2$-triple containing the nilpotent element n_λ. Let $\mu_i^r = i(r+1-i)$, and define the following three matrices:

$$H_{r+1} = \begin{pmatrix} r & & & & \\ & r-2 & & & \\ & & \cdots & & \\ & & & -r+2 & \\ & & & & -r \end{pmatrix}, \quad X_{r+1} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \cdots & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

$$Y_{r+1} = \begin{pmatrix} 0 & & & & \\ \mu_1^r & 0 & & & \\ & & \cdots & & \\ & & & 0 & \\ & & & \mu_r^r & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} H_{\lambda_1} & & & \\ & H_{\lambda_2} & & \\ & & \cdots & \\ & & & H_{\lambda_k} \end{pmatrix}, \qquad X = \begin{pmatrix} X_{\lambda_1} & & & \\ & X_{\lambda_2} & & \\ & & \cdots & \\ & & & X_{\lambda_k} \end{pmatrix}$$

$$Y = \begin{pmatrix} Y_{\lambda_1} & & & \\ & Y_{\lambda_2} & & \\ & & \cdots & \\ & & & Y_{\lambda_k} \end{pmatrix}$$

It is clear by calculation that $\{H_{r+1}, X_{r+1}, Y_{r+1}\}$ is an $\mathfrak{sl}_2$-triple, and hence that $\{H, X, Y\}$ is an $\mathfrak{sl}_2$-triple, with $X = n_\lambda$. This proves the Jacobson-Morozov Theorem for $\mathfrak{gl}_n(\mathbb{C})$. For future use, we prove the following lemma about $\mathfrak{sl}_2$-triples in symplectic groups.

Lemma 1.10. *Let V be a vector space with even dimension with a chosen symplectic form $\langle \cdot, \cdot \rangle$, and suppose $\{H, X, Y\}$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{sp}(V)$. Let $V = \bigoplus_{r=1}^{s} V_r$ be the decomposition of V into irreducible $\mathfrak{sl}_2$-modules. If dim$V_p \neq$ dimV_q, then $V_p \perp V_q$ with respect to the symplectic inner product $\langle \cdot, \cdot \rangle$.*

Proof. Let $\dim V_p = m, \dim V_q = n$, and without loss of generality assume that $m < n$. Let $\{v_1, v_2, \cdots, v_n\}$ denote a string basis of V_q, so that the actions of X and Y on V_q are given by the following (see Section 7.2 of [**5**] for an exposition of $\mathfrak{sl}_2$-theory):

$$Xv_i = (n+1-i)v_{i-1} \text{ and } Xv_i = 0; \quad v_i = \frac{X^{n-i}}{(n-i)!}v_n$$

$$Yv_i = iv_{i+1} \text{ and } Yv_n = 0; \quad v_i = \frac{Y^{i-1}}{(i-1)!}v_1$$

Similarly, let $\{w_1, w_2, \cdots, w_m\}$ denote a string basis of V_p, so that the actions of X and Y on V_p are given by the following:

$$Xw_i = (m+1-i)w_{i-1} \text{ and } Xw_1 = 0; \; w_i = \frac{X^{m-i}}{(m-i)!}w_m$$

$$Yw_i = iw_{i+1} \text{ and } Xw_m = 0; \; w_i = \frac{Y^{i-1}}{(i-1)!}w_1$$

Note the following two relations:

$$X^i w_i = \frac{X^m}{(m-i)!}w_m = 0$$

$$Y^{m-i+1}w_i = \frac{1}{(i-1)!}Y^m w_1 = 0$$

To prove that $V_p \perp V_q$, it suffices to prove $\langle v_i, w_j \rangle = 0$ for $1 \leq i \leq n, 1 \leq j \leq m$. If $n-i \leq j-1$ and $i-1 \leq m-j$, then $n \leq m$, which is a contradiction. So either $n-i > j-1$ or $i-1 > m-j$.
If $n-i > j-1$:

$$\begin{aligned} \langle v_i, w_j \rangle &= \left\langle \frac{X^{n-i}}{(n-i)!} v_n, w_j \right\rangle \\ &= \frac{(-1)^{n-i}}{(n-i)!} \left\langle v_n, X^{n-i} w_j \right\rangle \\ &= 0 \text{ since } n-i \geq j, X^j w_j = 0 \end{aligned}$$

If $i-1 > m-j$:

$$\begin{aligned} \langle v_i, w_j \rangle &= \left\langle \frac{Y^{i-1}}{(i-1)!} v_1, w_j \right\rangle \\ &= \frac{(-1)^{i-1}}{(i-1)!} \left\langle v_1, Y^{i-1} w_j \right\rangle \\ &= 0 \text{ since } i-1 \geq m-j+1, Y^{m-j+1} w_j = 0 \end{aligned}$$

Thus, in either case, $\langle v_i, w_j \rangle = 0$, as required. □

Proof. (of Proposition 1.7)
Given a nilpotent X of type λ, by the Jacobson-Morozov theorem, we can find an $\mathfrak{sl}_2$-triple $\{H, X, Y\}$ containing X. Decompose $\mathbb{C}^{2n}$ as the sum of irreducible $\mathfrak{sl}_2$-modules $V_1, V_2, \cdots V_j$, with dim $V_i = \mu_i$, and without loss of generality, we can assume that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_j$. By $\mathfrak{sl}_2$-theory, if $v_i \in V_i$ is a lowest weight vector in V_i, V_i is spanned by $\{v_i, Xv_i, \cdots, X^{\mu_i - 1} v_i\}$, and $X^{\mu_i} v_i = 0$. Thus the set $\{X^j v_i | 0 \leq j \leq \mu_i - 1\}$ is a Jordan basis for the nilpotent X, and so X has type μ, and so $\lambda = \mu$. We have just proven that the dimensions of the irreducible summands of $\mathbb{C}^{2n}$, considered as an $\mathfrak{sl}_2$-module, are $\lambda_1, \cdots, \lambda_k$ (so $j = k$ and dim $V_i = \lambda_i$).
Suppose r is odd. Make the following definition:

$$V^r = \bigoplus_{\lambda_i = r} V_i$$

Let H_{r-1} be the highest weight space in V^r, i.e. the $(r-1)$-weight space. It is clear that dim H_{r-1} is the multiplicity of r in the partition λ. Define a form $(\cdot, \cdot)$ on H_{r-1} by $(v, w) = \langle v, Y^{r-1} w \rangle$. Then the following calculation shows that $(\cdot, \cdot)$ is a skew-symmetric form.

$$\begin{aligned} (w, v) = \left\langle w, Y^{r-1} w \right\rangle &= (-1)^{r-1} \left\langle Y^{r-1} w, v \right\rangle \\ &= (-1)^r \left\langle v, Y^{r-1} w \right\rangle = -(v, w) \end{aligned}$$

Next we will prove that the form $(\cdot,\cdot)$ is non-degenerate. Suppose $v \in H_{r-1}$ and $(v,w) = 0\ \forall w \in H_{r-1}$. As w ranges over H_{r-1}, $Y^{r-1}w$ will range over the lowest weight space H_{1-r} in V^r, by $\mathfrak{sl}_2$-theory. Thus v is perpendicular to H_{1-r}, with respect to $\langle\cdot,\cdot\rangle$.

If $s \neq 1-r$, then v will be perpendicular to the s-weight space H_s in V^r by the following calculation. Let $w \in H_s$.

$$\begin{aligned} \langle Hv, w\rangle &= -\langle v, Hw\rangle \\ \langle (r-1)v, w\rangle &= -\langle v, sw\rangle \\ (r-1+s)\langle v, w\rangle &= 0 \\ \langle v, w\rangle &= 0 \end{aligned}$$

Since v is perpendicular to H_s for $s \neq 1-r$, and to H_{1-r}, v is perpendicular to all weight spaces in V^r, and so v is perpendicular to V^r, which is a direct sum of its weight spaces. Make the following definition:

$$V'^r = \bigoplus_{\lambda_i \neq r} V_i$$

By Lemma 1.10, v is perpendicular to V'^r, and hence v is perpendicular to $V^r \oplus V'^r = V$. Thus v lies in the kernel of the non-degenerate bilinear form $\langle\cdot,\cdot\rangle$, so $v = 0$.

Hence $(\cdot,\cdot)$ is a non-degenerate symplectic form on H_{r-1}, so dim H_{r-1} is even. Since dim H_{r-1} is equal to multiplicty of r in the partition λ, this proves that every odd part occurs with even multiplicity in λ. $\square$

Proof. (of Proposition 1.8)
First we prove the result for rectangle partitions, $\lambda = (d^i)$. Here if d is odd, i must be even.

Let V be a vector space of dimension i, with basis $\{v^{(1)}, v^{(2)}, \cdots, v^{(i)}\}$. If d is even, define a symmetric form on V via $(v^{(r)}, v^{(s)}) = \delta_{rs}$. If d is odd and i is even, define a symplectic form on V via $(v^{(r)}, v^{(s)}) = 0$ if $|r-s| \neq 1$; $(v^{(r)}, v^{(r+1)}) = 1$ if r is odd and 0 otherwise; $(v^{(r+1)}, v^{(r)}) = -1$ if r is odd and 0 otherwise.

We now construct a $2n$-dimensional vector space W, an $\mathfrak{sl}_2$-triple $\{X, Y, H\}$ for which W is a direct summand of i d-dimensional irreducible $\mathfrak{sl}_2$-submodules, and a symplectic form on W which is invariant under the $\mathfrak{sl}_2$ action. Let the highest weight space in W have basis $\{w^{(1)}_{d-1}, \cdots, w^{(i)}_{d-1}\}$. For $1 \leq m \leq d-1$, $1 \leq j \leq i$, define $w^{(j)}_{d-1-2m} = \frac{Y^m}{m!} w^{(j)}_{d-1}$, so that $\{w^{(j)}_{d-1}, \cdots, w^{(j)}_{1-d}\}$ is a basis for an irreducible $\mathfrak{sl}_2$ submodule. The superscript denotes which of the i submodules it lies in, and the subscript denotes the weight. We adopt the convention that $w^{(i)}_{d+1} = w^{(i)}_{-d-1} = 0$.

By $\mathfrak{sl}_2$-theory, the actions of X and Y are given by the following:

$$\begin{aligned} Xw^{(j)}_{d-1-2m} &= (d-m)w^{(j)}_{d+1-2m} \\ Yw^{(j)}_{d-1-2m} &= (m+1)w^{(j)}_{d-3-2m} \end{aligned}$$

Define the following form on W:

$$\left\langle w^{(i)}_{d-2m-1}, w^{(i')}_{d-2m'-1} \right\rangle = 0 \text{ unless } m+m' = d-1$$
$$\left\langle w^{(i)}_{d-2m-1}, w^{(i')}_{d-2m'-1} \right\rangle = (-1)^m \binom{d-1}{m} (v^{(i)}, v^{(i')}) \text{ if } m+m' = d-1$$

It is straightforward to check that $\langle \cdot, \cdot \rangle$ is a skew-symmetric form. If $m+m' \neq d-1$, then $\left\langle w^{(i)}_{d-2m-1}, w^{(i')}_{d-2m'-1} \right\rangle = -\left\langle w^{i'}_{d-2m'-1}, w^{i}_{d-2m-1} \right\rangle = 0$. If $m+m' = d-1$, then by construction of the form $(\cdot, \cdot)$, $(v^{(i)}, v^{(i')}) = (-1)^d (v^{(i')}, v^{(i)})$. Then we compute:

$$\begin{aligned} \left\langle w^{(i)}_{d-2m-1}, w^{(i')}_{d-2m'-1} \right\rangle &= (-1)^m \binom{d-1}{m} (v^{(i)}, v^{(i')}) \\ &= (-1)^m \binom{d-1}{m'} (-1)^d (v^{(i')}, v^{(i)}) \\ &= -(-1)^{d-1-m} \binom{d-1}{m'} (v^{(i')}, v^{(i)}) \\ &= -(-1)^{m'} \binom{d-1}{m'} (v^{(i')}, v^{(i)}) \\ &= -\left\langle w^{(i')}_{d-2m'-1}, w^{(i)}_{d-2m-1} \right\rangle \end{aligned}$$

Next we check that $\langle \cdot, \cdot \rangle$ is invariant under H.

Since $Hw^{(i')}_{d-2m-1} = (d-2m-1)w^{(i')}_{d-2m-1}$, if $m+m' \neq d-1$, $\left\langle Hw^{(i)}_{d-2m-1}, w^{(i')}_{d-2m'-1} \right\rangle + \left\langle w^{(i)}_{d-2m-1}, Hw^{(i')}_{d-2m'-1} \right\rangle = 0 + 0 = 0$.

If $m + m' = d - 1$:

$$\begin{aligned}&\left\langle Hw^{(i)}_{d-2m-1}, w^{(i')}_{d-2m'-1}\right\rangle + \left\langle w^{(i)}_{d-2m-1}, Hw^{(i')}_{d-2m'-1}\right\rangle \\ &= (d-2m-1+d-2m'-1)\left\langle w^{(i)}_{d-2m-1}, w^{(i')}_{d-2m'-1}\right\rangle \\ &= (2(d-1)-2(m+m'))\left\langle w^{(i)}_{d-2m-1}, w^{(i')}_{d-2m'-1}\right\rangle = 0\end{aligned}$$

Next we check that $\langle \cdot, \cdot \rangle$ is invariant under X.

$$\begin{aligned}&\left\langle Xw^{(i)}_{d-2m-3}, w^{(i')}_{d-2m'-1}\right\rangle + \left\langle w^{(i)}_{d-2m-3}, Xw^{(i')}_{d-2m'-1}\right\rangle \\ &= \left\langle (d-m-1)w^{(i)}_{d-2m-1}, w^{(i')}_{d-2m'-1}\right\rangle + \left\langle w^{(i)}_{d-1-2(m+1)}, (d-m')w^{(i')}_{d-1-2(m'-1)}\right\rangle\end{aligned}$$

If $m + m' \neq d - 1$, then both terms in the above sum are clearly 0. Otherwise, if $m + m' = d - 1$:

$$\begin{aligned}&\left\langle Xw^{(i)}_{d-2m-3}, w^{(i')}_{d-2m'-1}\right\rangle + \left\langle w^{(i)}_{d-2m-3}, Xw^{(i')}_{d-2m'-1}\right\rangle \\ &= (d-m-1)\left\langle w^{(i)}_{d-2m-1}, w^{(i')}_{d-2m'-1}\right\rangle + (d-m')\left\langle w^{(i)}_{d-1-2(m+1)}, w^{(i')}_{d-1-2(m'-1)}\right\rangle \\ &= (d-m-1)\left\langle w^{(i)}_{d-2m-1}, w^{(i')}_{d-2m'-1}\right\rangle + (m+1)\left\langle w^{(i)}_{d-1-2(m+1)}, w^{(i')}_{d-1-2(m'-1)}\right\rangle \\ &= (d-m-1)\binom{d-1}{m}(-1)^m(v^{(i)}, v^{(i')}) + (m+1)\binom{d-1}{m+1}(-1)^{m+1}(v^{(i)}, v^{(i')}) \\ &= 0, \text{ since } (d-m-1)\binom{d-1}{m} = (m+1)\binom{d-1}{m+1}\end{aligned}$$

Checking that $\langle \cdot, \cdot \rangle$ is invariant under Y is a similar calculation that we omit.

It is easy to see that the form $\langle \cdot, \cdot \rangle$ is non-degenerate by examining the matrix of the form with respect to the basis $\{w^{(j)}_{d-1-2m}\}$ of W. For a fixed m, j the expression $\left\langle w^{(j)}_{d-1-2m}, w^{(j')}_{d-1-2m'}\right\rangle$ will be non-zero for a unique choice of j', m' by construction of $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$. The matrix of the form $\langle \cdot, \cdot \rangle$ will have exactly one non-zero value in each row and column, and hence be invertible.

Hence $X \in \mathfrak{sp}(W)$, and X is a nilpotent of type (d^i) (this follows using the argument at the start of the proof of Proposition 1.7); this proves Proposition 1.9 in the case of rectangle partitions.

Now suppose $\lambda = (d_1^{i_1}, \cdots, d_r^{i_r})$. For $1 \leq j \leq r$, let W_j be a vector space of dimension $d_j i_j$, and let X_j be a nilpotent of type $d_j^{i_j}$ in End(W_j) which is invariant for a non-degenerate symplectic form $\langle \cdot, \cdot \rangle_j$ on W_j. Let W be the direct sum of the spaces W_j for $1 \leq j \leq r$, and let $\langle \cdot, \cdot \rangle$ be the non-degenerate symplectic form obtained as the orthogonal sum of the $\langle \cdot, \cdot \rangle_j$. Let X the sum of the matrices $\{X_j\}$. Then $X \in \mathfrak{sp}(W)$, and X will be a nilpotent of type λ, as required. □

Proof. (of Proposition 1.9)
Again we first prove the result for the case of rectangle partitions $\lambda = (d^i)$. Let X be a nilpotent element in $\mathfrak{sp}(W)$, for some vector space W of dimension di and a symplectic form $\langle \cdot, \cdot \rangle$. By the Jacobson-Morozov Theorem, we can find an $\mathfrak{sl}_2$-triple $\{X, Y, H\}$ in $\mathfrak{sp}(W)$. By the argument at the start of the proof of Proposition 1.7, W is the sum of i irreducible $\mathfrak{sl}_2$-modules of dimension d.

Let H_{d-1} denote the highest weight space in W, i.e. the $(d-1)$-weight space. Similarly to the proof of Proposition 1.7, define a form $(\cdot, \cdot)$ on H_{d-1} via $(v, w) = \left\langle v, \frac{Y^{d-1}}{(d-1)!} w \right\rangle$.

$$\begin{aligned}(w, v) = \left\langle w, \frac{Y^{d-1}}{(d-1)!} v \right\rangle &= (-1)^{d-1} \left\langle \frac{Y^{d-1}}{(d-1)!} w, v \right\rangle \\ = (-1)^d \left\langle v, \frac{Y^{d-1}}{(d-1)!} w \right\rangle &= (-1)^d (v, w)\end{aligned}$$

Thus $(\cdot, \cdot)$ is symmetric if d is even, and symplectic if d is odd. As proved in the proof of Proposition 1.7, $(\cdot, \cdot)$ is non-degenerate. In the case of d being even, pick an orthonormal basis of H_{d-1} for the form $(\cdot, \cdot)$, $\{w_{d-1}^{(1)}, \cdots, w_{d-1}^{(i)}\}$, so that $(w_{d-1}^{(i)}, w_{d-1}^{(j)}) = \delta_{ij}$. In the case of d being odd, pick a basis of H_{d-1}, $\{w_{d-1}^{(1)}, \cdots, w_{d-1}^{(i)}\}$, so that $(w_{d-1}^{(r)}, w_{d-1}^{(s)}) = 0$ if $|r - s| \neq 1$; $(w_{d-1}^{(r)}, w_{d-1}^{(r+1)}) = 1$ if r is odd and 0 otherwise; $(w_{d-1}^{(r+1)}, w_{d-1}^{(r)}) = -1$ if r is odd and 0 otherwise.

Define $w_{d-2m-1}^{(j)} = \frac{Y^m}{m!} w_{d-1}^{(j)}$ for $1 \leq m \leq d-1$, so that $\{w_{d-1}^{(j)}, \cdots, w_{1-d}^{(j)}\}$ is a string basis for an irreducible $\mathfrak{sl}_2$-module. Since the r-weight space is orthogonal under $\langle \cdot, \cdot \rangle$ to the s-weight space unless $r + s = 0$ (proven in the proof of Proposition 1.7):

$$\left\langle w_{d-2m-1}^{(j)}, w_{d-2m'-1}^{(j)} \right\rangle = 0 \text{ unless } m + m' = d - 1$$

If $m + m' = d - 1$:

$$\begin{aligned}\left\langle w^{(j)}_{d-2m-1}, w^{(j')}_{d-2m'-1}\right\rangle &= \left\langle \frac{Y^m}{m!} w^{(j)}_{d-1}, w^{(j')}_{d-2m'-1}\right\rangle \\ &= \frac{(-1)^m}{m!}\left\langle w^{(j)}_{d-1}, Y^m w^{(j')}_{1-d+2m}\right\rangle \\ &= \frac{(-1)^m (d-m)(d-m+1)\cdots(d-1)}{m!}\left\langle w^{(j)}_{d-1}, w^{(j')}_{1-d}\right\rangle \\ &= (-1)^m \binom{d-1}{m} (w^{(j)}_{d-1}, w^{(j')}_{d-1}) \text{ (since } w^{(j')}_{1-d} = \frac{Y^{d-1}}{(d-1)!} w^{(j')}_{d-1})\end{aligned}$$

In the above, $Y^m w^{(j')}_{1-d+2m} = (d-m)(d-m+1)\cdots(d-1)w^{(j')}_{1-d}$ since $Y w^{(j')}_{1-d+2m} = (d-m)w^{(j')}_{1-d+2m-2}$, $Y w^{(j')}_{1-d+2m-2} = (d-m+1)w^{(j')}_{1-d+2m-4}$, and so on until $Y w^{(j')}_{1-d+2} = (d-1)w^{(j')}_{1-d}$.

Hence we have proven above, that given two nilpotents X and X' of type (d^i) in $\mathfrak{sp}(W)$, we can construct bases of W, $\{w^{(j)}_{d-2m-1}\}$, $\{w'^{(j)}_{d-2m-1}\}$, with $0 \le m \le d-1, 1 \le j \le i$, such that the matrix of X with respect to the basis $\{w^{(j)}_{d-2m-1}\}$ is the same as the matrix of X' with respect to the basis $\{w'^{(j)}_{d-2m-1}\}$. Further, we have that:

$$\left\langle w^{(j)}_{d-2m-1}, w^{(j')}_{d-2m'-1}\right\rangle = \left\langle w'^{(j)}_{d-2m-1}, w'^{(j')}_{d-2m-1}\right\rangle$$

Let g be the matrix such that $g w^{(j)}_{d-2m-1} = w'^{(j)}_{d-2m-1}$. By the above condition, $g \in Sp(W)$; and the matrices X and X' are conjugate by g. This proves that any two nilpotents X and X' of type $(d^{(i)})$ are conjugate by g, proving the claim in the case of rectangle partitions.

In the case of an arbitrary partition $\lambda = (d_1^{i_1}, \cdots, d_r^{i_r})$, given a nilpotents of type λ, $X \in \mathfrak{sp}(W)$, by the Jacobson-Morozov Theorem we can find an $\mathfrak{sl}_2$-triple $\{X, Y, H\}$ in $\mathfrak{sp}(W)$. Let W_j be the sum of the i_j d_j-dimensional $\mathfrak{sl}_2$-submodules. By the Lemma 1.10, W_j is orthogonal under $\langle\cdot,\cdot\rangle$ to W_k if $j \neq k$. Now we can construct bases of the spaces W_j, and put them together to construct a basis of the space W, such that the matrix of X with respect to the basis, as well as the values of the form $\langle\cdot,\cdot\rangle$ applied to pairs of basis elements, depend solely on the partition λ (and not on X). Given another nilpotent X' of type λ, we can do the same thing. Then X' will be conjugate to X via the matrix g taking the first basis to the second, since the matrix of X with respect to the first basis is the same as the matrix of X' with respect to the second, and $g \in Sp(W)$ since the inner products of the corresponding pairs of basis elements is the same. □

Definition 1.11. *If $\lambda \in \mathcal{P}(2n)$ is a partition in which every odd part occurs with even multiplicity, then define $\mathcal{O}_\lambda$ to be the set of all nilpotents in $\mathfrak{sp}_{2n}(\mathbb{C})$ of type λ.*

1.2.3. Types B and D. Here we state the results without proofs. The proofs are similar in nature to that of type C.

Proposition 1.12. *Given any nilpotent $X \in \mathfrak{so}_{2n+1}(\mathbb{C})$, the Jordan type λ of X satisfies the condition that every even part in λ occurs with even multiplicity. Conversely, for any such partition λ of $2n+1$, there exists a nilpotent $X \in \mathfrak{so}_{2n+1}(\mathbb{C})$ with Jordan type λ. Any two such nilpotents $X, X' \in \mathfrak{so}_{2n+1}(\mathbb{C})$ are conjugate by an element of the special orthogonal group $SO_{2n+1}(\mathbb{C})$. Thus the nilpotent orbits in type B are in bijection with partitions of $2n+1$ in which every even part occurs with even multiplicity.*

Proposition 1.13. *Given any nilpotent $X \in \mathfrak{so}_{2n}(\mathbb{C})$, the Jordan type λ of X satisfies the condition that every even part in λ occurs with even multiplicity. Conversely, for any such partition λ of $2n$, there exists a nilpotent $X \in \mathfrak{so}_{2n}(\mathbb{C})$ with Jordan type λ. For such a partition λ of $2n$, the set of nilpotents of type λ are a single orbit under the action of $SO_{2n}(\mathbb{C})$; except when λ is a partition containing only even parts, in which case the set of nilpotents of type λ splits up into two orbits under the action of $SO_{2n}(\mathbb{C})$.*

1.3. Describing centralizers of nilpotent elements in type A

Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$. Recall that the representative n_λ for the nilpotent orbit $\mathcal{O}_\lambda$ takes the following form, where $N_i \in \mathrm{Mat}_n(\mathbb{C})$ is defined by $N_i = \sum_{j=1}^{n-1} e_{j,j+1}$ (as in Section 1.2.1).

$$n_\lambda = \begin{pmatrix} N_{\lambda_1} & & & \\ & N_{\lambda_2} & & \\ & & \cdots & \\ & & & N_{\lambda_k} \end{pmatrix}$$

First we compute the centralizer $C_{M_n(\mathbb{C})}(n_\lambda)$ of n_λ in the matrix algebra $M_n(\mathbb{C})$, then the centralizer $C_{GL_n(\mathbb{C})}(n_\lambda)$ in $GL_n(\mathbb{C})$.

Proposition 1.14. *Given $x \in M_n(\mathbb{C})$, let x consist of k^2 blocks $x_{i,j}$, where $x_{i,j}$ has size $\lambda_i \times \lambda_j$, as follows.*

$$x = \begin{pmatrix} x_{11} & & x_{1k} \\ x_{21} & & x_{2k} \\ & \cdots & \\ x_{k1} & & x_{kk} \end{pmatrix}$$

Then $x \in C_{M_n(\mathbb{C})}(n_\lambda)$ precisely when the blocks $x_{i,j}$ take the following form. If $\lambda_i \geq \lambda_j$, the entries must satisfy the following (here $1 \leq k \leq \lambda_i, 1 \leq l \leq \lambda_j$):

$$x_{ij,kl} = 0 \text{ if } k > l$$
$$x_{ij,kl} = x_{ij,1,l-k+1} \text{ if } k \leq l$$

If $\lambda_i < \lambda_j$, the entries must satisfy the following (here $1 \leq k \leq \lambda_i,\ 1 \leq l \leq \lambda_j$):

$$x_{ij,kl} = 0 \text{ if } l - k < \lambda_j - \lambda_i$$
$$x_{ij,kl} = x_{ij,1,l-k+1} \text{ otherwise}$$

Proof. Since $x \in C_{M_n(\mathbb{C})}(n_\lambda)$, we have $xn_\lambda = n_\lambda x$:

$$\begin{pmatrix} x_{11} & & x_{1k} \\ x_{21} & & x_{2k} \\ & ... & \\ x_{k1} & & x_{kk} \end{pmatrix} \begin{pmatrix} N_{\lambda_1} & & & \\ & N_{\lambda_2} & & \\ & & ... & \\ & & & N_{\lambda_k} \end{pmatrix} =$$
$$\begin{pmatrix} N_{\lambda_1} & & & \\ & N_{\lambda_2} & & \\ & & ... & \\ & & & N_{\lambda_k} \end{pmatrix} \begin{pmatrix} x_{11} & & x_{1k} \\ x_{21} & & x_{2k} \\ & ... & \\ x_{k1} & & x_{kk} \end{pmatrix}$$

Simplifying this expression,

$$\begin{pmatrix} N_{\lambda_1}x_{11} & & N_{\lambda_1}x_{1k} \\ N_{\lambda_2}x_{21} & & N_{\lambda_2}x_{2k} \\ & \dots & \\ N_{\lambda_k}x_{k1} & & N_{\lambda_k}x_{kk} \end{pmatrix} = \begin{pmatrix} x_{11}N_{\lambda_1} & & x_{1k}N_{\lambda_k} \\ x_{21}N_{\lambda_1} & & x_{2k}N_{\lambda_k} \\ & \dots & \\ x_{k1}N_{\lambda_1} & & x_{kk}N_{\lambda_k} \end{pmatrix}$$

Thus we require that $N_{\lambda_i}x_{ij} = x_{ij}N_{\lambda_j}$, for all $i, j \in \{1, \cdots, k\}$.

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & ... & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \begin{pmatrix} x_{ij,11} & & x_{ij,1\lambda_j} \\ & ... & \\ x_{ij,\lambda_i 1} & & x_{ij,\lambda_i\lambda_j} \end{pmatrix} =$$
$$\begin{pmatrix} x_{ij,11} & & x_{ij,1\lambda_j} \\ & ... & \\ x_{ij,\lambda_i 1} & & x_{ij,\lambda_i\lambda_j} \end{pmatrix} \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & ... & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

Expanding the above expression gives the following:

$$(1.15)\qquad \begin{pmatrix} x_{ij,21} & & x_{ij,2\lambda_j} \\ & \dots & \\ x_{ij,\lambda_i 1} & & x_{ij,\lambda_i\lambda_j} \\ 0 & & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_{ij,11} & & x_{ij,1,\lambda_j-1} \\ & & \dots & \\ 0 & x_{ij,\lambda_i 1} & & x_{ij,\lambda_i,\lambda_j-1} \end{pmatrix}$$

Case 1: If $\lambda_i \geq \lambda_j$, equating the entries in (1.15) now gives the following; here $1 \leq k \leq \lambda_i, 1 \leq l \leq \lambda_j$:

$$x_{ij,kl} = 0 \text{ if } k > l$$
$$x_{ij,kl} = x_{ij,1,l-k+1} \text{ if } k \leq l$$

To see this, we can illustrate with an example where $\lambda_i = 4, \lambda_j = 3$:

$$\begin{pmatrix} x_{ij,21} & x_{ij,22} & x_{ij,23} \\ x_{ij,31} & x_{ij,32} & x_{ij,33} \\ x_{ij,41} & x_{ij,42} & x_{ij,43} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_{ij,11} & x_{ij,12} \\ 0 & x_{ij,21} & x_{ij,22} \\ 0 & x_{ij,31} & x_{ij,32} \\ 0 & x_{ij,41} & x_{ij,42} \end{pmatrix}$$

$$x_{ij,11} = x_{ij,22} = x_{ij,33}, x_{ij,12} = x_{ij,23}$$
$$x_{ij,21} = x_{ij,32} = x_{ij,43} = 0, x_{ij,31} = x_{ij,42} = 0, x_{ij,41} = 0$$

$$x_{ij} = \begin{pmatrix} x_{ij,11} & x_{ij,12} & x_{ij,13} \\ 0 & x_{ij,11} & x_{ij,12} \\ 0 & 0 & x_{ij,11} \\ 0 & 0 & 0 \end{pmatrix}$$

Case 2: If $\lambda_i < \lambda_j$, equating the entries in (1.15) gives the following (here $1 \leq k \leq \lambda_i,\ 1 \leq l \leq \lambda_j$):

$$x_{ij,kl} = 0 \text{ if } l - k < \lambda_j - \lambda_i$$
$$x_{ij,kl} = x_{ij,1,l-k+1} \text{ otherwise}$$

To see this, consider the below example where $\lambda_i = 3, \lambda_j = 4$:

$$\begin{pmatrix} x_{ij,21} & x_{ij,22} & x_{ij,23} & x_{ij,24} \\ x_{ij,31} & x_{ij,32} & x_{ij,33} & x_{ij,34} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_{ij,11} & x_{ij,12} & x_{ij,13} \\ 0 & x_{ij,21} & x_{ij,22} & x_{ij,23} \\ 0 & x_{ij,31} & x_{ij,32} & x_{ij,33} \end{pmatrix}$$

$$x_{ij,11} = x_{ij,22} = x_{ij,33} = 0$$
$$x_{ij,21} = x_{ij,32} = 0, x_{ij,31} = 0$$
$$x_{ij,12} = x_{ij,23} = x_{ij,34}, x_{ij,13} = x_{ij,24}$$

$$x_{ij} = \begin{pmatrix} 0 & x_{ij,12} & x_{ij,13} & x_{ij,14} \\ 0 & 0 & x_{ij,12} & x_{ij,13} \\ 0 & 0 & 0 & x_{ij,12} \end{pmatrix}$$

□

Now that we know the structure of the matrices x_{ij} which form the matrix x, effectively we have computed the centralizer $C_{M_n(\mathbb{C})}(n_\lambda)$ of n_λ inside the matrix algebra $M_n(\mathbb{C})$. As an example of what this centralizer looks like in practice, we examine the example $\lambda = (3, 2^2, 1)$. A typical element $x \in C_{M_n(\mathbb{C})}(n_\lambda)$ has the following form:

$$\begin{pmatrix} x_{11,11} & x_{11,12} & x_{11,13} & x_{12,11} & x_{12,12} & x_{13,11} & x_{13,12} & x_{14,11} \\ & x_{11,11} & x_{11,12} & & x_{12,11} & & x_{13,11} & \\ & & x_{11,11} & & & & & \\ & x_{21,12} & x_{21,13} & x_{22,11} & x_{22,12} & x_{23,11} & x_{23,12} & x_{24,11} \\ & & x_{21,12} & & x_{22,11} & & x_{23,11} & \\ & x_{31,12} & x_{31,13} & x_{32,11} & x_{32,12} & x_{33,11} & x_{33,12} & x_{34,11} \\ & & x_{31,12} & & x_{32,11} & & x_{33,11} & \\ & & x_{41,13} & & x_{42,12} & & x_{43,12} & x_{44,11} \end{pmatrix}$$

Now we will compute the centralizer $C_{GL_n(\mathbb{C})}(n_\lambda)$ by describing precisely when a matrix $x \in C_{M_n(\mathbb{C})}(n_\lambda)$ is invertible.

Proposition 1.16. *Let $x \in C_{M_n(\mathbb{C})}(n_\lambda)$, and let $\lambda = (a_1^{b_1}, a_2^{b_2}, \cdots, a_t^{b_t})$. Define the following square matrices, where $c_s = b_1 + \cdots + b_s$, and $0 \leq s \leq t-1$. Then $x \in C_{GL_n(\mathbb{C})}(n_\lambda)$ precisely when the matrices x_s are invertible.*

$$x_{s+1} = \begin{pmatrix} x_{c_s+1,c_s+1,11} & \cdots & x_{c_s+1,c_s+b_{s+1},11} \\ \vdots & & \vdots \\ x_{c_s+b_{s+1},c_s+1,11} & \cdots & x_{c_s+b_{s+1},c_s+b_{s+1},11} \end{pmatrix}$$

Proof. We start with the above example $\lambda = (3, 2^2, 1)$. We may consider the rows and columns as being labelled by the set $\{e_{1,1}, e_{1,2}, e_{1,3}, e_{2,1}, e_{2,2}, e_{3,1}, e_{3,2}, e_{4,1}\}$. Consider re-labelling the rows and columns instead with the set $\{e_{1,1}, e_{2,1}, e_{3,1}, e_{4,1}, e_{1,2}, e_{2,2}, e_{3,2}, e_{1,3}\}$. The matrix now takes the following form:

$$\begin{pmatrix} x_{11,11} & x_{12,11} & x_{13,11} & x_{14,11} & x_{11,12} & x_{12,12} & x_{13,12} & x_{11,13} \\ & x_{22,11} & x_{23,11} & x_{24,11} & x_{21,12} & x_{22,12} & x_{23,12} & x_{21,13} \\ & x_{32,11} & x_{33,11} & x_{34,11} & x_{31,12} & x_{32,12} & x_{33,12} & x_{31,13} \\ & & & x_{44,11} & & x_{42,12} & x_{43,12} & x_{41,13} \\ & & & & x_{11,11} & x_{12,11} & x_{13,11} & x_{11,12} \\ & & & & & x_{22,11} & x_{23,11} & x_{21,12} \\ & & & & & x_{32,11} & x_{33,11} & x_{31,12} \\ & & & & & & & x_{11,11} \end{pmatrix}$$

The matrix is now block upper-triangular and will be invertible if and only if $x_{11,11} \neq 0, x_{44,11} \neq 0$, and the following matrix is invertible:

$$\begin{pmatrix} x_{22,11} & x_{23,11} \\ x_{32,11} & x_{33,11} \end{pmatrix}$$

The above example makes it clear that in the general case $\lambda = (a_1^{b_1}, a_2^{b_2}, \cdots, a_t^{b_t})$, after re-labeling the rows and columns of the matrix x as indicated above (so that $e_{i,k}$ precedes $e_{j,l}$ either if $k < l$ or if $k = l$ and $i < j$) to obtain a matrix x', the following square matrices (as defined above) will occur on the diagonal of x'. Here $c_s = b_1 + \cdots + b_s$, where $0 \leq s \leq t-1$.

$$x_{s+1} = \begin{pmatrix} x_{c_s+1,c_s+1,11} & \cdots & x_{c_s+1,c_s+b_{s+1},11} \\ \vdots & & \vdots \\ x_{c_s+b_{s+1},c_s+1,11} & \cdots & x_{c_s+b_{s+1},c_s+b_{s+1},11} \end{pmatrix}$$

The matrix x_s will occur a_s times. The entry in the row corresponding to $e_{i,k}$ and column corresponding to $e_{j,l}$ will occur in one of the copies of the matrix x_s precisely when $k = l$ and $\lambda_i = \lambda_j$. It remains to prove that all the other entries of the matrix x' will lie in the upper half of the matrix, so that x' will be block upper-triangular with blocks x_s; it will then follow that x is invertible precisely when all the matrices x_s are invertible.

First consider an entry $x_{ij,kl}$ of the matrix x' when $i \leq j$ (so $\lambda_i \geq \lambda_j$). The entry $x_{ij,kl}$ lies in the row corresponding to $e_{i,k}$, and the column corresponding to $e_{j,l}$. As shown above, the entry $x_{ij,kl}$ can only be non-zero if $k \leq l$. If $k < l$, then $e_{i,k}$ will precede $e_{j,l}$ in the labelling of the rows and columns in x', and hence $x_{ij,kl}$ will lie in the upper half of x'. If $k = l$ and $i < j$, again $e_{i,k}$ will precede $e_{j,l}$ in the labelling of the rows and columns, and $x_{ij,kl}$ will lie in the upper half of x'. If $k = l$ and $i = j$, then $e_{i,k} = e_{j,l}$, and $x_{ij,kl}$ will lie on the diagonal of the matrix x', and will lie in one of the matrices x_s.

Next consider an entry $x_{ij,kl}$ of the matrix x', when $i > j$ (so $\lambda_i \leq \lambda_j$). The entry $x_{ij,kl}$ will lie in the row corresponding to $e_{i,k}$, and the column corresponding to $e_{j,l}$. As proven above $x_{ij,kl}$ can only be nonzero if $l - k \geq \lambda_j - \lambda_i$. If $k < l$, then $e_{i,k}$ will precede $e_{j,l}$ in the labelling of the rows and columns in x', and hence $x_{ij,kl}$ will lie in the upper half of x'. If $k = l$, then $0 \geq \lambda_j - \lambda_i \geq 0$, so $\lambda_i = \lambda_j$. In this case the entry $x_{ij,kl}$ will occur in one of the matrices x_s.

This shows that every non-zero entry in x' occurs in the upper half, or in of the diagonal blocks x_s, proving that x' is block upper triangular with blocks x_s. It follows that x is invertible precisely when all the matrices x_s are invertible. □

We now compute the dimension of the orbit $\mathcal{O}_\lambda$. We first compute the dimension of the centralizer $C_{GL_n(\mathbb{C})}(n_\lambda)$ in the following two Lemmas. Given a partition λ, define its transpose partition μ by letting μ_i be $\#\{j|\lambda_j \geq i\}$. Diagramatically, the transpose partition μ is obtained from λ by reflecting λ along the diagonal containing the first box in the first row, the second box in the second row, and so on.

Lemma 1.17. *Given a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ and its transpose partition $\mu = (\mu_1, \mu_2, \cdots, \mu_l)$, the following identity holds:*

$$\sum_{j=1}^{k}(2j-1)\lambda_j = \sum_{j=1}^{l}\mu_j^2$$

Proof. Proof by induction on the number of parts of μ.

Suppose $\lambda_1 = \lambda_2 = \cdots = \lambda_i > \lambda_{i+1}$, so $\mu_l = i$. Consider the partition $\lambda' = (\lambda_1 - 1, \cdots, \lambda_i - 1, \lambda_{i+1}, \cdots, \lambda_k)$. By induction the result is true for λ':

$$(\lambda_1 - 1) + \cdots + (2i-1)(\lambda_i - 1) + (2i+1)\lambda_{i+1} + \cdots + (2k-1)\lambda_k = \sum_{j=1}^{l-1}\mu_j^2$$

Adding $\mu_l^2 = i^2 = 1 + 3 + \cdots + (2i-1)$ to each side now gives the required identity for λ':

$$\begin{aligned}\sum_{j=1}^{k}(2j-1)\lambda_j &= \lambda_1 + \cdots + (2i-1)\lambda_i + \cdots + (2k-1)\lambda_k \\ &= \mu_1^2 + \mu_2^2 + \cdots + \mu_l^2\end{aligned}$$

□

Lemma 1.18. *The dimension of the centralizer $C_{GL_n(\mathbb{C})}(n_\lambda)$ is $\sum_{j=1}^{l}\mu_j^2$.*

Proof. The dimension of the centralizer is clearly equal to the number of free variables $x_{ij,kl}$ in the description of the centralizer. We use the example $\lambda = (3, 2^2, 1)$ to illustrate the ideas. In this case, a typical element x of the centralizer $C_{GL_n(\mathbb{C})}(n_\lambda)$ takes the following form:

$$\begin{pmatrix}
x_{11,11} & x_{11,12} & x_{11,13} & x_{12,11} & x_{12,12} & x_{13,11} & x_{13,12} & x_{14,11} \\
 & x_{11,11} & x_{11,12} & & x_{12,11} & & x_{13,11} & \\
 & & x_{11,11} & & & & & \\
 & x_{21,12} & x_{21,13} & x_{22,11} & x_{22,12} & x_{23,11} & x_{23,12} & x_{24,11} \\
 & & x_{21,12} & & x_{22,11} & & x_{23,11} & \\
 & x_{31,12} & x_{31,13} & x_{32,11} & x_{32,12} & x_{33,11} & x_{33,12} & x_{34,11} \\
 & & x_{31,12} & & x_{32,11} & & x_{33,11} & \\
 & & x_{41,13} & & x_{42,12} & & x_{43,12} & x_{44,11}
\end{pmatrix}$$

The matrix x consists of blocks x_{ij} for $1 \leq i, j \leq k$. The number of free variables in the block $x_{i,j}$ is $\min(\lambda_i, \lambda_j)$. Hence the dimension of the centralizer is equal to the following sum:

$$\sum_{1 \leq i,j \leq k} \min(\lambda_i, \lambda_j)$$

In the above sum, the term λ_j occurs with multiplicity $2j-1$, since $\min(\lambda_i, \lambda_j) = \lambda_j$ when $i \leq j$, so $\lambda_j = \min\{\lambda_p, \lambda_q\}$ when $(p, q) \in \{(1, j), (2, j), \cdots, (j, j), (j, j-1), \cdots, (j, 1)\}$. Hence the dimension of the centralizer is equal to $\sum_{j=1}^{k}(2j-1)\lambda_j$; the conclusion now follows from Lemma 2.8. $\square$

Given the standard nilpotent $n_\lambda \in \mathcal{O}_\lambda$, clearly $\mathcal{O}_\lambda = G.n_\lambda$. Hence $\dim(G.n_\lambda) = \dim(G) - \dim(\mathrm{Stab}(n_\lambda))$. Since the stabilizer of n_λ in G is simply the centralizer $C_{GL_n(\mathbb{C})}(n_\lambda)$, using Lemma 1.18 we now compute:

$$\begin{aligned}
\dim(\mathcal{O}_\lambda) &= \dim(G) - \dim(C_{GL_n(\mathbb{C})}(n_\lambda)) \\
&= n^2 - \sum_{i=1}^{l} \mu_i^2
\end{aligned}$$

CHAPTER 2

A resolution of singularities and the closure ordering for nilpotent orbits $\mathcal{O}_\lambda$

In this section, given a nilpotent orbit $\mathcal{O}_\lambda$ we describe how to construct a resolution of singularities for the algebraic variety $\overline{\mathcal{O}_\lambda}$ (here $\overline{\mathcal{O}_\lambda}$ denotes the closure of the quasi-affine algebraic variety $\mathcal{O}_\lambda$). As a consequence of this resolution of singularities, we derive the closure ordering on the nilpotent orbits $\mathcal{O}_\lambda$. The resolution of singularities for the general linear group (type A) and for the symplectic group (type C) are somewhat different in nature.

2.1. Resolution of singularities in type A

Let $x \in \mathfrak{gl}_n(\mathbb{C})$ be nilpotent, corresponding to a partition $\lambda \in \mathcal{P}(n)$. In this section, we will construct a resolution of singularities for the algebraic variety $\overline{\mathcal{O}_\lambda} = \overline{Gx}$, where $G = GL_n(\mathbb{C})$.

Consider the following flag of vector spaces inside $V \cong \mathbb{C}^n$ (V being the vector space of dimension n on which $\mathfrak{gl}_n(\mathbb{C})$ acts):

$$0 \subset \ker(x) \subset \ker(x^2) \subset \cdots \subset \ker(x^m) = V$$

Define P to be the parabolic subgroup of the Lie group $GL_n(\mathbb{C})$ stabilizing the above flag:

$$P := \{u \in GL_n(\mathbb{C}) : u(\ker x^i) \subset \ker x^i\}$$

Define $\mathfrak{p}$ to be the Lie algebra of P, i.e. the parabolic Lie algebra stabilizing the above flag:

$$\mathfrak{p} := \{u \in \mathfrak{gl}_n(\mathbb{C}) : u(\ker x^i) \subset \ker x^i\}$$

Define $\mathfrak{n}$ to be the Lie algebra moving each vector space in the above flag into the one below:

$$\mathfrak{n} := \{u \in \mathfrak{gl}_n(\mathbb{C}) : u(\ker x^i) \subset \ker x^{i-1}\}$$

One can show that $\mathfrak{n}$ is the nil-radical of the parabolic Lie algebra P. The below example shows the structure of $\mathfrak{n}$ and $\mathfrak{p}$ for some specific nilpotent elements:

Example 2.1. Consider the partition $\lambda = (3,2,1)$. Pick x to be the following matrix, of type $(3,2,1)$. In this example will compute $\mathfrak{p}$ and $\mathfrak{n}$.

$$\begin{pmatrix} 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & 0 & & & \\ & & & 0 & & 1 \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

Then by calculation, we see the following:

$$\ker(x) = \left\{ \begin{pmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad \ker(x^2) = \left\{ \begin{pmatrix} * \\ * \\ * \\ * \\ * \\ 0 \end{pmatrix} \right\}$$

We claim that $\mathfrak{p}$ and $\mathfrak{n}$ are given by the following:

$$\mathfrak{p} = \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ & & & * & * & * \\ & & & * & * & * \\ & & & & & * \end{pmatrix}, \quad \mathfrak{n} = \begin{pmatrix} 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ & & & 0 & 0 & * \\ & & & 0 & 0 & * \\ & & & & & 0 \end{pmatrix}$$

In the above, the corresponding parabolic subgroup P of $GL_n(\mathbb{C})$ will be the intersection of $\mathfrak{p}$ with $GL_n(\mathbb{C})$, as sets.

To see that $\mathfrak{p}$ and $\mathfrak{n}$ have this form, it is clear by inspection that for all $u \in \mathfrak{p}$, $u(\ker(x)) \subset \ker(x)$, $u(\ker(x^2)) \subset \ker(x^2)$, and that for all $v \in \mathfrak{n}$, $v(\ker(x)) = 0, v(\ker(x^2)) \subset \ker(x)$. It remains to see that there are no other matrices in $\mathfrak{gl}_n(\mathbb{C})$ satisfying this property. The first column of a matrix u in $\mathfrak{p}$ is equal to ue_1, and hence must lie in $\ker(x)$. By a similar argument, the second and third columns of u must lie in $\ker(x)$, the fourth and fifth columns must lie in $\ker(x^2)$, and the last column of u must lie in $\ker(x^3)$. The first column of a matrix v in $\mathfrak{n}$ is equal to ve_1 and hence must be 0. By a similar argument, the second and third columns of v must be 0, the fourth and fifth columns of v must lie in $\ker(x)$, and the last column of v must lie in $\ker(x^2)$. This shows that $\mathfrak{p}$ and $\mathfrak{n}$ have the form specified above.

In general, given a nilpotent element x corresponding to a partition λ, the following result gives the dimensions of the spaces $\ker(x^i)$.

Lemma 2.2. *If $\lambda = (\lambda_1, \cdots, \lambda_k)$, suppose $\mu = \lambda^t$ is the transpose partition with parts $\mu = (\mu_1, \cdots, \mu_l)$. Then for $0 \le s \le l$, $\dim(\ker(x^s)) = \mu_1 + \cdots + \mu_s$, and for $s > l$, $\dim(\ker(x^s)) = n$.*

Proof. Since $x \in \mathcal{O}_\lambda$, there exists a basis $e_{i,j}$ of V, with $1 \leq i \leq k,\ 1 \leq j \leq \lambda_i$, such that $xe_{i,j} = e_{i,j-1}$ if $j \geq 1$, and $xe_{i,1} = 0$. Then it follows by a quick induction that $x^s e_{i,j} = 0$ if $j \leq s$, and $x^s e_{i,j} = e_{i,j-s}$ if $j > s$. We claim that ker(x^s) is spanned by $\{e_{i,j} | j \leq s\}$. To prove this, it is clear that the vectors $\{e_{i,j} | j \leq s\}$ are linearly independent and lie inside ker(x^s). Conversely, suppose $a \in$ ker(x^s), where $a = \sum_{i,j} a_{i,j} e_{i,j}$. Then:

$$\begin{aligned} x^s a &= \sum_{i,j} a_{i,j} x^s e_{i,j} \\ &= \sum_{i;j>s} a_{i,j} e_{i,j-s} = 0 \end{aligned}$$

Hence if $j > s$, $a_{i,j} = 0$, so $a = \sum_{i;j \leq s} a_{i,j} e_{i,j}$, and a lies in the span of the vectors $\{e_{i,j} | j \leq s\}$. This proves that ker(x^s) is spanned by $\{e_{i,j} | j \leq s\}$. It follows that the basis vectors spanning ker(x^s) consists of the basis vectors spanning ker(x^{s-1}), together with $\{e_{i,j} | j = s\}$. The cardinality of the set $\{e_{i,j} | j = s\}$ is equal to the maximum value of i for which $\lambda_i \geq s$; this is in turn equal to μ_s. Thus dim(ker(x^s)) $-$ dim(ker(x^{s-1})) $= \mu_s$. It then follows by a quick induction that dim(ker(x^s)) $= \mu_1 + \cdots + \mu_s$. If $s > l$, then $\mu_{l+1} = \cdots = \mu_s = 0$, so dim(ker$(x^s)$) $= \mu_1 + \cdots + \mu_l = n$.

□

Now we will compute dim $\mathfrak{p}$ and dim $\mathfrak{n}$. From Example 2.1 and Lemma 2.2, it is clear that for a suitable choice of $x \in \mathcal{O}_\lambda$ (so that ker(x^i) will consist of the vectors in V with the first $\mu_1 + \cdots + \mu_i$ co-ordinates non-zero), $\mathfrak{p}$ will consists of matrices that are block upper-triangular, with blocks of sizes $\mu_1, \cdots, \mu_l$. The subalgebra $\mathfrak{n}$ of $\mathfrak{p}$ will consist of those matrices in $\mathfrak{p}$ with all diagonal blocks equal zero.

$$\begin{aligned} \dim(\mathfrak{p}) &= \sum_{i \geq j} \mu_i \mu_j = \frac{1}{2} \sum_{i=1}^{l} \mu_i^2 + \frac{1}{2} (\sum_{i=1}^{l} \mu_i^2 + 2 \sum_{i>j} \mu_i \mu_j) \\ &= \frac{1}{2} \sum_{i=1}^{l} \mu_i^2 + \frac{1}{2} (\sum_{i=1}^{l} \mu_i)^2 = \frac{1}{2} \sum_{i=1}^{l} \mu_i^2 + \frac{1}{2} n^2 \\ \dim(\mathfrak{n}) &= \dim(\mathfrak{p}) - \sum_{i=1}^{l} \mu_i^2 = \frac{1}{2} n^2 - \frac{1}{2} \sum_{i=1}^{l} \mu_i^2 \end{aligned}$$

Definition 2.3. *Let $G \times_P \mathfrak{n}$ to be the quotient of the Cartesian product space $G \times \mathfrak{n}$ by the equivalence relation $(g, y) \sim (gp^{-1}, p.y) = (gp^{-1}, pyp^{-1})$, where $p \in P$.*

Recall that the quotient G/P is the partial flag variety, under the identification $gP \to \{0 \subset g$ ker$(x) \subset g$ ker$(x^2) \subset \cdots \subset g$ ker$(x^l)\}$:

$$G/P = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_l = V | \dim(V_i) = \dim(\ker(x^i))\}$$

The partial flag variety is a projective variety, since there is a natural inclusion of G/P into the product of the Grassmanians $Gr_{a_1}(V) \times Gr_{a_2}(V) \times \cdots \times Gr_{a_{l-1}}(V)$ (where $a_i = \dim(\ker(x^i))$), and each Grassmanian is a projective variety.

Proposition 2.4. *Define the map $\phi : G \times_P \mathfrak{n} \to G/P \times \mathfrak{gl}_n(\mathbb{C})$ by $\phi(g, y) = (gP, gyg^{-1})$. Then the map ϕ is well-defined and injective, hence giving $G \times_P \mathfrak{n}$ the structure of an algebraic variety (as a closed subvariety of $G/P \times \mathfrak{gl}_n(\mathbb{C})$). Further, im$(\phi) = \{((V_i), y) | yV_i \subset V_{i-1}\}$.*

Proof. To check that ϕ is well-defined, it suffices to check that $\phi(g, y) = \phi(gp^{-1}, pyp^{-1})$. This is a straightforward calculation:

$$\begin{aligned}\phi(gp^{-1}, pyp^{-1}) &= (gp^{-1}P, (gp^{-1})(pyp^{-1})(gp^{-1})^{-1}) \\ &= (gP, (gp^{-1})(pyp^{-1})(pg^{-1})) \\ &= (gP, gyg^{-1}) = \phi(g, y)\end{aligned}$$

To check that ϕ is injective, suppose $\phi(g_1, y_1) = \phi(g_2, y_2)$, so that $(g_1P, g_1y_1g_1^{-1}) = (g_2P, g_2y_2g_2^{-1})$. Since $g_1P = g_2P$, $p = g_2^{-1}g_1 \in P$. Since $g_1y_1g_1^{-1} = g_2y_2g_2^{-1}$, $g_2^{-1}g_1y_1g_1^{-1}g_2 = y_2$ so $py_1p^{-1} = y_2$. Hence $(g_2, y_2) = (g_1p^{-1}, py_1p^{-1}) \sim (g_1, y_1)$ by definition of the equivalence relation in $G \times_P \mathfrak{n}$.

It is clear from the definition of ϕ that im$(\phi) = \{(gP, y) : g^{-1}yg \in \mathfrak{n}\}$. Since this is a closed condition, this shows that $\mathrm{im}(\phi)$ is a closed subvariety of $G/P \times \mathfrak{gl}_n(\mathbb{C})$. The element $gP \in G/P$ corresponds to the partial flag $\{0 \subset g\ker(x) \subset \cdots \subset g\ker(x^l)\}$. Since $g^{-1}yg \in \mathfrak{n}$, $g^{-1}yg\ker(x^{i+1}) \subset \ker(x^i)$, so then $yg\ker(x^{i+1}) \subset g\ker(x^i)$. Hence any $(gP, y) \in \mathrm{im}(\phi)$ corresponds to an element $((V_i), y)$ with $yV_i \subset V_{i-1}$. Conversely, given some $((V_i), y)$ such that $yV_i \subset V_{i-1}$, suppose for each i, $V_i = g\ker(x^i)$ for some $g \in G$. Then $yg\ker(x^i) \subset g\ker(x^{i-1})$, so $g^{-1}yg\ker(x^i) \subset \ker(x^{i-1})$ for each i, and $g^{-1}yg \in \mathfrak{n}$. Hence $((V_i), y)$ corresponds to (gP, y) where $g^{-1}yg \in \mathfrak{n}$, and hence $((V_i), y)$ lies in im(ϕ) if $yV_i \subset V_{i-1}$. This proves that im$(\phi) = \{((V_i), y) | yV_i \subset V_{i-1}\}$. Since ϕ is injective, $G \times_P \mathfrak{n}$ can be identified with this variety.

□

Corollary 2.5. *$G \times_P \mathfrak{n}$ is a vector bundle over G/P, with fibres isomorphic to $\mathfrak{n}$.*

Proof. By Proposition 2.4, identify $G \times_P \mathfrak{n}$ with the variety $\{((V_i), y) |\ yV_i \subset V_{i-1}\}$. Let p_1 denote the projection onto the first factor; p_1 is clearly a surjective map onto G/P. Given a fixed element $gP = (V_i) \in G/P$, the fibre $p_1^{-1}(V_i)$ is $\{y \in \mathfrak{gl}_n(\mathbb{C}) | yV_i \subset V_{i-1}\}$. This fibre is simply a conjugate of the Lie algebra $\mathfrak{n}$, and hence is isomorphic to $\mathfrak{n}$ as vector spaces. □

Define the map $\pi : G \times_P \mathfrak{n} \to \mathfrak{gl}_n(\mathbb{C})$, by $\pi(g, y) = gyg^{-1}$. As proved in Proposition 2.4, π is well-defined. If we instead consider $G \times_P \mathfrak{n}$ as the set $\{((V_i), y) | yV_i \subset V_{i-1}\}$, the map π is projection onto the second factor.

Corollary 2.6. *The map π is a projective morphism of varieties.*

Proof. By definition, a map $\pi : X \to Y$ is a projective morphism if we can find a projective variety Z, and a closed embedding $\phi : X \to Z \times Y$, so that $\pi = p_2 \circ \phi$, where $p_2 : Z \times Y \to Y$ is the projection onto the second factor. In this case, let $Z = G/P$ be the partial flag variety, and let $\phi : G \times_P \mathfrak{n} \to G/P \times \mathfrak{gl}_n(\mathbb{C})$ be the map from Proposition 2.4. It is clear that Z is a projective variety. Since ϕ is an injective map (proven in Proposition 2.4), whose image is a closed-subvariety of $G/P \times \mathfrak{gl}_n(\mathbb{C})$ (namely the set $\{(gP, y) | g^{-1}yg \in \mathfrak{n}\}$), ϕ is a closed embedding. It is clear that $p_2 \circ \phi = \pi$, since $p_2(\phi(g, y)) = p_2(gP, gyg^{-1}) = gyg^{-1} = \pi(g, y)$. This proves that π is a projective morphism. □

Now we can state the first main result of this section: the map π is a resolution of singularities for the orbit closure $\overline{\mathcal{O}_\lambda}$. At this point, it is hardly clear the above map even surjects onto $\overline{\mathcal{O}_\lambda}$; this, among a couple other things are what needs to be verified in order to check that this map is a resolution of singularities.

Theorem 2.7. *The map $\pi : G \times_P \mathfrak{n} \to \mathfrak{gl}_n(\mathbb{C})$ is a resolution of singularities for the orbit closure $\mathcal{O}_\lambda$. Equivalently, the following four statements are true:*

(i) The image of the map π in $\mathfrak{gl}_n(\mathbb{C})$ is precisely the orbit closure $\overline{\mathcal{O}_\lambda}$.

(ii) The map π is injective when restricted to $\pi^{-1}(\mathcal{O}_\lambda)$. The inverse map, when restricted to $\mathcal{O}_\lambda$, $\pi^{-1} : \mathcal{O}_\lambda \to G \times_P \mathfrak{n}$ is a morphism of algebraic varieties.

(iii) As an algebraic variety, $G \times_P \mathfrak{n}$ is smooth and irreducible.

(iv) The map π is a proper morphism of algebraic varieties.

Proof. (of Theorem 2.7 (iii), (iv))
The variety $G \times_P \mathfrak{n}$ is irreducible, since it is a vector bundle over the partial flag variety G/P, and the partial flag variety is an irreducible variety. The variety $G \times_P \mathfrak{n}$ is smooth, since it is a vector bundle over the G/P, and G/P is smooth as it is a homogenous space.

The fact that π is a proper morphism of varieties follows from the fact that it is a projective morphism of varieties (Corollary 2.6).

□

Proof. (of Theorem 2.7 part (i))
The image of the map π contains $\mathcal{O}_\lambda$, since given $n \in \mathcal{O}_\lambda$, $n = gxg^{-1}$ for some $g \in GL_n(\mathbb{C})$, so if we know that $x \in \mathfrak{n}$, then $\pi(g, x) = n$. To verify that $x \in \mathfrak{n}$, using the definition of $\mathfrak{n}$, it suffices to verify that $x(\ker(x^i)) \subset (\ker(x^{i-1}))$. This is

clear, since if $u \in (\ker(x^i))$, $x^i u = x^{i-1}(xu) = 0$; i.e. $xu \in (\ker(x^{i-1}))$.

Since the map π is a projective morphism, the image of π is a closed subvariety of $\mathfrak{gl}_n(\mathbb{C})$. Since the image of π contains $\mathcal{O}_\lambda$, it follows that the image of π contains $\overline{\mathcal{O}_\lambda}$. As proven above, the variety $G \times_P \mathfrak{n}$ is an irreducible variety, so its image under the morphism ϕ will be an irreducible variety.

Next, it is clear that the image of the map π is stable under conjugation by elements of G. If $v \in \text{im}(\pi)$, i.e. $v = gyg^{-1}$ for $g \in G, y \in \mathfrak{n}$, and if $w = hvh^{-1}$ (i.e. w lies in the same G-orbit as v), it follows that $w = h(gyg^{-1})h^{-1} = (hg)y(hg)^{-1}$, so $w \in \text{im}(\pi)$. It is also clear that every element in the image of the map π is nilpotent, since if $v = gyg^{-1}$ lies in the image of π, y is nilpotent since every element in $\mathfrak{n}$ is nilpotent, and hence v is nilpotent. This means that the image of the map $\text{im}(\pi)$ is a union of nilpotent orbits of G. However, we already know that $\text{im}(\pi)$ is a closed algebraic variety, which means that $\text{im}(\pi)$ is a union of some family of nilpotent orbit closures. Since $\text{im}(\pi)$ is an irreducible algebraic variety, it follows that $\text{im}(\pi)$ is a single orbit closure.

Since $\text{im}(\pi)$ contains $\mathcal{O}_\lambda$, and since the dimension of the image of the morphism π is at most the dimension of the domain $G \times_P \mathfrak{n}$, we have the following chain of inequalities:

$$\dim(\overline{\mathcal{O}_\lambda}) \leq \dim(\text{im}(\pi)) \leq \dim(G \times_P \mathfrak{n})$$

Using the fact that the variety $G \times_P \mathfrak{n}$ is a vector bundle over the partial flag variety G/P with fibres isomorphic to $\mathfrak{n}$, we can compute its dimension. The dimension of G/P is equal to $\dim(G) - \dim(P)$, where G and P are quasi-affine algebraic varieties (following the notation of Hartshorne). The dimension of the Lie groups G and P will be equal to the dimension of the corresponding Lie algebras $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{p}$. Since we have computed $\dim(\mathfrak{p})$ and $\dim(\mathfrak{n})$ previously, we can now compute $\dim(G \times_P \mathfrak{n})$:

$$\begin{aligned}
\dim(G \times_P \mathfrak{n}) &= \dim(G) - \dim(P) + \dim(\mathfrak{n}) \\
&= \dim(\mathfrak{gl}_n(\mathbb{C})) - \dim(\mathfrak{p}) + \dim(\mathfrak{n}) \\
&= n^2 - (\frac{1}{2}n^2 + \frac{1}{2}\sum_{i=1}^{l} \mu_i^2) + (\frac{1}{2}n^2 - \frac{1}{2}\sum_{i=1}^{l} \mu_i^2) \\
&= n^2 - \sum_{i=1}^{l} \mu_i^2
\end{aligned}$$

On the other hand, $\dim(\overline{\mathcal{O}_\lambda}) = \dim(\mathcal{O}_\lambda)$ (since the dimension of a closure of an algebraic variety is equal to the dimension of the algebraic variety). As proven in

Section 1.3, $\dim(\mathcal{O}_\lambda) = n^2 - \sum_{i=1}^{l} \mu_i^2$. Hence $\dim(\overline{\mathcal{O}_\lambda}) = n^2 - \sum_{i=1}^{l} \mu_i^2$.

Hence $\dim(G \times_P \mathfrak{n}) = \dim(\overline{\mathcal{O}_\lambda})$, which means we must have equality in the chain of inequalities $\dim(\overline{\mathcal{O}_\lambda}) \leq \dim(\mathrm{im}(\pi)) \leq \dim(G \times_P \mathfrak{n})$. Hence $\mathrm{im}(\pi)$ is a single nilpotent orbit closure containing $\overline{\mathcal{O}_\lambda}$, and has the same dimension, so $\mathrm{im}(\pi)$ is equal to $\overline{\mathcal{O}_\lambda}$.

□

Proof. (of Theorem 2.7 part (ii))
First we need to check that the map π is injective when restriced to $\pi^{-1}(\mathcal{O}_\lambda)$. Consider $G \times_P \mathfrak{n}$ as being the variety $\{((V_i), y) | yV_i \subset V_{i-1}\}$ and the map π as being projection onto the second factor. We are required to prove that given a $y \in \mathcal{O}_\lambda$, there exists a unique flag (V_i) with $\dim(V_i) = \dim(\ker(x^i))$ such that $yV_i \subset V_{i-1}$. Suppose that (V_i) is such a flag. Since $yV_1 = 0$, $V_1 \subset \ker(y)$; since $y^2V_2 \subset yV_1 = 0$, $V_2 \subset \ker(y^2)$; by a quick induction it follows that $V_i \subset \ker(y^i)$. But $\dim(V_i) = \dim(\ker(x^i)) = \dim(\ker(y^i))$, since $x, y \in \mathcal{O}_\lambda$. Hence $V_i = \ker(y^i)$. Conversely, it is clear that $(\ker(y^i))$ is a flag with the desired properties. This shows that π is injective when restricted to $\pi^{-1}(\mathcal{O}_\lambda)$.

Next we need to show that the map $\pi^{-1} : \mathcal{O}_\lambda \to G \times_P \mathfrak{n}$ is a morphism of varieties. From the above paragraph, when restricted to $\mathcal{O}_\lambda$, $\pi^{-1}(y) = \{((\ker(y^i)), y)\}$. Hence we must prove that the map $\alpha : \mathcal{O}_\lambda \to G/P$, defined by $\alpha(y) = (\ker(y^i))$ is a morphism of varieties.

Let $G^x = C_{GL_n(\mathbb{C})}(x)$. We claim that $G^x \subset P$. If $v \in G^x$ then $vx = xv$, and a quick induction shows that $vx^i = x^i v$ for all i. Then $v(\ker(x^i)) \subset \ker(x^i)$, since if $w \in \ker(x^i)$ for some $w \in \mathbb{C}^n$, then $x^i.w = 0$ so $x^i.(vw) = v(x^i.w) = 0$, i.e. $v.w \in \ker(x^i)$. Since $v(\ker(x^i)) \subset \ker(x^i)$ for all i, $v \in P$; thus $G^x \subset P$. This means that now we can define a morphism of varieties $\theta : G/G^x \to G/P$ by declaring that $\theta(uG^x) = uP$.

Since $\mathcal{O}_\lambda = Gx$, the orbit-stabilizer theorem gives a bijection between $\mathcal{O}_\lambda$ and G/G^x, via $uxu^{-1} \to uG^x$. This bijection is an isomorphism since we are working over $\mathbb{C}$. We claim now that the maps α and θ can be identified under this bijection. To prove this, under the map α, the element uxu^{-1} is sent to the flag $(\ker(uxu^{-1})^i) = (\ker(ux^iu^{-1}))$. Under the map θ, the corresponding element uG^x is sent to uP, which corresponds to the flag $(u\ker(x^i))$. Thus we need to prove that $\ker(ux^iu^{-1}) = u\ker(x^i)$. This is a straightforward check: $v \in \ker(ux^iu^{-1}) \iff ux^iu^{-1}v = 0 \iff x^iu^{-1}v = 0 \iff u^{-1}v \in \ker(x^i) \iff v \in u\ker(x^i)$. Hence α and θ can be identified, proving that α is a morphism of varieties.

□

2.2. Closure ordering in Type A

In this section, given two nilpotent orbits $\mathcal{O}_\lambda$ and $\mathcal{O}_{\lambda'}$, we give a criterion for $\mathcal{O}_{\lambda'} \subset \overline{\mathcal{O}_\lambda}$. Recall the definition of the dominance partial ordering on the set of partitions $\mathcal{P}(n)$ of n:

Definition 2.8. *$\lambda \geq \lambda'$ if for each i, $\lambda_1 + \cdots + \lambda_i \geq \lambda'_1 + \cdots + \lambda'_i$.*

Theorem 2.9. *$\mathcal{O}_{\lambda'} \subset \overline{\mathcal{O}_\lambda}$ precisely when $\lambda' \leq \lambda$.*

Proof. By Theorem 2.7, $\overline{\mathcal{O}_\lambda}$ is the image of the π, so $y \in \overline{\mathcal{O}_\lambda}$ if and only if there exists a flag (V_i) with dimV_i = dim(ker(x^i)) $= \mu_1 + \cdots + \mu_i$ such that $yV_i \subset V_{i-1}$.

Suppose $\mathcal{O}_{\lambda'} \subset \overline{\mathcal{O}_\lambda}$. Pick $y \in \mathcal{O}_{\lambda'}$; then there exists a flag (V_i) with dimV_i $= \mu_1 + \cdots + \mu_i$ such that $yV_i \subset V_{i-1}$. It is clear that $y^iV_i = 0$, so $V_i \subset$ ker(y^i), so dim$V_i \leq$ dim(ker(y^i)). If μ' is the transpose partition to λ', dim(ker(y^i)) $= \mu'_1 + \cdots + \mu'_i$, as seen in Proposition 2.2. Hence $\mu_1 + \cdots + \mu_i \leq \mu'_1 + \cdots + \mu'_i$ for each i, so $\mu \leq \mu'$. Since the operation of transposition is an order-reversing involution on $\mathcal{P}(n)$ (see pg 6-7 of [**7**]), it follows that $\lambda \geq \lambda'$.

Conversely, suppose $y \in \mathcal{O}_{\lambda'}$, for some $\lambda' \leq \lambda$; we will prove that $y \in \overline{\mathcal{O}_\lambda}$, by constructing a flag of subspaces (V_i) with dim$V_i = \mu_1 + \cdots + \mu_i$, such that $yV_i \subset V_{i-1}$. Let μ' denote the transpose partition for λ'. To construct this flag of subspaces, pick a Jordan basis for y, and represent it by a diagram for the partition λ', so that y moves each box to the box immediately to the left of it. As an example, let $\lambda' = (4^2, 2^2), \lambda = (5, 3, 2), \mu' = (4, 2^3), \mu = (3^2, 2, 1^2)$. In the below diagram, we have filled up the partition diagram for λ' with μ_1 1-s, μ_2 2-s, μ_3 3-s, μ_4 4-s and μ_5 5-s, so that the numbers strictly increase across the rows. Now let V_i be spanned by the boxes filled with numbers smaller than or equal to i. Then the spaces V_i will have the correct dimensions; and will have the property that $yV_i \subset V_{i-1}$, due to strict increase across rows.

1	2	3	4
1	2	3	5
1			
2			

The above example shows that it is sufficient to prove that there exists a way of filling up the diagram for the partition λ' with μ_1 1-s, μ_2 2-s, and so on, such that the numbers strictly increase across the rows. By transposing the diagram, this is equivalent to proving that there exists a way of filling up the diagram for the partition μ' with μ_1 1-s, μ_2 2-s, and so on, such that the numbers strictly increase down the columns. The slightly stronger statement, that there exists a semistandard tableaux of shape μ' and content μ if $\mu \leq \mu'$ is a standard fact; see pg 26 of Fulton [**4**]. Here a semistandard tableaux of shape μ' and content μ is a diagram of the partition μ' filled with μ_1 1-s, μ_2 2-s, and so on, such that the numbers weakly increase across rows and strictly increase down columns. $\square$

2.3. Resolution of singularities in type C

Here we use the theory of the Jacobson-Morozov resolution to construct a resolution of singularities for the nilpotent orbit closure $\overline{\mathcal{O}_\lambda}$ in the symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$. Denote $G = Sp_{2n}(\mathbb{C}), \mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$. Let X be a nilpotent of type λ in $\mathcal{O}_\lambda$. By the Jacobson-Morozov Theorem, we can find an $\mathfrak{sl}_2$-triple $\{H, X, Y'\}$ in $\mathfrak{g}$. Then we can consider $\mathfrak{g}$ as $\mathfrak{sl}_2$-module, via the adjoint action. By $\mathfrak{sl}_2$-theory, we have the following:

$$\mathfrak{g} = \bigoplus_{a\in\mathbb{Z}} \mathfrak{g}_a, \text{ where } \mathfrak{g}_a = \{Y \in \mathfrak{g} | [H, Y] = aY\}$$

The following calculation shows that $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$. Let $u \in \mathfrak{g}_a, v \in \mathfrak{g}_b$:

$$\begin{aligned}
[H, [u, v]] &= [[H, u], v] + [[v, H], u] \\
&= [au, v] + [-bv, u] \\
&= (a + b)[u, v] \\
[u, v] &\in \mathfrak{g}_{a+b}
\end{aligned}$$

Let $\mathfrak{g}_{\geq i} = \bigoplus_{a\geq i} \mathfrak{g}_a$, and let $\mathfrak{p} = \mathfrak{g}_{\geq 0}$. Since $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$, $\mathfrak{p}$ is closed under the Lie bracket, and hence is a Lie sub-algebra of $\mathfrak{g}$. Let P be the corresponding connected Lie subgroup of G.

Since there is a natural action of G on a vector space V of dimension $2n$, we may also consider V as an $\mathfrak{sl}_2$-module. By the argument at the start of Proposition 1.7, the dimensions of the irreducible $\mathfrak{sl}_2$-submodules of V are $\lambda_1, \lambda_2, \cdots, \lambda_k$. The weights of V as an $\mathfrak{sl}_2$-submodule will thus be $\lambda_1 - 1, \lambda_1 - 3, \cdots, 3 - \lambda_1, 1 - \lambda_1, \lambda_2 - 1, \lambda_2 - 3, \cdots, 3 - \lambda_2, 1 - \lambda_2, \cdots$. By $\mathfrak{sl}_2$-theory, we have the following:

$$V = \bigoplus_{a\in\mathbb{Z}} V_a, \text{ where } V_a = \{v \in V | H \cdot v = av\}$$

The following calculation shows that $\mathfrak{g}_a \cdot V_b \subseteq V_{a+b}$. Let $u \in \mathfrak{g}_a, v \in V_b$, so that $[H, u] = au, H \cdot v = bv$.

$$\begin{aligned}
a(u \cdot v) = [H, u] \cdot v &= H(u \cdot v) - u(H \cdot v) \\
&= H(u \cdot v) - u(bv) \\
&= H(u \cdot v) - b(u \cdot v) \\
H(u \cdot v) &= (a + b)(u \cdot v), \text{ so } u \cdot v \in V_{a+b}
\end{aligned}$$

We now construct a partial flag in V as follows:

$$V_{\geq b} = \bigoplus_{a \geq b} V_a$$
$$0 \subseteq \cdots \subseteq V_{\geq 2} \subseteq V_{\geq 1} \subseteq V_{\geq 0} \subseteq V_{\geq -1} \subseteq V_{\geq -2} \subseteq \cdots \subseteq V$$
$$\dim V_{\geq b} = \sum_{1 \leq i \leq k, \lambda_i > b} \left\lceil \frac{\lambda_i - b}{2} \right\rceil = d_b(\lambda)$$

We now claim that this partial flag is isotropic, i.e. that $V_{\geq b}^{\perp} = V_{\geq 1-b}$. To prove this, first recall from the proof of Proposition 1.7 that the r-weight space V_r will be orthogonal to the s-weight space V_s unless $r + s = 0$. This shows that V_r will be perpendicular to V_s if $r \geq b, s \geq 1 - b$ since then $r + s \geq 1 > 0$, and hence $V_{\geq b}$ is perpendicular to $V_{\geq 1-b}$. Further, $\dim V_{\geq b}$ is equal to the number of weights in V which are at least b, counted with multiplicity, and $\dim V_{\geq 1-b}$ is equal to the number of weights in V which are at least $1 - b$, counted with multiplicity. Since the weights in V are symmetric about 0, the number of weights in V which are at least $1 - b$ is equal to the number of weights in V which are at most $b - 1$. Since every weight is either at most $b - 1$ or at least b, and since there are $2n$ weights in total, this shows that $\dim V_{\geq b} + \dim V_{\geq 1-b} = 2n$. It now follows that $V_{\geq b}^{\perp} = V_{\geq 1-b}$.

Proposition 2.10. *P is the stabilizer of this partial isotropic flag in $Sp_{2n}(\mathbb{C})$.*

Proof. It suffices to prove that the stabilizer of this flag in $\mathfrak{sp}_{2n}(\mathbb{C})$ is $\mathfrak{p}$. To see why this is true, if A stabilizes this flag, then for each t, $\exp(tA)$ will stabilize this flag. Since P is generated is by the one-parameter subgroups $\exp(tA)$ as A ranges over $\mathfrak{p}$, it follows that P will stabilize this flag.

Conversely we must show that the stabilizer of this flag in $Sp_{2n}(\mathbb{C})$ is not larger than P; so suppose that it is P', and has Lie algebra $\mathfrak{p}'$. Since the stabilizer of an isotropic flag is connected, and the Lie algebra determines the Lie group if the Lie group is connected, it suffices to prove that $\mathfrak{p}' \subseteq \mathfrak{p}$, which will imply $P' \subseteq P$ and hence $P' = P$. Given any $A \in \mathfrak{p}'$, for all t, $\exp(tA) \in P'$, so $\exp(tA)$ will stabilize the flag, and so will $\exp(tA) - 1$. Defining the norm $||X||$ of a matrix to be the maximum of the absolute values of its entries, recall that the power series for $\ln(1 + X)$ will converge if $||X|| < 1$. Choosing t arbitrarily small so that $||\exp(tA) - 1|| < 1$, then $tA = \ln(1 + \exp(tA) - 1)$ will stabilize the flag; hence A will stabilize the flag, so $A \in \mathfrak{p}$. This shows that $\mathfrak{p}' \subseteq \mathfrak{p}$; hence it is sufficient to show that the stabilizer of the isotropic flag in $\mathfrak{sp}_{2n}(\mathbb{C})$ is $\mathfrak{p}$.

Let $\mathfrak{q}$ be the stabilizer of the flag in $\mathfrak{sp}_{2n}(\mathbb{C})$.

$$\mathfrak{q} = \{Y \in \mathfrak{g} | Y V_{\geq b} \subseteq V_{\geq b} \ \forall b\}$$
$$= \{Y \in \mathfrak{g} | Y V_b \subseteq V_{\geq b} \ \forall b\}$$

To see the equality of the above two lines, clearly $YV_{\geq b} \subseteq V_{\geq b}$ implies $YV_b \subseteq V_{\geq b}$. Conversely, if $YV_b \subseteq V_{\geq b}$ for all b, then if $a \geq b$, $YV_a \subseteq V_{\geq a} \subset V_{\geq b}$, so $YV_{\geq b} \subseteq V_{\geq b}$.

Since $\mathfrak{g}_i V_b \subseteq V_{b+i} \subset V_{\geq b}$ if $i \geq 0$, it follows that $\mathfrak{g}_i \subset \mathfrak{q}$ for each $i \geq 0$, and thus $\mathfrak{p} \subseteq \mathfrak{q}$. Conversely, suppose $x \in \mathfrak{q}, x \notin \mathfrak{p}$. Let $x = \sum_i x_i$, with $x_i \in \mathfrak{g}_i$. Since $x \notin \mathfrak{p}$, for some $j < 0$, $x_j \neq 0$. Then $x_j v_l \neq 0$ for some $v_l \in V_l$; we know that $x_j v_l \in V_{j+l}$. Then $xv_l = \sum_i x_i v_l$, and $x_i v_l \in V_{i+l}$; since $x_j v_l \neq 0$, and $j + l < l$, it follows $xv_l \notin V_{\geq l}$. This contradicts the fact that $xV_l \subseteq V_l$. Hence $\mathfrak{q} \subseteq \mathfrak{p}$, implying that $\mathfrak{p} = \mathfrak{q}$, as required. □

Since P is the stabilizer of an isotropic flag, it follows that P is a parabolic subgroup of $Sp_{2n}(\mathbb{C})$. Since $[\mathfrak{g}_a, \mathfrak{g}_b] \subset \mathfrak{g}_{a+b}$, $[\mathfrak{p}, \mathfrak{g}_{\geq 2}] \subseteq \mathfrak{g}_{\geq 2}$. It follows that P acts by conjugation on $\mathfrak{g}_{\geq 2}$. To see this, given $X \in \mathfrak{p}$, $\mathfrak{g}_{\geq 2}$ is stable under $\mathrm{ad}(X)$, and hence under $\exp(\mathrm{ad}(X))$. But we have that $\exp(\mathrm{ad}(X)) = \mathrm{Ad}(\exp(X))$, so $\mathfrak{g}_{\geq 2}$ is stable under conjugation by elements of the form $\exp(X)$ for $X \in \mathfrak{p}$. Since P is generated by elements of the form $\exp(X)$ for $X \in \mathfrak{p}$, it follows that P acts by conjugation on $\mathfrak{g}_{\geq 2}$.

Definition 2.11. *Let $G \times_P \mathfrak{g}_{\geq 2}$ be the quotient of the Cartesian product $G \times \mathfrak{g}_{\geq 2}$ by the equivalence relation $(g, y) \sim (gp^{-1}, p \cdot y) = (gp^{-1}, pyp^{-1})$ where $p \in P$.*

P is the stabilizer of the isotropic flag $\{\cdots \subseteq V_{\geq 1} \subseteq V_{\geq 0} \subseteq V_{\geq -1} \subseteq \cdots\}$, and G acts transitively on the set $\{(W_b) | \dim W_b = d_b(\lambda), W_b^{\perp} = W_{1-b}\}$ (this follows from the stronger fact that $Sp_{2n}(\mathbb{C})$ acts transitively on the set of flags $\{0 = V_0 \subset V_1 \subset \cdots \subset V_{2n-1} \subset V_{2n} = V | V_i^{\perp} = V_{2n-i}\}$). Thus under the identification $gP \to \{\cdots \subseteq gV_{\geq 1} \subseteq gV_{\geq 0} \subseteq gV_{\geq -1} \subseteq \cdots\}$, G/P is the partial flag variety:

$$G/P = \{(W_b) | \dim W_b = d_b(\lambda), W_b^{\perp} = W_{1-b}\}$$

The partial flag variety G/P is a projective variety, since there is a natural inclusion of G/P into the product of the Grassmanians $\prod_b Gr_{d_b(\lambda)}(V)$, and each Grassmanian is a projective variety.

Proposition 2.12. *Define the map $\phi : G \times_P \mathfrak{g}_{\geq 2} \to G/P \times \mathfrak{g}$ by $\phi(g, Y) = (gP, gYg^{-1})$. Then ϕ is well-defined and injective. Then we have that $\mathrm{im}(\phi) = \{((W_b), U) | UW_b \subseteq W_{b+2}\}$. Thus $\mathrm{im}\,\phi$ is a closed subvariety of $G/P \times \mathfrak{g}$, giving $G \times_P \mathfrak{g}_{\geq 2}$ the structure of an algebraic variety.*

Proof. The proof that ϕ is a well-defined and injective map is the same as in the proof of the Proposition 2.4, and is omitted.

It is clear by definition of ϕ that $\mathrm{im}(\phi) = \{(gP, U) | U \in g\mathfrak{g}_{\geq 2}g^{-1}\}$. The condition that $U \in g\mathfrak{g}_{\geq 2}g^{-1}$ is a closed condition, and thus $\mathrm{im}(\phi)$ is a closed subvariety of $G/P \times \mathfrak{g}$. Suppose $(gP, U) \in \mathrm{im}(\phi)$, so that $U = gYg^{-1}$ for some $Y \in \mathfrak{g}_{\geq 2}$. The element $gP \in G/P$ corresponds to the flag $(gV_{\geq b})$, so $gYg^{-1}(gV_{\geq b}) = gYV_{\geq b} \subseteq gV_{\geq b+2}$ (here $YV_{\geq b} \subseteq V_{\geq b+2}$ since $Y \in \mathfrak{g}_{\geq 2}$). Hence the (gP, U) corresponds to an element $((W_b), U)$ such that $UW_b \subseteq W_{b+2}$.

Conversely, suppose we have an element $(gP, U) = (gP, gYg^{-1})$ which corresponds to an element $((W_b), U)$ such that $UW_b \subset W_{b+2}$; we will prove that $Y \in \mathfrak{g}_{\geq 2}$. Since $W_b = gV_{\geq b}$, we have that $gYg^{-1}(gV_{\geq b}) \subset gV_{\geq b+2}$, i.e. $gYV_{\geq b} \subset gV_{\geq b+2}$, or $YV_{\geq b} \subset V_{\geq b+2}$. In the proof of Proposition 2.10, we proved that $YV_{\geq b} \subseteq V_{\geq b}$ for each b implies that $Y \in \mathfrak{g}_{\geq 0}$. A similar argument will show that if $YV_{\geq b} \subset V_{\geq b+2}$ for each b, then $Y \in \mathfrak{g}_{\geq 2}$. □

Corollary 2.13. *$G \times_P \mathfrak{g}_{\geq 2}$ is a vector bundle over G/P, with fibres isomorphic to $\mathfrak{g}_2$.*

Proof. By Proposition 2.12, identify $G \times_P \mathfrak{g}_{\geq 2}$ with the variety $\{(gP, U) | U \in g\mathfrak{g}_{\geq 2}g^{-1}\}$. Let p_1 denote the projection onto the first factor; p_1 is clearly a surjective map onto G/P. Given a fixed element $gP \in G/P$, the fibre $p_1^{-1}(gP)$ is $\{U \in g\mathfrak{g}_{\geq 2}g^{-1}\}$. This fibre is simply a conjugate of the vector space $\mathfrak{g}_{\geq 2}$, and hence is isomorphic to $\mathfrak{g}_{\geq 2}$ as vector spaces. □

Define the map $\pi : G \times_P \mathfrak{g}_{\geq 2} \to \mathfrak{g}$ by $\pi(g, y) = gyg^{-1}$. As proved in Proposition 2.12, π is a well-defined map. If we instead consider $G \times_P \mathfrak{g}_{\geq 2}$ as $\{((W_b), U) | UW_b \subseteq W_{b+2}\}$, π is projection onto the second factor. By the exact same argument used in Corollary 2.6, we may prove that π is a projective morphism of varieties. Now we state the main result of this section: the map π is a resolution of singularities for the orbit closure $\mathcal{O}_\lambda$.

Theorem 2.14. *The map $\pi : G \times_P \mathfrak{g}_{\geq 2} \to \mathfrak{g}$ is a resolution of singularities for the orbit closure $\mathcal{O}_\lambda$. Equivalently, the following four statements are true:*

(i) The image of the map π in $\mathfrak{g}$ is precisely the orbit closure $\overline{\mathcal{O}_\lambda}$.

(ii) The map π is injective when restricted to $\pi^{-1}(\mathcal{O}_\lambda)$. The inverse map, when restricted to $\mathcal{O}_\lambda$, $\pi^{-1} : \mathcal{O}_\lambda \to G \times_P \mathfrak{g}_{\geq 2}$ is a morphism of algebraic varieties.

(iii) As an algebraic variety, $G \times_P \mathfrak{g}_{\geq 2}$ is smooth and irreducible.

(iv) The map π is a proper morphism of algebraic varieties.

Proof. (of Theorem 2.14 (iii), (iv)) The variety $G \times_P \mathfrak{g}_{\geq 2}$ is smooth and irreducible, since it is a vector bundle over the partial flag variety G/P, which is smooth and irreducible since it is a homogeneous space.

The fact that π is a proper morphism of varieties follows from the fact that it is a projective morphism of varieties. □

Proof. (of Theorem 2.14 (i))
We first prove that $\mathrm{im}(\pi)$ contains $\overline{\mathcal{O}_\lambda}$. Since π is a projective morphism, its image is a closed subvariety of $\mathfrak{g}$, so it suffices to prove that $\mathrm{im}(\pi)$ contains $\mathcal{O}_\lambda$. Since

$X \in \mathfrak{g}_2 \subset \mathfrak{g}_{\geq 2}$, $X = \pi(1, X)$, so X is contained in $\mathrm{im}(\pi)$. If $Y \in \mathcal{O}_\lambda$, then $Y = gXg^{-1}$ for some $g \in G$, so $Y = \pi(g, X)$ and so Y is contained in $\mathrm{im}(\pi)$. This proves that $\mathrm{im}(\pi)$ contains $\overline{\mathcal{O}_\lambda}$.

Next, it is clear that $\mathrm{im}(\pi)$ is stable under conjugation by elements of G. If $v \in \mathrm{im}(\pi)$, i.e. $v = gyg^{-1}$ for $g \in G, y \in \mathfrak{g}_{\geq 2}$, and if $w = hvh^{-1}$ (i.e. w lies in the same G-orbit as v), it follows that $w = h(gyg^{-1})h^{-1} = (hg)y(hg)^{-1}$, so $w \in \mathrm{im}(\pi)$. It is also clear that every element of $\mathfrak{g}_{\geq 2}$ is nilpotent. To see this, $\mathfrak{g} = \mathfrak{g}_{\geq i}$ for some i sufficiently small; $[\mathfrak{g}_{\geq 2}, \mathfrak{g}_{\geq i}] \subset [\mathfrak{g}_{\geq i+2}]$, so if $x \in \mathfrak{g}_{\geq 2}$, $\mathrm{ad}(x)^j \mathfrak{g}_{\geq i} \in \mathfrak{g}_{\geq i+2j} = 0$ if j is picked sufficiently large; this proves that $\mathrm{ad}(x)$ is a nilpotent endomorphism of $\mathfrak{g}$, and hence x is nilpotent. Since every element in $\mathrm{im}(\pi)$ is conjugate to an element in $\mathfrak{g}_{\geq 2}$, every element in $\mathrm{im}(\pi)$ is nilpotent. Since $\mathrm{im}(\pi)$ is stable under conjugation, it follows that $\mathrm{im}(\pi)$ is a union of nilpotent orbits. Since $\mathrm{im}(\pi)$ is closed, it is a union of nilpotent orbit closures. As $\mathrm{im}(\pi)$ is an irreducible variety, it is a single nilpotent orbit closure.

As $\mathrm{im}(\pi)$ contains $\overline{\mathcal{O}_\lambda}$, and since the dimension of the image of the morphism π is at most the dimension of its domain $G \times_P \mathfrak{g}_{\geq 2}$, we have the following chain of inequalities:

$$\dim(\overline{\mathcal{O}_\lambda}) \leq \dim(\mathrm{im}(\pi)) \leq \dim(G \times_P \mathfrak{g}_{\geq 2})$$

Since $G \times_P \mathfrak{g}_{\geq 2}$ is a vector bundle over G/P with fibres isomorphic to $\mathfrak{g}_{\geq 2}$, we may compute its dimension as follows:

$$\begin{aligned}
\dim(G \times_P \mathfrak{g}_{\geq 2}) &= \dim(G) - \dim(P) + \dim(\mathfrak{g}_{\geq 2}) \\
&= \dim(\mathfrak{g}) - \dim(\mathfrak{p}) + \dim(\mathfrak{g}_{\geq 2}) \\
&= \dim(\mathfrak{g}) - \dim(\mathfrak{g}_{\geq 0}) + \dim(\mathfrak{g}_{\geq 2}) \\
&= \dim(\mathfrak{g}) - \dim(\mathfrak{g}_0) - \dim(\mathfrak{g}_1)
\end{aligned}$$

To compute $\dim(\overline{\mathcal{O}_\lambda})$, first note that $\dim(\overline{\mathcal{O}_\lambda}) = \dim(\mathcal{O}_\lambda) = \dim(G.X) = \dim(G) - \dim(G^X)$. Here G^X denotes the centralizer of X in G, which has the same dimension as its Lie algebra, the centralizer of X in $\mathfrak{g}$. To compute this dimension, consider $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module, under the adjoint action. Then $\mathfrak{g}$ has a basis of weight vectors for H, with the weights consisting of the strings $\{\mu_1 - 1, \mu_1 - 3, \cdots, 3 - \mu_1, 1 - \mu_1\}, \{\mu_2 - 1, \cdots, 1 - \mu_2\}$, and so-on (where μ_i are the dimensions of the irreducible summands of $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module). By $\mathfrak{sl}_2$-theory, X will annihilate the highest weight vector in each string, and act non-trivially on all other vectors in each string. Thus the centralizer of X in $\mathfrak{g}$ will be spanned by the highest weight vectors in each string, and its dimension will be equal to the number of weight strings. Each weight string will either pass through 0 (if the highest weight in the string is even) or pass through 1 (if the highest weight in the string is odd). Thus the dimension of the centralizer of X in $\mathfrak{g}$ will be equal to $\dim(\mathfrak{g}_0) + \dim(\mathfrak{g}_1)$. Thus we have:

$$\begin{aligned}\dim(\overline{\mathcal{O}_\lambda}) &= \dim(\mathcal{O}_\lambda)\\ &= \dim(G) - \dim(G^X)\\ &= \dim(G) - \dim(\mathfrak{g}^X)\\ &= \dim(G) - \dim(\mathfrak{g}_0) - \dim(\mathfrak{g}_1)\end{aligned}$$

Hence $\dim(\overline{\mathcal{O}_\lambda}) = \dim(G \times_P \mathfrak{g}_{\geq 2})$, and we must have equality in the chain of inequalities $\dim(\overline{\mathcal{O}_\lambda}) \leq \dim(\mathrm{im}(\pi)) \leq \dim(G \times_P \mathfrak{g}_{\geq 2})$. Since $\mathrm{im}(\pi)$ is a single nilpotent orbit closure containing $\overline{\mathcal{O}_\lambda}$, and with the same dimension, $\mathrm{im}(\pi) = \overline{\mathcal{O}_\lambda}$. □

Proof. (of Theorem 2.14 (ii))
We first need to check that the map π is injective when restricted to $\pi^{-1}(\mathcal{O}_\lambda)$. Considering $G \times_P \mathfrak{g}_{\geq 2}$ as the set $\{((W_b), U) | UW_b \subseteq W_{b+2}\}$ and π as the projection onto the second factor, we must prove that given $U \in \mathcal{O}_\lambda$, there exists a unique flag (W_b) such that $UW_b \subseteq W_{b+2}$. It suffices to prove this for $U = X$; indeed suppose we know the statement for $U = X$ and suppose for some $U = gXg^{-1}$ with $g \in G$, we have two flags (W_b) and (W'_b) with the required property. Then the flags $(g^{-1}W_b)$ and $(g^{-1}W'_b)$ both have the required property with respect to X, and must be the same (since there is only one flag which has the property for X). This then forces (W_b) and (W'_b) to be the same, proving the statement for U.

So we need to show that if (W_b) is an isotropic flag with $XW_b \subseteq W_{b+2}$, then $W_b = V_{\geq b}$. We first illustrate with the example $\lambda = (3^2, 2)$. The below partition diagram shows a basis of V, where X takes each box to the box on its left, and the numbers represent the weights of the basis vectors. Reading off the diagram, we have that $d_3(\lambda) = 0, d_2(\lambda) = 2, d_1(\lambda) = 3, d_0(\lambda) = 5, d_{-1}(\lambda) = 6, d_{-2}(\lambda) = 8$. Since $XV = XW_{-2} \subset W_0$, and $\dim XV = 5$, $\dim W_0 = d_0(\lambda) = 5$, it follows that $XV = W_0$; by inspection we also have that $XV = V_{\geq 0}$, so $W_0 = V_{\geq 0}$. Similarly, since $X^2V = X^2W_{-2} \subset W_2$, but $\dim X^2V = 2$ and $\dim W_2 = d_2(\lambda) = 2$; by inspection we also have that $X^2V = V_{\geq 2}$, so $W_2 = V_{\geq 2}$. Now that we have $W_2 = V_{\geq 2}$ and $W_0 = V_{\geq 0}$, since both flags are isotropic we have $W_{-1} = V_{\geq -1}$ and $W_1 = V_{\geq 1}$. This now proves that the two flags are the same, as required.

2	0	-2
2	0	-2
1	-1	

To show the statement in general, we proceed by induction on the size of the largest part λ_1. The base case is when $\lambda = (1^l)$, for some l even. Here, $d_0(\lambda) = l$ and $d_1(\lambda) = 0$, so we are dealing with a flag $W_1 \subset W_0$ with $\dim W_1 = 0, \dim W_0 = l$. This forces $W_1 = 0 = V_{\geq 1}, W_0 = V = V_{\geq 0}$, proving the statement in this case.

We will first prove that $W_{\lambda_1 - 1} = V_{\geq \lambda_1 - 1} = \mathrm{im}(X^{\lambda_1 - 1})$ (note W_{λ_1-1} is the smallest non-zero subspace in the flag). Since $V = W_{1-\lambda_1}$, applying the fact that $XW_b \subseteq W_{b+2}$ $\lambda_1 - 1$ times, we get that $X^{\lambda_1-1}V \subseteq W_{\lambda_1-1}$. But $\dim X^{\lambda_1-1}$ is equal to the multiplicity of λ_1 in λ (this can be seen by looking at how X^{λ_1-1} acts on a Jordan basis for X); and $\dim W_{\lambda_1-1}$ is equal to the number of weights which are greater than or equal to $\lambda_1 - 1$, which is also equal to the multiplicity of λ_1 in λ. This shows that $X^{\lambda_1-1}V = W_{\lambda_1-1}$. The same argument also proves that $X^{\lambda_1-1}V = V_{\geq\lambda_1-1}$, and hence $W_{\lambda_1-1} = V_{\geq\lambda_1-1}$. Since the flag is isotropic, $W_{\lambda_1-1}^{\perp} = W_{2-\lambda_1}$ and $V_{\geq\lambda_1-1}^{\perp} = V_{\geq 2-\lambda_1}$, and so $W_{2-\lambda_1} = V_{\geq 2-\lambda_1}$.

Consider $V' = V_{\geq 2-\lambda_1}/V_{\geq\lambda_1-1}$ (as proved in the last paragraph, this is equal to $W_{2-\lambda_1}/W_{\lambda_1-1}$). Define a symplectic form on this space as follows:

$$\langle v + V_{\geq\lambda_1-1}, v' + V_{\geq\lambda_1-1}\rangle = \langle v, v'\rangle$$

This form is well-defined since $V_{\geq 2-\lambda_1}$ is perpendicular to $V_{\geq\lambda_1-1}$, so v and v' will be orthogonal to any vector in $V_{\geq\lambda_1-1}$. This form is clearly skew-symmetric. It is non-degenerate, since if $v + V_{\geq\lambda_1-1}$ lies in the kernel of the form, then v will be orthogonal to everything in $V_{\geq 2-\lambda_1}$, so $v \in V_{\geq 2-\lambda_1}^{\perp} = V_{\geq\lambda_1-1}$, so then $v + V_{\geq\lambda_1-1} = 0$.

The endomorphism X will induce a nilpotent endomorphism X' in V' of type λ', where λ' is the partition with parts $\lambda_1 - 2, \cdots, \lambda_r - 2, \lambda_{r+1}, \cdots, \lambda_k$, arranged in decreasing order (where r is the multiplicity of λ_1 in λ). To see this, suppose V has basis $e_{i,j}$, where $1 \leq i \leq k, 1 \leq j \leq \lambda_i$, such that $Xe_{i,j} = e_{i,j-1}$ if $j > 1$ and 0 otherwise. Then $V_{\geq\lambda_1-1} = \mathrm{im}(X^{\lambda_1-1})$ will be spanned by $\{e_{1,1}, \cdots, e_{r,1}\}$, while $V_{\geq 2-\lambda_1} = \mathrm{im}(X^{\lambda_1-1})^{\perp}$ will be spanned by all of the $e_{i,j}$-s excluding $\{e_{1,\lambda_1}, \cdots, e_{r,\lambda_1}\}$. Thus V' is spanned by the images of all $e_{i,j}$-s satisfying $2 \leq j \leq \lambda_1 - 1$ if $1 \leq i \leq r$, and X acts as before; this shows that X' has type λ'. Consider the two flags induced by $V_{\geq b}$ and W_b in V':

$$0 \subset W_{\lambda_1-2}/V_{\geq\lambda_1-1} \subset \cdots \subset W_{3-\lambda_1}/V_{\geq\lambda_1-1} \subset V'$$
$$0 \subset V_{\geq\lambda_1-2}/V_{\geq\lambda_1-1} \subset \cdots \subset V_{\geq 3-\lambda_1}/V_{\geq\lambda_1-1} \subset V'$$

It is easy to check that the above flags are isotropic; we will check that the first flag is isotropic (the same method will work for the second flag). Given $3 - \lambda_1 \leq b \leq \lambda_1 - 2$, we need to check that $(W_b/V_{\geq\lambda_1-1})^{\perp} = W_{1-b}/V_{\geq\lambda_1-1}$. Since $W_b^{\perp} = W_{1-b}$, the two spaces are orthogonal. It suffices to check that $\dim(W_b/V_{\geq\lambda_1-1}) + \dim(W_{1-b}/V_{\geq\lambda_1-1}) = \dim V'$. This is true because $\dim(V_{\geq\lambda_1-1}) = r$, so $LHS =$

$$\dim(W_{\geq b}) + \dim(W_{\geq 1-b}) - 2r = \dim V - 2r = \dim V' = RHS.$$

Since $X(W_b) \subseteq W_{b+2}$, we have $X'(W_b/V_{\geq \lambda_1 - 1}) \subseteq W_{b+2}/V_{\geq \lambda_1 - 1}$, and similarly $X'(V_{\geq b}/V_{\geq \lambda_1-1}) \subseteq V_{\geq b+2}/V_{\geq \lambda_1 - 1}$. By the induction hypothesis applied to X', it now follows that $W_b/V_{\geq \lambda_1 - 1} = V_{\geq b}/V_{\geq \lambda_1 - 1}$, and hence $W_b = V_{\geq b}$. This gives the required result, proving that the map π is injective when restricted to $\pi^{-1}(\mathcal{O}_\lambda)$.

Next we need to prove that the map $\pi^{-1} : \mathcal{O}_\lambda \to G \times_P \mathfrak{g}_{\geq 2}$ is a morphism of varieties. Given $U = gXg^{-1} \in \mathcal{O}_\lambda$, from the above paragraphs, there exists a unique flag (W_b) with $UW_b \subseteq W_{b+2}$, and it is clear that $(gV_{\geq b})$ is such a flag. Thus $\pi^{-1}(gXg^{-1}) = ((gV_{\geq b}), gXg^{-1})$. To prove that π^{-1} is a morphism, it suffices to prove that the mapα, defined by $\alpha(gXg^{-1}) = (gV_{\geq b})$ is a morphism of varieties.

Let $G^X = C_{Sp_{2n}(\mathbb{C})}(X)$. Since $\mathcal{O}_\lambda = Gx$, the orbit-stabilizer theorem gives a bijection between $\mathcal{O}_\lambda$ and G/G^X via $uXu^{-1} \to uG^X$. This bijection is an isomorphism of varieties since we are working over $\mathbb{C}$. We also have that $G^X \subset P$; to see this, given $g \in G^X$, then $g.(V_{\geq b})$ will be an isotropic flag with $X(gV_{\geq b}) \subset gV_{\geq b+2}$. Since there is a unique flag with this property, this forces $g \in P$, and hence $G^X \subset P$. This means we have a morphism $\theta : G/G^X \to G/P$, defined via $\theta(uG^X) = uP$. We claim that θ and α can be identified under the bijection between $\mathcal{O}_\lambda$ and G/G^X. To prove this, under the map α, the element gXg^{-1} is sent to $(gV_{\geq b})$. Under the map θ, the corresponding element gG^X is sent to gP, which corresponds to the flag $(gV_{\geq b})$, as required. This proves that α is a morphism of varieties. $\square$

CHAPTER 3

The enhanced nilpotent cone and some variations

3.1. The enhanced nilpotent cone

The enhanced nilpotent cone for $GL_n(\mathbb{C})$ (see Achar-Henderson, [**1**]), $V \times \mathcal{N}$, is defined to be the product of the vector space $V = \mathbb{C}^n$ with the nilpotent cone $\mathcal{N}$; there is a natural action of $GL_n(\mathbb{C})$ on V, which gives a natural action of $GL_n(\mathbb{C})$ on $V \times \mathcal{N}$. In this section, we classify the orbits of $GL_n(\mathbb{C})$ on $V \times \mathcal{N}$, and prove that the orbits are in $1-1$ correspondence with bi-partitions of n, defined below.

Definition 3.1. *An ordered pair of partitions (μ, ν) with $|\mu| + |\nu| = n$ is called a bi-partition of n. The set of all bi-partitions of n will be denoted $\mathcal{Q}(n)$. Call (μ, ν) a bi-partition of λ, if $\mu + \nu = \lambda$.*

Definition 3.2. *Given a vector $v \in V$, for $1 \leq j \leq k$, define the j-th **"portion"** of v to be the vector obtained by considering the $\lambda_1 + \cdots + \lambda_{j-1} + 1$-th, $\cdots, \lambda_1 + \cdots \lambda_{j-1} + \lambda_j$-th coordinates of V.*

The below theorem, which re-phrases Proposition 2.3 from [**1**], is the main Theorem in this section. The proof given here is a variation of the proof given in [**1**].

Theorem 3.3. *The orbits of $GL_n(\mathbb{C})$ on $V \times \mathcal{N}$ are in one-to-one correspondence with the set $\mathcal{Q}(n)$. The partition $\mu + \nu = \lambda$ of n specifies the Jordan type of the nilpotent element $n \in \mathcal{N}$. Suppose $\lambda = (\lambda_1, \cdots, \lambda_k), \mu = (\mu_1, \cdots, \mu_k), \nu = (\nu_1, \cdots, \nu_k)$. Then (v_μ, n_λ) is an orbit representative for the orbit corresponding to (μ, ν), where n_λ is as described in section 2.2.1, and v_μ is constructed as follows: for each $1 \leq j \leq k$, the j-th portion has a 1 in the μ_j-th coordinate, and 0-s elsewhere.*

The problem of computing the orbits of $GL_n(\mathbb{C})$ on $V \times \mathcal{N}$ is equivalent to the problem of computing the orbits of $C_{GL_n(\mathbb{C})}(n_\lambda)$ on V. To see this, if two elements $(v_1, n_1), (v_2, n_2) \in V \times \mathcal{N}$ are in the same $GL_n(\mathbb{C})$-orbit, then n_1 and n_2 are nilpotent elements with the same Jordan type, say corresponding to the partition λ. Then it suffices to check when $(v_1, n_\lambda), (v_2, n_\lambda)$ are in the same $GL_n(\mathbb{C})$-orbit. This happens when there exists a g with $gv_1 = v_2$ with $gn_\lambda g^{-1} = n_\lambda$, i.e. if v_1, v_2 lie in the same $C_{GL_n(\mathbb{C})}(n_\lambda)$-orbit.

Example 3.4. Let $\lambda = (3, 2, 2)$. A typical element $g \in C_{GL_7(\mathbb{C})}(n_\lambda)$ takes the following form, using the structure of the centralizer $C_{GL_n(\mathbb{C})}(n_\lambda)$ as described in Section 2.3.1. In the case of the bi-partition $(\mu, \nu) = ((2, 1, 1), (1, 1, 1))$, v_μ will be as follows.

$$g = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 \\ & \lambda_1 & \lambda_2 & & \lambda_4 & & \lambda_6 \\ & & \lambda_1 & & & & \\ & \lambda_8 & \lambda_9 & a_{11} & \lambda_{12} & a_{12} & \lambda_{13} \\ & & \lambda_8 & & a_{11} & & a_{12} \\ & \lambda_{10} & \lambda_{11} & a_{21} & \lambda_{14} & a_{22} & \lambda_{15} \\ & & \lambda_{10} & & a_{21} & & a_{22} \end{pmatrix}, v_\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$gv_\mu = \begin{pmatrix} \lambda_2 + \lambda_4 + \lambda_6 \\ \lambda_1 \\ 0 \\ \lambda_8 + a_{11} + a_{12} \\ 0 \\ \lambda_{10} + a_{21} + a_{22} \\ 0 \end{pmatrix}$$

Definition 3.5. *Given a vector v, say that it is of type (α, β), where $\alpha = (\alpha_1, \cdots, \alpha_k)$, $\beta = (\beta_1, \cdots, \beta_k)$ and $\alpha_i + \beta_i = \lambda_i$ for all $1 \le i \le k$, if the maximal t such that the t-th component in the i-th portion is non-zero, is α_i for each $1 \le i \le k$. Let W_α be the set of vectors of type (α, β). Let v_α be constructed as follows: for each $1 \le j \le k$, the j-th portion has a 1 in the α_j-th coordinate, and 0-s elsewhere.*

Definition 3.6. *Define the ordering $\dot{>}$ on pairs of compositions (α, β) such that $\alpha + \beta = \lambda$, by declaring $(\alpha, \beta) \dot{>} (\alpha', \beta')$ iff $\alpha_j \ge \alpha'_j$ for all $1 \le j \le k$. In particular this ordering restricts to bi-partitions λ. Define a map ϕ from pairs of compositions adding up to λ, to bi-partitions of λ as follows: let $\phi(\alpha, \beta)$ be the unique minimal bi-partition lying above the pair of compositions (α, β).*

In the above definition, it is not immediately clear why there must exist a unique minimal bi-partition above the pair of compositions (α, β). We justify this in the following proposition, by explicitly constructing $\phi(\alpha, \beta)$ in terms of (α, β).

Proposition 3.7. *The unique minimal bi-partitition $(\gamma, \delta) = \phi(\alpha, \beta)$ lying above the pair (α, β) is given by the following:*

$\gamma_i = \max(\{\alpha_j \mid j \ge i\} \cup \{\lambda_i - \beta_j \mid j < i\}),\ \delta_i = \min(\{\lambda_i - \alpha_j \mid j \ge i\} \cup \{\beta_j \mid j < i\})$

Proof. In order to prove the proposition, we must check that:

a) $(\gamma, \delta) \in \mathcal{Q}(n)$ and $\gamma + \delta = \lambda$,
b) $(\alpha, \beta) \dot{<} (\gamma, \delta)$
c) If $(\alpha, \beta) \dot{<} (\gamma', \delta')$ for some $(\gamma', \delta') \in \mathcal{Q}(n)$ with $\gamma' + \delta' = \lambda$, then $(\gamma, \delta) \dot{<} (\gamma', \delta')$.

To check (a), it is clear from the definitions that $\gamma + \delta = \lambda$, so it suffices to check that $\gamma_i \ge \gamma_{i+1}$ and $\delta_i \ge \delta_{i+1}$ for all $1 \le i \le k-1$. To show that $\gamma_i \ge \gamma_{i+1}$, we must show that

$\max\{\alpha_i, \cdots, \alpha_k, \lambda_i - \beta_1, \cdots, \lambda_i - \beta_{i-1}\} \ge \max\{\alpha_{i+1}, \cdots, \alpha_k, \lambda_{i+1} - \beta_1, \cdots, \lambda_{i+1} - \beta_i\}$

To do this, for every element u in the set $\{\alpha_{i+1}, \cdots, \alpha_k, \lambda_{i+1} - \beta_1, \cdots, \lambda_{i+1} - \beta_i\}$, we will show that there exists some element u' in the set $\{\alpha_i, \alpha_{i+1}, \cdots, \alpha_k, \lambda_i - \beta_1, \cdots, \lambda_i - \beta_{i-1}\}$ with $u' \geq u$. If $u = \alpha_j$ for some $i+1 \leq j \leq k$, we can simply pick $u' = u$. If $u = \lambda_{i+1} - \beta_j$ for some $1 \leq j \leq i-1$, then we can pick $u' = \lambda_i - \beta_j$. If $u = \lambda_{i+1} - \beta_i$, then we can pick $u' = \alpha_i$ (since $\lambda_{i+1} \leq \lambda_i = \beta_i + \alpha_i$). This concludes the proof that $\gamma_i \geq \gamma_{i+1}$. To show that $\delta_i \geq \delta_{i+1}$, we must show that

$$\min\{\lambda_i - \alpha_i, \cdots, \lambda_i - \alpha_k, \beta_1, \cdots, \beta_{i-1}\} \geq \min\{\lambda_{i+1} - \alpha_{i+1}, \cdots, \lambda_{i+1} - \alpha_k, \beta_1, \cdots, \beta_i\}$$

To do this we will prove that for any v in the set $\{\lambda_i - \alpha_i, \cdots, \lambda_i - \alpha_k, \beta_1, \cdots, \beta_{i-1}\}$, we can find a v' in the set $\{\lambda_{i+1} - \alpha_{i+1}, \cdots, \lambda_{i+1} - \alpha_k, \beta_1, \cdots, \beta_i\}$ with $v \geq v'$. If $v = \lambda_i - \alpha_j$ with $i+1 \leq j \leq k$, then we can simply pick $v' = \lambda_{i+1} - \alpha_j$. If $v = \lambda_i - \alpha_i = \beta_i$, we can pick $v' = \beta_i$. If $v = \beta_j$ for $1 \leq j \leq i-1$, we can pick $v' = \beta_j$. This concludes the proof that $\delta_i \geq \delta_{i+1}$, and hence that of (a).

To check (b), it suffices to show that $\alpha_i \leq \gamma_i$ for all $1 \leq i \leq k$. This follows from the definition $\gamma_i = \max\{\alpha_i, \alpha_{i+1}, \cdots, \alpha_k, \lambda_i - \beta_1, \cdots, \lambda_i - \beta_{i-1}\}$.

To check (c), we must show that $(\gamma, \delta) \dot{<} (\gamma', \delta')$, i.e. that

$$\gamma_i' \geq \gamma_i = \max\{\alpha_i, \alpha_{i+1}, \cdots, \alpha_k, \lambda_i - \beta_1, \cdots, \lambda_i - \beta_{i-1}\}$$

Thus we must first show that $\gamma_i' \geq \alpha_j$ for $i \leq j \leq k$. This is true because $\gamma_i' \geq \gamma_j' \geq \alpha_j$. We must also show that $\gamma_i' \geq \lambda_i - \beta_j$ for $1 \leq j \leq i-1$. This is equivalent to showing that $\beta_j \geq \lambda_i - \gamma_i' = \delta_i'$. The reason that this is true is because $\beta_j \geq \delta_j' \geq \delta_i'$, (where $\beta_j \geq \delta_j'$ is true since $\gamma_j' \geq \alpha_j$ and $\gamma_j' + \delta_j' = \alpha_j + \beta_j = \lambda_j$). □

Definition 3.8. *Let V_μ be the set of vectors $v \in V$ such that the type (α, β) of v satisfies $\phi(\alpha, \beta) = (\mu, \nu)$.*

It is clear that $v_\mu \in W_\mu \subset V_\mu$. We claim that $V_\mu = C_{GL_n(\mathbb{C})}(n_\lambda).v_\mu$. This is sufficient to prove the Theorem 3.3, since V is the disjoint union of all V_μ, as (μ, ν) ranges over $\mathcal{Q}(n)$. The fact V_μ is a single $C_{GL_n(\mathbb{C})}(n_\lambda)$-orbit is a consequence of the following three Lemmas. The reason for this is that Lemma 3.9 implies that V_μ is a union of $C_{GL_n(\mathbb{C})}(n_\lambda)$-orbits on V, while Lemma 3.10 applied to $\alpha = \mu$, along with Lemma 3.11, imply that $C_{GL_n(\mathbb{C})}(n_\lambda)$ acts transitively on V_μ.

Lemma 3.9. *The set V_μ is stable under the action of $C_{GL_n(\mathbb{C})}(n_\lambda)$.*

Lemma 3.10. *Any vector in W_α can be mapped to any other vector in W_α by some element of $C_{GL_n(\mathbb{C})}(n_\lambda)$.*

Lemma 3.11. *Given a vector v of type (α, β), there exists another v' of type $\phi(\alpha, \beta)$ in the same $C_{GL_n(\mathbb{C})}(n_\lambda)$-orbit as v.*

Proof. (of Lemma 3.10)
It suffices to show that v_α can be mapped to any other vector in W_α (since it is clear that $v_\alpha \in W_\alpha$). To illustrate the idea behind this proof, we revisit Example 3.4 with $\lambda = (3, 2, 2)$, $\alpha = (2, 1, 1)$. Consider the following choice of $g \in C_{GL_n(\mathbb{C})}(n_\lambda)$:

$$g = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & & & & \\ & \lambda_1 & \lambda_2 & & & & \\ & & \lambda_1 & & & & \\ & & & a_{11} & \lambda_{12} & & \\ & & & & a_{11} & & \\ & & & & & a_{22} & \lambda_{15} \\ & & & & & & a_{22} \end{pmatrix}, v_\alpha = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, gv_\alpha = \begin{pmatrix} \lambda_2 \\ \lambda_1 \\ 0 \\ a_{11} \\ 0 \\ a_{22} \\ 0 \end{pmatrix}$$

The only constraints on the matrix $g \in C_{GL_n(\mathbb{C})}(n_\lambda)$ above is that $\lambda_1, a_{11}, a_{12} \neq 0$, so the image gv_α ranges across all vectors of the form above subject to this constraint. But this is precisely how W_α is defined, concluding the proof in the case of this example. It is clear how to generalize this proof for $\lambda = (3, 2, 2)$ to arbitrary $\lambda = (\lambda_1, \cdots, \lambda_k)$. ☐

Proof. (of Lemma 3.11)
If $\phi(\alpha, \beta) = (\mu, \nu)$, we shall prove that v_α is in the same orbit as some vector $v' \in V_\mu$; this suffices since by Lemma 3.10 any vector of type (α, β) is in the same $C_{GL_n(\mathbb{C})}(n_\lambda)$-orbit as v_α. We will first prove that there exists $g \in C_{M_n(\mathbb{C})}(n_\lambda)$ such that $gv_\alpha \in W_\mu$.

By Proposition 3.7, $\mu_i = \max(\{\alpha_j | j \geq i\} \cup \{\lambda_i - \beta_j | j < i\})$. We will show that, for any $s \in \{\alpha_j | j \geq i\} \cup \{\lambda_i - \beta_j | j \leq i\}$, we can choose g, so that gv_α has a non-zero value in the s-th co-ordinate in the i-th portion, and all higher entries in the i-th portion are 0.

If $s = \alpha_j$, for some $j \geq i$, pick any g with a non-zero entry in the α_j-th row and α_j-th column of the (i, j)-th block of g. Since $j \geq i, \lambda_i \geq \lambda_j$, so this will be possible, since it permissible to have a non-zero entry in the r-th row and s-th column of the (i, j)-th block of g when $r \leq s$ in this case, by Proposition 1.14. We stipulate also that for all j', all other entries in the (i, j')-th blocks of g, apart from those forced to be equal to this entry, be 0. It follows then that gv_α will have a non-zero entry in the α_j-th coordinate of the i-th portion, as required. The fact that all higher entries in the i-th portion are 0 follows from the fact that all entries higher than α_j in the j-th portion of v_α are 0.

Otherwise if $s = \lambda_i - \beta_j$ for some $j \leq i$, pick a g with a non-zero entry in the $(\lambda_i - \beta_j)$-th row and α_j-th column of the (i, j)-th block of g. Since $j \leq i, \lambda_i \leq \lambda_j$, it is permissible to have a non-zero entry in the r-th row and s-th column of the (i, j)-th block only if $r - s \leq \lambda_i - \lambda_j$; thus this is permissible since $\lambda_i - \beta_j - \alpha_j = \lambda_i - \lambda_j$. We stipulate also that all other entries in the (i, j')-th blocks of g, apart from those already forced to be equal to this entry, be 0. It follows then that gv_α will have a non-zero entry in the $(\lambda_i - \beta_j)$-th coordinate of the i-th portion, as required. The fact that all higher entries in the i-th portion are 0 follows from the fact that all entries higher than α_j in the j-th portion of v_α are 0.

In the above paragraphs, we have shown by imposing certain restrictions on the coordinates in the (i, j)-th blocks of g, for fixed i, we can ensure that gv_α has a non-zero value in the μ_i-th co-ordinate in the i-th portion, and all higher entries in the i-th portion are 0. Since there are no dependencies between the entries in the (i, j)-th block and (i', j')-th block of g unless $i = i', j = j'$, we can impose all of these restrictions simultaneously, and find a g so that gv_α will have a non-zero value in the μ_i-th coordinate of the i-th portion, and zero values in all higher co-ordinates for each i. Equivalently, we have found a g so that gv_α has type (μ, ν).

Now note that $g + aI \in C_{M_n(\mathbb{C})}(n_\lambda)$, for all $a \in \mathbb{C}$, and $(g + aI)(v_\alpha) = gv_\alpha + av_\alpha$. It is clear that for all a with a finite number of exceptions, $gv_\alpha + av_\alpha$ will have type (μ, ν), and that for all a with a finite number of exceptions, $g + aI$ will be invertible. Taking a value of a outside the union of these two finite sets of exceptions, $g + aI \in C_{GL_n(\mathbb{C})}(n_\lambda)$ will have the required property.

□

Proof. (of Lemma 3.9)
The proof of Lemma 3.9 uses Lemmas 3.10 and Lemma 3.11. Suppose that V_μ is not stable under the action of $C_{GL_n(\mathbb{C})}(n_\lambda)$; i.e. suppose that for given some $v_1 \in V_{\mu_1}$ for some bi-partition (μ_1, ν_1), we can find some $v_2 \in V_{\mu_2}$ for some bi-partition (μ_2, ν_2) with v_1, v_2 being in the same $C_{GL_n(\mathbb{C})}(n_\lambda)$ orbit. Then by Lemmas 3.10 and 3.11, v_1 is in the same orbit as v_{μ_1}, and v_2 is in the same orbit as v_{μ_2}. Thus v_{μ_1} and v_{μ_2} are in the same orbit; we will now show that this is not possible.

We claim that for $g \in C_{GL_n(\mathbb{C})}(n_\lambda)$, the l^{th} co-ordinate of the i^{th} portion of $g.v_\mu$ is 0 for $l > \mu_i$. (The computation in Example 3.4 of gv_μ illustrates this claim.) This will then imply that v_{μ_1} and v_{μ_2} cannot be in the same $C_{GL_n(\mathbb{C})}(n_\lambda)$-orbit, since for some i, since $\mu_1 \neq \mu_2$, WLOG we must have $(\mu_1)_i > (\mu_2)_i$. If $v_{\mu_1} = g.v_{\mu_2}$, then the vector v_{μ_1} will have a non-zero entry, 1, in the $(\mu_1)_i$-th position, which contradicts the above claim since $(\mu_1)_i > (\mu_2)_i$.

To show that the l^{th} co-ordinate of the i^{th} portion of $g.v_\mu$ is 0 for $l > \mu_i$, we must examine the structure of the centralizer $C_{GL_n(\mathbb{C})}(n_\lambda)$. We are required to prove that the $(\lambda_1 + \cdots + \lambda_{i-1} + l)$-th coordinate of $g.v_\mu$ is 0; accordingly, examine the entries in the $(\lambda_1 + \cdots + \lambda_{i-1} + l)$-th row of the matrix $g \in C_{GL_n(\mathbb{C})}(n_\lambda)$. The non-zero entries in the column vector v_μ occur in the co-ordinates $(\lambda_1 + \cdots + \lambda_{j-1} + \mu_j)$-th positions for $j = 1, \cdots, k$. Thus to prove that the $(\lambda_1 + \cdots + \lambda_{i-1} + l)$-th coordinate of $g.v_\mu$ is 0, it suffices to prove that the $(\lambda_1 + \cdots + \lambda_{j-1} + \mu_j)$-th coordinate in the $(\lambda_1 + \cdots + \lambda_{i-1} + l)$-th row of g is 0 for all $l > \mu_i$ and all j. To do this, we must separately examine the cases $i \leq j$ and $i > j$.

If $i \leq j$, note that in the (i, j)-th block of the matrix g, which is of size $\lambda_i \times \lambda_j$, the (u, v)-th coordinate is 0 if $u > v$, by Proposition 1.14. Here we are interested in the co-ordinate in the $(\lambda_1 + \cdots + \lambda_{i-1} + l)$ row and the $(\lambda_1 + \cdots + \lambda_{j-1} + \mu_j)$ column of g. This lies in the l-th row and μ_j-th column of the (i, j)-th block of the matrix

g. But $l > \mu_i \geq \mu_j$ since $i < j$, which concludes the proof of the claim in this case.

If $i > j$, then note that in the (i, j)-th block of the matrix g, which is of size $\lambda_i \times \lambda_j$, the (u, v)-th coordinate is 0 if $v - u < \lambda_j - \lambda_i$, by Proposition 1.14. Again, here we are interested in the co-ordinate in the $(\lambda_1 + \cdots + \lambda_{i-1} + l)$-th row and the $(\lambda_1 + \cdots + \lambda_{j-1} + \mu_j)$-th column of g, which lies in the l-th row and the μ_j-th column of the (i, j)-th block of g. But $\mu_j - l < \mu_j - \mu_i \leq \lambda_j - \lambda_i$; where $\mu_j - \mu_i \leq \lambda_j - \lambda_i$ since $\lambda_i - \mu_i \leq \lambda_j - \mu_j$ (i.e. $\nu_i \leq \nu_j$, which is true since $i > j$). This concludes the proof of the claim in the second case, and hence the proof of Lemma 3.9.

□

3.2. The exotic nilpotent cone

In this section, we study Kato's exotic nilpotent cone, $\mathfrak{N}$. We state the classification of the orbits in the exotic nilpotent cone, and describe part of the proof of this classification. Let $V \cong \mathbb{C}^{2n}$ be a vector space of dimension $2n$, with a symplectic form $\langle \cdot, \cdot \rangle$, $\langle v, w \rangle = v^t J w$ corresponding to the following $2n \times 2n$ matrix J:

$$\begin{pmatrix} & & & & & 1 \\ & & & & \cdots & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdots & & & & \\ -1 & & & & & \end{pmatrix}$$

Following [**2**], define $S, \mathfrak{N}$ as follows (here $\mathcal{N}$ denotes $\mathcal{N}(\mathfrak{gl}(V))$, the set of nilpotent matrices in End(V)). $\mathfrak{N}$ is the exotic nilpotent cone.

$$\begin{aligned} S &= \{x \in \text{End}(V) | \langle xv, w \rangle = \langle v, xw \rangle, \forall v, w \in V\} \\ \mathfrak{N} &= V \times (S \cap \mathcal{N}) \end{aligned}$$

The following calculation shows that there is a well-defined action of $Sp(V)$ on S by conjugation. Let $g \in Sp(V)$, so that $\langle gv, gw \rangle = \langle v, w \rangle$ for all v, w, and let $x \in S$:

$$\begin{aligned} \langle gxg^{-1}v, w \rangle &= \langle gxg^{-1}v, gg^{-1}w \rangle = \langle xg^{-1}v, g^{-1}w \rangle \\ &= \langle g^{-1}v, xg^{-1}w \rangle = \langle gg^{-1}v, gxg^{-1}w \rangle \\ &= \langle v, gxg^{-1}w \rangle \end{aligned}$$

Since $Sp(V)$ clearly acts by conjugation on $\mathcal{N}$, $Sp(V)$ acts by conjugation on $S \cap \mathcal{N}$. There is also a natural action of $Sp(V)$ on V, and hence there is an action of $Sp(V)$ on the exotic nilpotent cone $\mathfrak{N} = V \times (S \cap \mathcal{N})$.

There is a natural inclusion map $\phi_2 : \mathfrak{N} \to V \times \mathcal{N}$, of the exotic nilpotent cone into the enhanced nilpotent cone of size $2n$. We next construct a map $\phi_1 : V_1 \times \mathcal{N}_1 \to \mathfrak{N}$, where $V_1 \cong \mathbb{C}^n$ is a vector space of dimension n, and $\mathcal{N}_1 = \mathcal{N}(\mathfrak{gl}(V_1))$ is the set of nilpotent matrices in End(V_1); thus $V_1 \times \mathcal{N}_1$ is the enhanced nilpotent cone of size n.

Lemma 3.12. *The following map $\phi_1 : V_1 \times \mathcal{N}_1 \to \mathfrak{N}$ is well-defined. Here $x \in \mathcal{N}_1$, and x^t denotes its transpose along the skew diagonal, so that $x^t_{i,j} = x_{n+1-j,n+1-i}$.*

$$\phi_1(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, x) = (\begin{pmatrix} v_1 \\ \vdots \\ v_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} x & \\ & x^t \end{pmatrix})$$

Proof. Let $X = \left(\begin{smallmatrix} x & \\ & x^t \end{smallmatrix}\right)$. It is clear that if x is nilpotent, then x^t is nilpotent, and so X is nilpotent, i.e. $X \in \mathcal{N}$. Thus it suffices to prove that $X \in S$, i.e. $\langle Xv, w\rangle = \langle v, Xw\rangle$ for all $v, w \in V$. Letting e_i be the vector with a 1 in the i-th co-ordinate and zeroes elsewhere, since $\{e_1, \cdots, e_{2n}\}$ is a basis for V, it sufficient to prove that $\langle Xe_i, e_{i'}\rangle = \langle e_i, Xe_{i'}\rangle$. By construction of $\langle \cdot, \cdot\rangle$, recall that:

$$\langle e_i, e_{i'}\rangle = \begin{cases} 0 & \text{if } i + i' \neq 2n+1, \\ 1 & \text{if } i < i', i + i' = 2n+1, \\ -1 & \text{if } i > i', i + i' = 2n+1 \end{cases}$$

If $1 \le i, i' \le n$, then $Xe_i, Xe_{i'} \in \text{Span}\{e_1, \cdots, e_n\}$, and $\langle Xe_i, e_{i'}\rangle = \langle e_i, Xe_{i'}\rangle = 0$. Similarly if $n+1 \le i, i' \le 2n$, $\langle Xe_i, e_{i'}\rangle = \langle e_i, Xe_{i'}\rangle = 0$. If $1 \le i \le n, n+1 \le i' \le 2n$, then we compute $\langle Xe_i, e_{i'}\rangle, \langle e_i, Xe_{i'}\rangle$ as follows. Let $i' = j + n$, with $1 \le j \le n$.

$$\langle Xe_i, e_{j+n}\rangle = \langle x_{1,i}e_1 + \cdots + x_{n,i}e_n, e_{j+n}\rangle = x_{n+1-j,i}$$
$$\langle e_i, Xe_{j+n}\rangle = \left\langle e_i, x^t_{1,j}e_{n+1} + \cdots + x^t_{n,j}e_{2n}\right\rangle = x^t_{n+1-i,j}$$

By the definition of x^t, $x_{n+1-j,i} = x^t_{n+1-i,j}$, so $\langle Xe_i, e_{j+n}\rangle \langle e_i, Xe_{j+n}\rangle$ in this case. A similar argument works in the case when $n+1 \le i \le 2n, 1 \le i' \le n$. Hence $X \in S$, as required. $\square$

Lemma 3.13. *Suppose (v, n_1) and (w, n_2) are in the same $GL(V_1)$ orbit of $V_1 \times \mathcal{N}_1$. Then $\phi(v, n_1)$ and $\phi(w, n_2)$ are in the same $Sp(V)$-orbit of $\mathfrak{N}$.*

Proof. Suppose $g(v, n_1) = (w, n_2)$. Define $g_1 = \left(\begin{smallmatrix} g & \\ & (g^t)^{-1} \end{smallmatrix}\right)$. We will first check that $g_1 \in Sp(V)$, and then verify that $g_1(\phi(v, n_1)) = \phi(w, n_2)$.

To check that $g_1 \in Sp(V)$, it is necessary to check that $g_1^T J g_1 = J$ (g_1^T denotes the ordinary transpose). Here $J = \left(\begin{smallmatrix} & I' \\ -I' & \end{smallmatrix}\right)$; where I' denotes the $n \times n$ matrix with 1's along the skew diagonal. To help with this computation, note first that $g^T I' = I' g^t$, since by inspection both matrices will have $g_{n+1-j,i}$ as their (i,j)-th co-ordinate. Then we obtain that $g^T I'(g^t)^{-1} = I'$; and transposing the equation we also have $I'g = {g^t}^T I'$, so $({g^t}^T)^{-1} I' g = I'$. Now we compute:

$$\begin{aligned}
&\begin{pmatrix} g & \\ & (g^t)^{-1} \end{pmatrix}^T \begin{pmatrix} & I' \\ -I' & \end{pmatrix} \begin{pmatrix} g & \\ & (g^t)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} g^T & \\ & ({g^t}^T)^{-1} \end{pmatrix} \begin{pmatrix} & I'(g^t)^{-1} \\ -I'g & \end{pmatrix} \\
&= \begin{pmatrix} & g^T I'(g^t)^{-1} \\ -({g^t}^T)^{-1} I' g & \end{pmatrix} = \begin{pmatrix} & I' \\ -I' & \end{pmatrix}
\end{aligned}$$

Hence $g_1 \in Sp(V)$. Next, it is clear that if $gv = w$, then $\left(\begin{smallmatrix} g & \\ & (g^t)^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right)$. Since $g(v, n_1) = (w, n_2)$, we have $g n_1 g^{-1} = n_2$. Since $(AB)^t = B^t A^t$ (one way of seeing this is that, from above, $x^T I' = I' x^t$ so $x^t = I'^{-1} x^T I'$, and this identity holds for the ordinary transpose), taking the transpose of $g n_1 g^{-1} = n_2$ gives $(g^t)^{-1} n_1^t g^t = n_2^t$. Now we compute:

$$\begin{aligned}
&\begin{pmatrix} g & \\ & g^{t-1} \end{pmatrix} \begin{pmatrix} n_1 & \\ & n_1^t \end{pmatrix} \begin{pmatrix} g & \\ & g^{t-1} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} g & \\ & g^{t-1} \end{pmatrix} \begin{pmatrix} n_1 & \\ & n_1^t \end{pmatrix} \begin{pmatrix} g^{-1} & \\ & g^t \end{pmatrix} \\
&= \begin{pmatrix} g n_1 g^{-1} & \\ & (g^t)^{-1} n_1^t g^t \end{pmatrix} = \begin{pmatrix} n_2 & \\ & n_2^t \end{pmatrix}
\end{aligned}$$

This proves that $g_1(\phi(v, n_1)) = \phi(w, n_2)$, concluding the proof. □

From results in section 3.1, we know that the orbits of $GL(V_1)$ on $V_1 \times \mathcal{N}_1$ are in bijection with bi-partitions of n; call $\mathcal{O}_{\mu,\nu}$ the orbit corresponding to the bi-partition (μ, ν). By the above Lemma, any two elements in $\phi(\mathcal{O}_{\mu,\nu})$ will be in the same $Sp(V)$ orbit of $\mathfrak{N}$.

Definition 3.14. *Define $\mathbb{O}_{\mu,\nu}$ to be the $Sp(V)$-orbit in $\mathfrak{N}$ containing $\phi_1(\mathcal{O}_{\mu,\nu})$.*

Given a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$, let $\lambda \cup \lambda$ denote the partition $(\lambda_1, \lambda_1, \cdots, \lambda_k, \lambda_k)$.

Proposition 3.15. *We have that $\phi_2(\phi_1(\mathcal{O}_{\mu,\nu})) \subseteq \mathcal{O}_{\mu\cup\mu,\nu\cup\nu}$.*

Proof. As in the proof of Theorem 3.3, let (v_μ, n_λ) be an orbit representative for $\mathcal{O}_{\mu,\nu}$ in $V_1 \times \mathcal{N}_1$. It suffices to prove that $\phi_2(\phi_1(v_\mu, n_\lambda)) \in \mathcal{O}_{\mu\cup\mu,\nu\cup\nu}$. Consider the following example, where $\lambda = (2, 1), \mu = (1, 1), \nu = (1, 0)$.

$$\phi_2(\phi_1(v_\mu, n_\lambda)) = (\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix})$$

This example makes it clear that while n_λ consists of the blocks $N_{\lambda_1}, \cdots, N_{\lambda_k}$ arranged in that order, n_λ^t will consists of the same blocks arranged in the reverse order: $N_{\lambda_k}, \cdots, N_{\lambda_1}$. Since n_λ and n_λ^t have the same Jordan type, it follows then that we can find a matrix g, such that $gn_\lambda g^{-1} = n_\lambda^t$. Let g' be a $2n \times 2n$ matrix consisting of the four $n \times n$ quadrants, with the top left quadrant being the identity matrix, the bottom right quadrant being g, and the other two quadrants empty. Acting on the above pair by the matrix g', g' will stabilize the vector and transform the matrix into a matrix with n_λ occurring twice:

$$(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & 1 & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix})$$

Now act on the above pair by the matrix h which will conjugate the matrix above to the standard nilpotent $n_{\lambda\cup\lambda}$. It is clear what effect this will have on the vector.

$$(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & & & \\ & 0 & & & & \\ & & 0 & 1 & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix})$$

It is clear how this example generalizes: we have shown that $\phi_2(\phi_1(v_\mu, n_\lambda))$ is in the same orbit as $(v'_\mu, n_{\lambda\cup\lambda})$, where v'_μ is a vector with its $(2i-1)$-st portion consisting of a 1 in the μ_i-th coordinate and 0-s elsewhere, and its $2i$-th portion zero for all $1 \leq i \leq k$. We can now use the results in Section 3.1 to compute the orbit in $V \times \mathcal{N}$ that it lies in.

The type of the vector v'_μ is (α, β), where

$$\alpha = (\mu_1, 0, \mu_2, 0, \cdots, \mu_k, 0), \quad \beta = (\nu_1, \lambda_1, \nu_2, \lambda_2, \cdots, \nu_k, \lambda_k).$$

Using Proposition 3.7 and Lemmas $3.8 - 3.10$, the orbit which $(v'_\mu, n_{\lambda\cup\lambda})$ lies in will be given by the bi-partition $\phi(\alpha,\beta) = (\gamma,\delta)$. We have that

$$\gamma_{2i} = \max(\{\alpha_j \mid j \geq 2i\} \cup \{\lambda_i - \beta_j | j < 2i\})$$

Inspecting the values of $\{\alpha_j | j \geq 2i\}$, it is clear that $\max\{\alpha_j | j \geq 2i\} = \mu_{i+1}$. Inspecting the values of $\{\beta_j | j < 2i\}$, it is clear that we have $\min\{\beta_j | j < 2i\} = \min(\nu_1, \lambda_1, \nu_2, \lambda_2, \ldots, \nu_i) = \nu_i$, so $\max\{\lambda_i - \beta_j | j < 2i\} = \lambda_i - \nu_i = \mu_i$. Hence $\gamma_{2i} = \max(\mu_{i+1}, \mu_i) = \mu_i$. Next, we have

$$\gamma_{2i-1} = \max(\{\alpha_j | j \geq 2i-1\} \cup \{\lambda_i - \beta_j | j < 2i-1\})$$

Inspecting the values of $\{\alpha_j | j \geq 2i-1\}$, it is clear that $\max\{\alpha_j | j \geq 2i-1\} = \mu_i$. Inspecting the values of $\{\beta_j | j < 2i-1\}$, it is clear that $\min\{\beta_j | j < 2i-1\} = \min(\nu_1, \lambda_1, \nu_2, \lambda_2, \ldots, \nu_{i-1}, \lambda_{i-1}) = \nu_{i-1}$, so $\max\{\lambda_i - \beta_j | j \leq 2i-1\} = \lambda_i - \nu_{i-1}$. Hence $\gamma_{2i-1} = \max(\mu_i, \lambda_i - \nu_{i-1}) = \mu_i$, since $\nu_{i-1} \geq \nu_i = \lambda_i - \mu_i$, so $\mu_i \geq \lambda_i - \nu_{i-1}$. Thus $\gamma_{2i-1} = \gamma_{2i} = \mu_i$, proving that $\gamma = \mu \cup \mu$, and hence $\delta = \nu \cup \nu$. This concludes the proof that $\phi_2(\phi_1(v_\mu, n_\lambda)) \in \mathcal{O}_{\mu\cup\mu,\nu\cup\nu}$.

□

Corollary 3.16. *The orbits $\mathbb{O}_{\mu,\nu}$ and $\mathbb{O}_{\mu',\nu'}$ are distinct if $(\mu,\nu) \neq (\mu',\nu')$ are distinct bi-partitions.*

Proof. This follows from the above Lemma, since $\phi_2(\mathbb{O}_{\mu,\nu})$ and $\phi_2(\mathbb{O}_{\mu',\nu'})$ belong to $\mathcal{O}_{\mu\cup\mu,\nu\cup\nu}$ and $\mathcal{O}_{\mu'\cup\mu',\nu'\cup\nu'}$, respectively. □

Theorem 3.17. *The orbits of $Sp(V)$ on $\mathcal{N}'$ are in bijection with the bi-partitions of n, with the bipartition (μ,ν) corresponding to the orbit $\mathbb{O}_{\mu,\nu}$.*

Proof. All that remains to be proved is that every point in $\mathfrak{N}$ lies in one of the orbits $\mathbb{O}_{\mu,\nu}$. For this, we refer the reader to Theorem 1.14 in [**6**]. □

3.3. The 2-enhanced nilpotent cone

Here we describe some partial results regarding the orbits of $G = GL_n(\mathbb{C})$ on the 2-enhanced nilpotent cone, defined below.

Definition 3.18. *The 2-enhanced nilpotent cone is defined to be $V \times V \times \mathcal{N}$.*

The action of G on the 2-enhanced nilpotent cone is clear, since G acts naturally on V, and acts by conjugation on $\mathcal{N}$. This is the first instance so far in which we will find that there are infinitely many orbits. The problem of computing the orbits of $GL_n(\mathbb{C})$ on $V \times V \times \mathcal{N}$ is equivalent to the problem of computing the orbits of $C_{GL_n(\mathbb{C})}(n_\lambda)$ on $V \times V$. To see this, if two elements $(v_1, w_1, n_1), (v_2, w_2, n_2) \in V \times V \times \mathcal{N}$ are in the same $GL_n(\mathbb{C})$-orbit, then n_1 and n_2 are nilpotent elements with the same Jordan form, say corresponding to the partition λ. Thus it suffices to check when two vectors (v'_1, w'_1, n_λ) and (v'_2, w'_2, n_λ) are in the same $GL_n(\mathbb{C})$-orbit. This happens iff there exists g with $g^{-1}n_\lambda g = n_\lambda$ such that $g(v'_1, w'_1) = (v'_2, w'_2)$; i.e. iff $(v'_1, w'_1), (v'_2, w'_2)$ lie in the same $C_{GL_n(\mathbb{C})}(n_\lambda)$ orbit.

Example 3.19. Let $\lambda = (4, 2, 1)$, and let $\mu = (2, 2, 1), \nu = (2, 0, 0)$, so that (μ, ν) is a bi-partition of λ. Pick $v \in V$ arbitrary, and $g \in C_{GL_n(\mathbb{C})}(n_\lambda)$. In the following calculation, v should be considered as the union of the 1st portion (v_1, v_2, v_3, v_4), the 2nd portion (v_5, v_6), and the 3rd portion (v_7). It is clear that the 7 coordinate functions specifying the vector gv can be expressed as bilinear functions in the variables λ_i specifying the coordinates of g, and the variables v_i describing the vector v. For $1 \leq j \leq 7$, define f_j to be the coordinate function describing the j-th coordinate of the vector gv. Using the below example as a starting point, we will make an observation about which variables precisely the functions f_j depend upon.

$$g = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ & \lambda_1 & \lambda_2 & \lambda_3 & & \lambda_{12} & \\ & & \lambda_1 & \lambda_2 & & & \\ & & & \lambda_1 & & & \\ & & \lambda_8 & \lambda_9 & \lambda_5 & \lambda_6 & \lambda_{15} \\ & & & \lambda_8 & & \lambda_5 & \\ & & & \lambda_{10} & & \lambda_{11} & \lambda_7 \end{pmatrix}, v = \begin{pmatrix} * \\ * \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix}$$

$$gv = \begin{pmatrix} * \\ * \\ \lambda_1 v_3 + \lambda_2 v_4 \\ \lambda_1 v_4 \\ \lambda_8 v_3 + \lambda_9 v_4 + \lambda_5 v_5 + \lambda_6 v_6 + \lambda_{15} v_7 \\ \lambda_8 v_4 + \lambda_5 v_6 \\ \lambda_{10} v_4 + \lambda_{11} v_6 + \lambda_7 v_7 \end{pmatrix}$$

In the above example, we note that the functions f_3, f_4, f_5, f_6, f_7 do not depend on the variables v_1 and v_2 (the variables v_1 and v_2 have been left out to emphasize this fact). Equivalently, the 3rd, 4th, 5th, 6th and 7th coordinates of gv can be expressed as linear functions of the 3rd, 4th, 5th, 6th and 7th coordinates of v. This leads to the question: what is so special about those particular co-ordinates which results in this phenomena? To answer this question, the co-ordinate v_3 and v_4 are the last 2 coordinates in the 1st portion of v, v_5 and v_6 are the last 2 co-ordinates in the 2nd portion of v, and the coordinate v_7 is the last 1 coordinate in the 3rd portion of v; and the triple $(2, 2, 1)$ specifies the bi-partition (μ, ν) of λ. The generalization of this phenomenon is stated in the below Proposition.

Proposition 3.20. *Let the vector v have portions of sizes $(\lambda_1, \cdots, \lambda_k)$, let (μ, ν) be a bi-partition of λ, and let $S_{\mu,\nu} = \{\mu_1+1, \cdots, \lambda_1, \lambda_1+\mu_2+1, \cdots, \lambda_1+\lambda_2, \cdots, \lambda_1+ \cdots + \lambda_{k-1} + \mu_k + 1, \cdots, \lambda_1 + \cdots + \lambda_k\}$ (here $|S_{\mu,\nu}| = |\nu|$). If $g \in C_{GL_n(\mathbb{C})}(n_\lambda)$, then for $j \in S_{\mu,\nu}$, the j-th coordinate of the vector gv can be expressed as a sum of products of coefficients of g and co-ordinates v_i as i ranges over S_μ.*

Proof. Since $j \in S_{\mu,\nu}$, let $j = \lambda_1 + \cdots + \lambda_{s-1} + p$, for some $\mu_s + 1 \leq p \leq \lambda_s$ (here $1 \leq s \leq k$). We will examine the j-th row of the matrix g. It is sufficient to prove that the entry in the i-th column and j-th row of the matrix g is 0 if $i \notin S_{\mu,\nu}$.

The required result will then follow since the j-th coordinate of gv, which is the product of the row vector formed by the j-th row of g with the column vector v, will only contain terms involving the co-ordinates v_i for $i \in S_{\mu,\nu}$. Since $i \notin S_{\mu,\nu}$, let $i = \lambda_1 + \cdots + \lambda_{t-1} + q$, for some $1 \leq q \leq \mu_t$. If we divide up the matrix g into k^2 blocks (with the (u,v)-th block having dimension $\lambda_u \times \lambda_v$), as described in Section 2.3.1, the entry in the j-th row and i-th column will be the p-th row and q-th column in the (s,t)-th block. We consider the two cases where $s \leq t$ and $s > t$.

If $s \leq t$, we have that $\lambda_s \geq \lambda_t$; and the co-ordinate in the p-th row and the q-th column will be zero if $p > q$. We know that $p \geq \mu_s + 1 > \mu_t \geq q$; here $p \geq \mu_s$ and $q \leq \mu_t$ are from the above paragraph, while $\mu_s \geq \mu_t$ is because $s \leq t$. This completes the proof when $s \leq t$.

If $s > t$, we have that $\lambda_s \leq \lambda_t$; and the co-ordinate in the p-th row and the q-th column will be zero if $q - p < \lambda_t - \lambda_s$. Since $q \leq \mu_t$, $\mu_s + 1 \leq p$, $q - p < \mu_t - \mu_s \leq \lambda_t - \lambda_s$. The last inequality is because $\nu_s \leq \nu_t$ (since $s > t$), so $\lambda_s - \mu_s \leq \lambda_t - \mu_t$, so $\mu_t - \nu_s \leq \lambda_t - \lambda_s$. This completes the proof when $s > t$. □

Definition 3.21. *Given a bi-partition (μ, ν) of λ, let $T_{\mu,\nu} = \{v \in V | v_i = 0$ for all $i \in S_{\mu,\nu}\}$.*

Definition 3.22. *Let $X_{\mu,a_0,a_1,\cdots,a_{\nu_1-1}} = \{(v,w) \in V \times V | (a_0 + a_1 n_\lambda + \cdots + a_{\nu_1-1} n_\lambda^{\nu_1-1})v - w \in T_{\mu,\nu}\}$.*

Proposition 3.23. *The set $X_{\mu,a_0,a_1,\cdots,a_{\nu_1-1}} \subset V \times V$ is stable under the action of $C_{GL_n(\mathbb{C})}(n_\lambda)$.*

Example 3.24. Let $\lambda = (4,2,1)$. To illustrate the above concepts, the following is a description of the sets $T_{\mu,\nu}$ and $X_{\mu,a_0,\cdots,a_{\nu_1-1}}$, for the bi-partition $\mu = (2,0,0), \nu = (2,2,1)$. In the below examples, all vectors should be consisted as consisting of the first portion (the first 4 co-ordinates), the second portion (the following 2 co-ordinates), and the third portion (the last co-ordinate).

$$T_{\mu,\nu} = \{ \begin{pmatrix} * \\ * \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \}, a_0 + a_1 n_\lambda = \begin{pmatrix} a_0 & a_1 & & & & & \\ & a_0 & a_1 & & & & \\ & & a_0 & a_1 & & & \\ & & & a_0 & & & \\ & & & & a_0 & a_1 & \\ & & & & & a_0 & \\ & & & & & & a_0 \end{pmatrix}$$

$$\text{If } v = \begin{pmatrix} * \\ * \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix}, (a_0 + a_1 n_\lambda)v = \begin{pmatrix} * \\ * \\ a_0v_3 + a_1v_4 \\ a_0v_4 \\ a_0v_5 + a_1v_6 \\ a_0v_6 \\ a_0v_7 \end{pmatrix}$$

$$\text{Hence } (v, w) \in X_{\mu,a_0,a_1} \text{ iff } w = \begin{pmatrix} w_1 \\ w_2 \\ a_0v_3 + a_1v_4 \\ a_0v_4 \\ a_0v_5 + a_1v_6 \\ a_0v_6 \\ a_0v_7 \end{pmatrix} \text{ for some } w_1, w_2.$$

Now we return to the proof of Proposition 3.23.

Proof. (of Proposition 3.23)
Suppose $(v, w) \in X_{\mu,a_0,a_1,\cdots,a_{\nu_1-1}}$; we need to prove that $(gv, gw) \in X_{\mu,a_0,a_1,\cdots,a_{\nu_1-1}}$ for all $g \in C_{GL_n(\mathbb{C})}(n_\lambda)$. This is equivalent to saying that $(a_0 + a_1n_\lambda + \cdots + a_{\mu_1-1}n_\lambda^{\nu_1-1})gv - gw \in T_{\mu,\nu}$.

Since g commutes with n_λ, g commutes with all powers of n_λ, so $(a_0 + a_1n_\lambda + \cdots + a_{\nu_1-1}n_\lambda^{\nu_1-1})gv = g(a_0 + a_1n_\lambda + \cdots + a_{\nu_1-1}n_\lambda^{\nu_1-1})v$. Thus we must prove that $g[(a_0 + a_1n_\lambda + \cdots + a_{\nu_1-1}n_\lambda^{\nu_1-1})v - w] \in T_{\mu,\nu}$. Since $(v, w) \in X_{\mu,a_0,a_1,\cdots,a_{\nu_1-1}}$, $(a_0 + a_1n_\lambda + \cdots + a_{\mu_1-1}n_\lambda^{\mu_1-1})v - w \in T_{\mu,\nu}$. Thus it is sufficient for us to show that $T_{\mu,\nu}$ is stable under the action of $C_{GL_n(\mathbb{C})}(n_\lambda)$.

The fact that $T_{\mu,\nu}$ is stable under the action of $C_{GL_n(\mathbb{C})}(n_\lambda)$ is clear using Proposition 3.20 by the following argument. Suppose $v \in T_{\mu,\nu}$, i.e. that $v_i = 0$ for all $i \in S_{\mu,\nu}$. Given $g \in G$, the i-th co-ordinate of gv is a linear combination of co-ordinates from g and co-ordinates v_j for $j \in S_{\mu,\nu}$, and since $v_j = 0$ for $j \in S_{\mu,\nu}$, the i-th co-ordinate of gv is 0 for $i \in S_{\mu,\nu}$. Hence $gv \in T_{\mu,\nu}$, completing the proof. □

Thus we have found certain sets which are stable under the action of $C_{GL_n(\mathbb{C})}(n_\lambda)$. Using a variant of this method, we next describe the orbits of $C_{GL_n(\mathbb{C})}(n_\lambda)$ on $V \times V$ in the case where $\lambda = (p^q)$ is a rectangle partition. (Here $(p^q) = (p, p, \cdots, p)$, where p occurs q times.)

Using Theorem 3.3, we can assume that the second vector is in standard form, v_μ for some bi-partition $(\mu, \nu) = (r^q, (p - r)^q)$ of λ. Then the problem is equivalent to classifying the orbits of $G^{(v_\mu,n_\lambda)}$ on V, where $G^{(v_\mu,n_\lambda)}$ is the stabilizer in $G = GL_n(\mathbb{C})$ of the pair (v_μ, n_λ). For convenience, let $G^r = G^{(n_\lambda,v_\mu)}$. Below we describe the structure of G^r. Recall from Proposition 1.14 that for any $g \in G^{n_\lambda}$, g can be divided into q^2 $p \times p$ blocks $g_{i,j}$, with $1 \leq i, j \leq q$, with $g_{i,j}$ upper triangular with entries on the diagonals equal. Call the entries in the first row of the block $g_{i,j}$, $g_1^{i,j}, g_2^{i,j}, \cdots, g_p^{i,j}$. The matrix g must satisfy the condition that the matrix $(g_1^{i,j})$ is invertible.

Lemma 3.25. *A matrix g of the form above, will lie in G^r if and only if it satisfies the following additional conditions, for $1 \leq i \leq q$:*

$$\sum_{j=1}^{q} g_k^{i,j} = 1 \text{ if } k = 1, \sum_{j=1}^{q} g_k^{i,j} = 0 \text{ if } 2 \leq k \leq r$$

If $r = 0$ there are no additional restrictions on G^r.

Proof. First consider the example $p = 4, q = 2, r = 3$. For convenience, we use different notation for the entries of g to that above.

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & a_2 & b_2 & c_2 & d_2 \\ & a_1 & b_1 & c_1 & & a_2 & b_2 & c_2 \\ & & a_1 & b_1 & & & a_2 & b_2 \\ & & & a_1 & & & & a_2 \\ a_3 & b_3 & c_3 & d_3 & a_4 & b_4 & c_4 & d_4 \\ & a_3 & b_3 & c_3 & & a_4 & b_4 & c_4 \\ & & a_3 & b_3 & & & a_4 & b_4 \\ & & & a_3 & & & & a_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ b_1 + b_2 \\ a_1 + a_2 \\ 0 \\ c_3 + c_4 \\ b_3 + b_4 \\ a_3 + a_4 \\ 0 \end{pmatrix}$$

Equating the coefficients gives $a_1 + a_2 = a_3 + a_4 = 1, b_1 + b_2 = b_3 + b_4 = 0, c_1 + c_2 = c_3 + c_4 = 0$. This proves it in the case of this example, and it is clear how this example will generalize. □

Definition 3.26. *Let $v \in V$, $v = (v_{1,1}, \cdots, v_{1,p}, v_{2,1}, \cdots, v_{2,p}, \cdots, v_{q,1}, \cdots, v_{q,p})$. For $1 \leq i \leq p$, let $v_i = (v_{1,i}, v_{2,i}, \cdots, v_{q,i})$. Define the height h of v to be the maximal h such that v_h is non-zero (so $v_{h+1} = \cdots = v_p = 0, v_h \neq 0$).*

It is clear the set of all vectors of a fixed height h will be stable under the action of G^{n_λ}, and hence G^r. To see this, the set of all vectors of a fixed height h is the set $T_{\mu_1,\nu_1} - T_{\mu_2,\nu_2}$, where $(\mu_1, \nu_1) = (h^q, (p-h)^q), (\mu_2, \nu_2) = ((h-1)^q, (p-h+1)^q)$. The result then follows from the fact that T_{μ_1,ν_1} and T_{μ_2,ν_2} are stable under the action of G^{n_λ}, which is shown in the proof of Proposition 3.23.

Definition 3.27. *Let the type of v be a sequence $(\lambda_1, \cdots, \lambda_j)$ defined as follows. Let $j = min(j', r)$, where $j' \leq h$ is maximal such that $v_h, v_{h-1}, \cdots, v_{h+1-j'} \in \mathbb{C}w_0$; here $w_0 = (1, 1, \cdots, 1)$ (with q '1's). For $1 \leq i \leq j$, let λ_i be the scalar such that $v_{h+1-i} = \lambda_i w_0$.*

Note that in the above definition, $j = 0$ is allowed, in which case the type will be empty. If $j > 0$, then $\lambda_1 \neq 0$, since $v_h = \lambda_1 w_0$ and $v_h \neq 0$ by definition of h.

Lemma 3.28. *The set of all vectors of height h and type $(\lambda_1, \cdots, \lambda_j)$ is stable under the action of G^r.*

Proof. It suffices to show that the set of all vectors with height h and satisfying $v_h = \lambda_1 w_0, \cdots, v_{h+1-j} = \lambda_j w_0$ is stable under the action of G^r. The reason why this suffices is as follows: if we have a vector v_1 of type $(\lambda_1, \cdots, \lambda_j)$ in the same orbit as a vector v_2, then if v_2 does not have type $(\lambda_1, \cdots, \lambda_j)$ then v_2 is forced to have type $(\lambda_1, \cdots, \lambda_j, \lambda_{j+1}, \cdots)$; but then v_1, which is in the same orbit as v_2, will now be forced to have type $(\lambda_1, \cdots, \lambda_j, \lambda_{j+1}, \cdots)$, which is a contradiction.

Let us consider the example $p = 4, q = 2, r = 3, h = 3, j = 2$. Here $y = \lambda_1, z = \lambda_2$.

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & a_2 & b_2 & c_2 & d_2 \\ & a_1 & b_1 & c_1 & & a_2 & b_2 & c_2 \\ & & a_1 & b_1 & & & a_2 & b_2 \\ & & & a_1 & & & & a_2 \\ a_3 & b_3 & c_3 & d_3 & a_4 & b_4 & c_4 & d_4 \\ & a_3 & b_3 & c_3 & & a_4 & b_4 & c_4 \\ & & a_3 & b_3 & & & a_4 & b_4 \\ & & & a_3 & & & & a_4 \end{pmatrix} \begin{pmatrix} * \\ z \\ y \\ 0 \\ * \\ z \\ y \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} * \\ (a_1 + a_2)z + (b_1 + b_2)y \\ (a_1 + a_2)y \\ 0 \\ * \\ (a_3 + a_4)z + (b_3 + b_4)y \\ (a_3 + a_4)y \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ z \\ y \\ 0 \\ * \\ z \\ y \\ 0 \end{pmatrix}$$

Above we have used the relations in G^r, $a_1 + a_2 = a_3 + a_4 = 1, b_1 + b_2 = b_3 + b_4 = 0, c_1 + c_2 = c_3 + c_4 = 0$. It is fairly clear how this example will generalize. The reason it is necessary to define j so that $j \leq r$, is that by Lemma 3.25 there are precisely r equations among its entries which govern its structure, and if $j > r$ then these equations will run out, and there is no assurance that the conditon $v_{h-r} = \lambda_{r+1} w$ will be preserved by the action of G^r. $\square$

Lemma 3.29. *If $j < r$, then any two vectors of height h and type $(\lambda_1, \cdots, \lambda_j)$ are in the same G^r-orbit.*

Proof. Let v be the vector with height h such that $v_1 = v_2 = \cdots = v_{h-j-1} = 0$, $v_{h-j} = (0, \cdots, 0, 1)$ (here there are $q - 1$ 0-s), $v_{h+1-j} = \lambda_j w_0, \cdots, v_h = \lambda_1 w_0$. Then v clearly has height h and type $(\lambda_1, \cdots, \lambda_j)$. We will prove that any other vector of height h and type $(\lambda_1, \cdots, \lambda_j)$ lies in the same G^p-orbit as v; this will suffice, since by Lemma 3.25, $G^p \subset G^r$. Let us consider the example with $\lambda = (4^2), \mu = (3^2), j = 2, h = 4$.

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & a_2 & b_2 & c_2 & d_2 \\ & a_1 & b_1 & c_1 & & a_2 & b_2 & c_2 \\ & & a_1 & b_1 & & & a_2 & b_2 \\ & & & a_1 & & & & a_2 \\ a_3 & b_3 & c_3 & d_3 & a_4 & b_4 & c_4 & d_4 \\ & a_3 & b_3 & c_3 & & a_4 & b_4 & c_4 \\ & & a_3 & b_3 & & & a_4 & b_4 \\ & & & a_3 & & & & a_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \lambda_2 \\ \lambda_1 \\ 0 \\ 1 \\ \lambda_2 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} b_2 \\ a_2 \\ \lambda_2 \\ \lambda_1 \\ b_4 \\ a_4 \\ \lambda_2 \\ \lambda_1 \end{pmatrix}$$

Here we have used the relations in G^p, $a_1+a_2 = 1, b_1+b_2 = c_1+c_2 = d_1+d_2 = 0$. In the vector on the right above, the value of the pair (a_2, a_4) can be an arbitrary element of $\mathbb{C}^2$ not lying in $\mathbb{C}w_0$. To see this, the invertible 2×2 matrix $\left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ acts on $\mathbb{C}^2 \times \mathbb{C}^2$, and can take any pair of vectors forming a rank 2 matrix, to any other pair of vectors forming a rank 2 matrix. In particular, it can take the pair of vectors $(0, 1), (1, 1)$ to $(a_2, a_4), (1, 1)$ if $a_2 \neq a_4$. The condition that the matrix stabilizes $(1, 1)$ imposes the conditions that we have on the matrix, $a_1 + a_2 = a_3 + a_4 = 1$, and the matrix will take $(0, 1)$ to (a_2, a_4). The variables b_2 and b_4 are free to vary, because the equations relate them to b_1 and b_3 which do not occur. This then means the set of vectors on the right ranges over all vectors of height 4 and type (x, y), proving the statement in the case of the example.

It is clear how this example will generalize. Here we require that $j < r$; if $j = r$, in the above example for instance if $r = 2$ instead, there will be vectors of type (λ_1, λ_2) with $a_2 = a_4$, which is not accounted for in the above argument.

□

Lemma 3.30. *If $j = r$, then any two vectors of height h and type $(\lambda_1, \cdots, \lambda_r)$ are in the same G^r-orbit.*

Proof. Let v be the vector with height h such that $v_1 = v_2 = \cdots = v_{h-r} = 0$, $v_{h+1-r} = \lambda_r w_0, \cdots, v_h = \lambda_1 w_0$. Then v is clearly of type $(\lambda_1, \cdots, \lambda_r)$, so it suffices to prove that any other vector of type $(\lambda_1, \cdots, \lambda_r)$ lies in the same orbit as v. Let us consider the example with $\lambda = (4^2), \mu = (2^2), h = 4$. Here $x = \lambda_1, y = \lambda_2$.

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & a_2 & b_2 & c_2 & d_2 \\ & a_1 & b_1 & c_1 & & a_2 & b_2 & c_2 \\ & & a_1 & b_1 & & & a_2 & b_2 \\ & & & a_1 & & & & a_2 \\ a_3 & b_3 & c_3 & d_3 & a_4 & b_4 & c_4 & d_4 \\ & a_3 & b_3 & c_3 & & a_4 & b_4 & c_4 \\ & & a_3 & b_3 & & & a_4 & b_4 \\ & & & a_3 & & & & a_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ y \\ x \\ 0 \\ 0 \\ y \\ x \end{pmatrix} = \begin{pmatrix} (c_1+c_2)y + (d_1+d_2)x \\ (c_1+c_2)x \\ y \\ x \\ (c_3+c_4)y + (d_3+d_4)x \\ (c_3+c_4)x \\ y \\ x \end{pmatrix}$$

Here we have used the relations in G^r, $a_1 + a_2 = 1, b_1 + b_2 = 0$. In the above, the vector on the right can be made to be an arbitrary vector of type (x, y). To see this, since $x \neq 0$, and there are no restrictions on c_1, c_2, c_3 and c_4, the values of $(c_1+c_2)x$

and $(c_3 + c_4)x$ can be made arbitrary. Since there are no restrictions on d_1, d_2, d_3 and d_4, the values of $(c_1 + c_2)y + (d_1 + d_2)x$ and $(c_3 + c_4)y + (d_3 + d_4)x$ can be made arbitrary. This proves the statement in the case of the example. It is clear that this example will generalize. □

Combining the previous three Lemmas, we can now state the classification of orbits of G^r on V.

Theorem 3.31. *The orbits of G^r on V are given by the height h of the vector v, and its type $(\lambda_1, \cdots, \lambda_j)$.*

As an example to illustrate the theorem, the following are the orbits of G^r on V in the case where $\lambda = (3^2), \mu = (2^2)$.

$$\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} a \\ 0 \\ 0 \\ b \\ 0 \\ 0 \end{pmatrix} |a \neq b\right\};\left\{\begin{pmatrix} \lambda \\ 0 \\ 0 \\ \lambda \\ 0 \\ 0 \end{pmatrix}\right\}, \lambda \neq 0$$

$$\left\{\begin{pmatrix} * \\ a \\ 0 \\ * \\ b \\ 0 \end{pmatrix} |a \neq b\right\};\left\{\begin{pmatrix} a \\ \lambda \\ 0 \\ b \\ \lambda \\ 0 \end{pmatrix} |a \neq b\right\}, \lambda \neq 0;\left\{\begin{pmatrix} \lambda_2 \\ \lambda_1 \\ 0 \\ \lambda_2 \\ \lambda_1 \\ 0 \end{pmatrix}\right\}, \lambda_1 \neq 0$$

$$\left\{\begin{pmatrix} * \\ * \\ b \\ * \\ * \\ a \end{pmatrix} |a \neq b\right\};\left\{\begin{pmatrix} * \\ a \\ \lambda \\ * \\ b \\ \lambda \end{pmatrix} |a \neq b\right\}, \lambda \neq 0;\left\{\begin{pmatrix} * \\ \lambda_2 \\ \lambda_1 \\ * \\ \lambda_2 \\ \lambda_1 \end{pmatrix}\right\}, \lambda_1 \neq 0$$

CHAPTER 4

Springer fibres

In this section, we will define Springer fibres, and study some properties of these Springer fibres. We describe the irreducible components of the Springer fibres associated to a nilpotent of type λ, and prove that they are in bijection with the set of standard tableaux of type λ. Here we follow the presentation in Spaltenstein, [**8**].

4.1. Irreducible components of Springer fibres

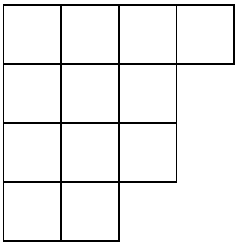
Recall that in Section 2.1, we constructed a map $\pi : G \times_P \mathfrak{n} \to \overline{\mathcal{O}_\lambda}$. Consider the case when $\lambda = (n)$. Using results in Section 2.2, $\mathcal{O}_{\lambda'} \subset \overline{\mathcal{O}_\lambda}$ when $\lambda' \leq \lambda$. Since $\lambda' \leq (n)$ for any partition λ', it follows that $\overline{\mathcal{O}_{(n)}} = \mathcal{N}$ in our case. The space $G \times_P \mathfrak{n}$ can be viewed as $\{(x, (V_i)) | xV_i \subseteq V_{i-1}\}$ with $\dim V_i = \mu_1 + \cdots + \mu_i$ (where μ is the transpose partition of (n)); here since $\mu = (1^n)$, $\mu_1 + \cdots + \mu_i = i$, and the parabolic subgroup P will become a Borel subgroup B. Hence the space $G \times_B \mathfrak{n}$ can be viewed as $\{(x, (0 = V_0 \subset V_1 \subset \cdots \subset V_n = V)) | x(V_i) \subseteq V_{i-1}\}$, with $\dim V_i = i$, and the map π, which is known as the Springer resolution, is projection onto the first factor.

Definition 4.1. *Given $x \in \mathcal{O}_\lambda$, the Springer fibre $\mathcal{F}_x$ is defined as follows:*

$$\mathcal{F}_x = \pi^{-1}(x) = \{0 = V_0 \subset V_1 \subset \cdots \subset V_n = V | x(V_i) \subset V_{i-1}\}$$

Let S_λ denote the set of standard tableaux associated to the partition λ; here a "standard tableau" is a way of filling up the Young diagram for the partition λ with the numbers $1, 2, 3, \cdots, n$ in such a way that the numbers are strictly increasing along rows and down columns. Given a $\sigma \in S_\lambda$, let σ_i denote the number of the column which i lies in. It is fairly clear that σ can be reconstructed from the numbers $\sigma_1, \sigma_2, \cdots, \sigma_n$, since proceeding inductively, the entry i must be inserted into the highest entry in the σ_i-th column which has not yet been occupied by the entries $1, 2, \cdots, i-1$ (to ensure that entries are increasing down columns). Define a total ordering on S_λ as follows (known as the "reverse lexicographic" ordering): given $\sigma, \sigma' \in S_\lambda$, suppose that $\sigma < \sigma'$ if for some i, $\sigma_i < \sigma'_i$ but $\sigma_j = \sigma'_j$ if $i < j \leq n$.

Suppose we have a Jordan basis $e_{i,j}$ for x, with $1 \leq i \leq \mu_1$, $1 \leq j \leq \lambda_i$, and $xe_{i,j} = e_{i,j-1}$ if $j \geq 1$, and 0 otherwise. Consider an example with $\lambda = (4, 3^2, 2)$. The diagram below shows the partition λ, and the basis elements $e_{i,j}$ can be identified with the boxes of λ in an obvious way. Let $W_i = \text{span}(e_{1,1}, e_{2,1}, \cdots, e_{i,1})$. In the case of our example, the below diagram shows that $\ker x = W_4$, $\ker x \cap \text{im}\, x^3 = W_1, \ker x \cap \text{im}\, x^2 = W_3, \ker x \cap \text{im}\, x = W_4$. This shows that in general $\ker x \cap \text{im}\, x^{i-1} = W_{\mu_i}$ (here μ_i is the length of the i-th column of λ). In particular, W_{μ_i} is independent of the choice of Jordan basis (although W_i for general i is not).



Lemma 4.2. *Consider the flag $(V_1/V_1, V_2/V_1, \cdots, V_n/V_1)$. The nilpotent x clearly induces a nilpotent transformation x' of V_n/V_1, and moves each element of this flag into the previous flag. Then the Jordan type of the nilpotent x' corresponds to the partition obtained by deleting the corner at the bottom of the j-th column of λ, where j is such that $V_1 \in W_{\mu_j}$ but $V_1 \notin W_{\mu_{j+1}}$.*

Proof. We first consider the example above, $\lambda = (4, 3^2, 2)$, with $j = 3$. Then $V_1 \in \text{span}\,(e_{1,1}, e_{2,1}, e_{3,1})$ but $V_1 \notin \text{span}\,(e_{1,1})$. The first case is when $V_1 = \text{span}\,(ae_{1,1} + e_{2,1})$. Then the following set will be a second Jordan basis for x: $\{e_{1,1}, e_{1,2}, e_{1,3}, e_{1,4}\}, \{ae_{1,1}+e_{2,1}, ae_{1,2}+e_{2,2}, ae_{1,3}+e_{2,3}\}, \{e_{3,1}, e_{3,2}, e_{3,3}\}, \{e_{4,1}, e_{4,2}\}$. Letting the image of $e_{i,j}$ in V/V_1 be $e'_{i,j}$, note that in V/V_1, $ae'_{1,1} + e'_{2,1} = 0$. It follows that a possible Jordan basis for x on V/V_1 is given by $\{e'_{1,1}, e'_{1,2}, e'_{1,3}, e'_{1,4}\}$, $\{ae'_{1,2} + e'_{2,2}, ae'_{1,3} + e'_{2,3}\}$, $\{e'_{3,1}, e'_{3,2}, e'_{3,3}\}$, $\{e'_{4,1}, e'_{4,2}\}$. Thus the Jordan type of x' on V/V_1 is $(4, 3, 2^2)$, which is obtained from λ by deleting the last box in the third column, proving the claim in the first case of this example.

The second case is when $V_1 = \text{span}\,(ae_{1,1} + be_{2,1} + e_{3,1})$. Then the following set will be a second Jordan basis for x: $\{e_{1,1}, e_{1,2}, e_{1,3}, e_{1,4}\}$, $\{e_{2,1}, e_{2,2}, e_{2,3}\}$, $\{ae_{1,1} + be_{2,1} + e_{3,1}, ae_{1,2} + be_{2,2} + e_{3,2}, ae_{1,3} + be_{2,3} + e_{3,3}\}$, $\{e_{4,1}, e_{4,2}\}$. Letting the image of $e_{i,j}$ in V/V_1 be $e'_{i,j}$, note that in V/V_1, $ae'_{1,1} + be'_{2,1} + e'_{3,1} = 0$. It follows that one possible Jordan basis for x' on V/V_1 is given by $\{e'_{1,1}, e'_{1,2}, e'_{1,3}, e'_{1,4}\}$, $\{e'_{2,1}, e'_{2,2}, e'_{2,3}\}$, $\{ae'_{1,2} + be'_{2,2} + e'_{3,2}, ae'_{1,3} + be'_{2,3} + e'_{3,3}\}$, $\{e_{4,1}, e_{4,2}\}$. Thus the Jordan type of x' on V/V_1 is $(4, 3, 2^2)$, which is obtained from λ by deleting the last box in the third column, proving the claim in the second case of the example, and finishing the example.

In the general case, it is straightforward to see how to generalize this construction and construct the necessary Jordan basis for x' on V/V_1. □

By induction on dim V, we will show that there exists a map $\theta : \mathcal{F}_x \to S_\lambda$. Since by induction this is possible for the smaller flag $(V_1/V_1 \subset V_2/V_1 \subset \cdots \subset V_n/V_1)$, we have a way of associating a standard tableau σ^1 of size $n-1$ to the action of x' on this flag, where σ^1 has shape λ', which is obtained by deleting the corner at the bottom of the j-th row of λ (by the above Lemma). Then, to the action of x on the flag $(V_0 \subset V_1 \subset \cdots \subset V_n)$, we will associate the standard tableaux σ of shape λ, by adding to σ^1 the number n in the corner at the bottom of the j-th row (this is a standard tableaux since this box is a corner). This gives us the required map $\theta : \mathcal{F}_x \to S_\lambda$. By the construction of $\theta : \mathcal{F}_x \to S_\lambda$, given $F \in \mathcal{F}_x$, the shape of the standard tableau containing the entries $1, \cdots, n-s$ and ignoring all higher entries in $\theta(F)$, corresponds to the Jordan type of x on V/V_s. For $\sigma \in S_\lambda$, let $\mathcal{F}_\sigma = \theta^{-1}(\sigma)$. It is clear by induction that $\mathcal{F}_\sigma$ is non-empty. The following theorem is the main result of this section.

Theorem 4.3. *The irreducible components of the Springer fibre $\mathcal{F}_x$ are the closures $\overline{\mathcal{F}_\sigma}$, and hence are in bijection with the set of standard tableaux S_λ.*

Given a vector space W, let $\mathbb{P}(W)$ denote the set of all one-dimensional subspaces of W; then $\mathbb{P}(W)$ is a projective algebraic variety. Let $B_i = \mathbb{P}(W_i) - \mathbb{P}(W_{i-1})$ and $B_1 = \mathbb{P}(W_1)$, so that $\mathbb{P}(\ker x)$ is the disjoint union of B_i with $1 \leq i \leq \mu_1$. Define a map $p : \mathcal{F}_x \to \mathbb{P}(\ker x)$, which takes a flag $(V_0 \subset V_1 \subset \cdots \subset V_n)$ to V_1. Define $X_i = p^{-1}(B_i)$. Then $\mathcal{F}_x$ is the disjoint union of $X_1, X_2, \cdots, X_{\mu_1}$. Let $X'_i = p^{-1}(\mathbb{C}e_{i,1})$. The variety X'_i can be considered as a "$n-1$ dimensional version" of $\mathcal{F}_x$, since if x' is the induced nilpotent in $V/\mathbb{C}e_{i,1}$, X'_i can be identified with the set of all complete flags in this quotient space with x' moving each flag into the previous flag. Keeping in mind that X_i is the set of all flags with their 1-dimensional component in B_i, and X'_i is the set of all flags with their 1 dimensional component being a specific element of B_i, the following Proposition is plausible:

Proposition 4.4. *We can find an isomorphism of algebraic varieties $f : X_i \to B_i \times X'_i$, such that if $f(F) = (b_i, F')$ for some flags F, F' and $b_i \in B_i$, then $\theta(F) = \theta(F')$.*

Proof. First note that W_{i-1} is isomorphic to $B_i = \mathbb{P}(W_i) - \mathbb{P}(W_{i-1})$. To see this, $W_i = W_{i-1} \oplus \mathbb{C}e_{i,1}$, so every one-dimensional subspace in B_i can be written as $\mathbb{C}(e_{i,1} + w_{i-1})$ for some $w_{i-1} \in W_{i-1}$, uniquely. It follows that the map $w_{i-1} \to \mathbb{C}(e_{i,1} + w_{i-1})$ is an isomorphism from W_{i-1} to B_i. This means it suffices to prove the Proposition with B_i replaced by W_{i-1}.

Suppose $w \in W_{i-1}$, so $w = \sum_{1 \leq s \leq i-1} a_s e_{s,1}$ for some constants a_s. For $j \leq \lambda_{i-1}$, let $w_j = \sum_{1 \leq s \leq i-1} a_s e_{s,j}$. Note that $xw_j = w_{j-1}$ (here define $w_0 = 0$ for convenience). Define a linear map $g_w : V \to V$ by letting $g_w(e_{i',j}) = e_{i',j}$ if $i' \neq i$, and $g_w e_{i,j} = e_{i,j} + w_j$. It is clear that g_w is invertible, since the matrix of g_w with respect to the Jordan basis for x is unitriangular. We next check that $g_w x = x g_w$. If $i' \neq i$, then $g_w x e_{i',j} = g_w e_{i',j-1} = e_{i',j-1}$, while $x g_w e_{i',j} = x e_{i',j} = e_{i',j-1}$, so $g_w x e_{i',j} = x g_w e_{i',j}$ (for all i', let $e_{i',0} = 0$ to deal with the boundary case). Next, $g_w x e_{i,j} = g_w e_{i,j-1} = e_{i,j-1} + w_{j-1}$, while $x g_w e_{i,j} = x(e_{i,j} + w_j) = e_{i,j-1} + w_{j-1}$ since $xw_j = w_{j-1}$, so $g_w x e_{i,j} = x g_w e_{i,j}$.

Consider the map $g : W_{i-1} \times X'_i \to X_i$ defined by $g(w, F) = g_w(F)$. We first need to show that g is a well-defined map, i.e. that $g_w(F)$ lies in X_i. If $F = (V_0 \subset V_1 \subset \cdots \subset V_n)$, then $g_w(F) = (V_0 \subset g_w(V_1) \subset \cdots \subset g_w(V_{n-1}) \subset V_n)$, so $x g_w(V_i) = g_w x(V_i) \subset g_w(V_{i-1})$, which means $g_w(F) \in \mathcal{F}_x$. If $F = (V_0 \subset V_1 \subset \cdots \subset V_n)$ lies in X'_i, then $p(g_w(F)) = g_w(V_1) = g_w(\mathbb{C}e_{i,1}) = \mathbb{C}(e_{i,1} + w_1) = \mathbb{C}(e_{i,1} + w)$. Since $\mathbb{C}(e_{i,1} + w) \in B_i$, it follows that $g_w(F) \in X_i$; and hence g is a well defined map.

To check that g is surjective, suppose $F' = (V'_0 \subset V'_1 \subset \cdots \subset V'_n)$ is some flag in X_i. Suppose $V'_1 = \mathbb{C}(e_{i,1} + w)$. Since $g_w(\mathbb{C}(e_{i,1})) = \mathbb{C}(e_{i,1} + w)$ we have that $g_w^{-1}(V_1') = \mathbb{C}(e_{i,1})$. Since $xg_w = g_w x$, $xg_w^{-1} = g_w^{-1}x$, and so $xg_w^{-1}(V_i') = g_w^{-1}x(V_i') = g_w^{-1}V_{i-1}'$, which means $g_w^{-1}(F') \in X_i'$. It then follows that $F' = g(w, g_w^{-1}(F'))$, proving that g is surjective. To prove that g is injective, suppose we have that $g_w(F) = F'$ for some $w \in W_{i-1}, F \in X_i', F' \in X_i$. If $F' = (V_0' \subset V_1' \subset \cdots \subset V_n')$, $F = (V_0 \subset V_1 \subset \cdots \subset V_n)$, then $g_w(V_1) = g_w(\mathbb{C}(e_{i,1})) = \mathbb{C}(e_{i,1} + w_1) = V_1'$. The condition $\mathbb{C}(e_{i,1} + w_1) = \mathbb{C}(e_{i,1} + w) = V_1'$ determines w uniquely from F'. Then $g_w(F) = F'$, so $F = g_w^{-1}(F')$, which determines F uniquely from F'. This shows that g is injective. If we define f to be the inverse of g, f will an isomorphism between X_i and $W_{i-1} \times X_i'$.

To show the remainder of the proposition, it is enough to show that $\theta(F) = \theta(g_w(F))$, for $F = (V_0 \subset V_1 \subset \cdots \subset V_n) \in X_i'$. Recall that the shape of the standard tableau containing the entries $1, \cdots, n-s$ and ignoring all higher entries in $\theta(F)$, corresponds to the Jordan type of x on V/V_s. It thus suffices to prove that the Jordan type of x on V/V_s is the same as the Jordan type of x on $V/g_w(V_s)$ (here x really means the induced transformation on the quotient). To see this, suppose $a_{m,n} + g_w(V_s)$ is a Jordan basis for x on $V/g_w(V_s)$. Then consider the basis $g_w^{-1}a_{m,n} + V_s$ for V/V_s. Clearly $xV_s \subset V_s$, and $g_w^{-1}xg_w = x$, so $x(g_w^{-1}a_{m,n} + V_s) = (g_w^{-1}xg_w)(g_w^{-1}a_{m,n}) + V_s = g_w^{-1}xa_{m,n} + V_s = g_w^{-1}a_{m,n-1} + V_s$; hence $g_w^{-1}a_{m,n} + V_s$ is a Jordan basis for x on V/V_s. This shows that $\theta(F) = \theta(g_w(F))$, as required. □

Proposition 4.5. a) *The set $\mathcal{F}_\sigma$ is a locally closed subset of $\mathcal{F}_x$, and thus we can give $\mathcal{F}_\sigma$ the structure of an algebraic variety.*
b) *The variety $\mathcal{F}_\sigma$ is irreducible, and has dimension $\sum_i \frac{\mu_i(\mu_i-1)}{2}$.*

Proof. (a) It suffices to prove that the set $\cup_{\sigma' \geq \sigma}\mathcal{F}_{\sigma'}$ is closed for each σ. To see this, since the ordering $>$ is a total ordering, we can find a standard tableaux σ^+ which comes immediately before σ, so that $\sigma' > \sigma$ means $\sigma' \geq \sigma^+$. Then $\mathcal{F}_\sigma = \cup_{\sigma' \geq \sigma}\mathcal{F}_{\sigma'} - \cup_{\sigma' \geq \sigma^+}\mathcal{F}_{\sigma'}$. It would then follow that $\mathcal{F}_\sigma$ is an intersection of an open set and a closed set, and is hence locally closed.

Let $\sigma_n = j$. Make the following definitions:

$$A = \bigcup_{\sigma_n' \geq j+1} \mathcal{F}_{\sigma'}$$

$$Z = \bigcup_{\sigma' \geq \sigma} \mathcal{F}_{\sigma'}$$

$$B = \bigcup_{\sigma_n' \geq j} \mathcal{F}_{\sigma'}$$

Then by the way the ordering on S_λ is defined, we have $A \subset Z \subset B$. It is clear that $B = p^{-1}(\mathbb{P}(W_{\mu_j}))$, and hence B is closed in $\mathcal{F}_x$. Similarly A is closed in $\mathcal{F}_x$. Since B is closed in $\mathcal{F}_x$, to show that Z is closed in $\mathcal{F}_x$ it suffices to prove that Z is closed in B. Equivalently, it is sufficient to prove that $B - Z$ is open in B. Since A is closed, A is closed in B, so $B - A$ is open in B. Thus to show $B - Z$ is open in

B, it will suffice to show that $B - Z$ is open in $B - A$, since $B - A$ is open in B. The spaces $B - Z$ and $B - A$ are the following:

$$B - Z = \bigcup_{\sigma' < \sigma, \sigma'_n = j} \mathcal{F}_{\sigma'}$$

$$B - A = \bigcup_{\sigma'_n = j} \mathcal{F}_{\sigma'}$$

Pick an element F in $B - Z$; so suppose $F \in \mathcal{F}_\tau$ for some standard tableaux τ with $\tau < \sigma$ and $\tau_n = j$. We will construct an open neighbourhood of F in $B - A$.

Consider the isomorphism $X_{\mu_j} \cong B_{\mu_j} \times X'_{\mu_j}$ defined in Proposition 4.4, and view X'_{μ_j} as $\mathcal{F}'_{x'}$, an "$n-1$ dimensional version" of $\mathcal{F}_x$, corresponding to the action of x' on the flag variety $\mathcal{F}'$ consisting of quotient spaces (the Jordan type of x' is the partition obtained by deleting the last box in the j-th column of λ). If the flag in X_{μ_j} lies in $\mathcal{F}_{\sigma'}$ for some $\sigma' < \sigma, \sigma'_n = j$, then by the property stated in Proposition 4.4, the flag in X'_{μ_j} will also lie in $\mathcal{F}_{\sigma'}$, and hence will lie in $\mathcal{F}'_{\sigma'^1}$ when X'_{μ_j} is considered as $\mathcal{F}'_{x'}$ (here σ'^1 is obtained by deleting the n at the end of the j-th column of σ'). We thus have the following:

$$\text{Under the isomorphism } X_{\mu_j} \cong B_{\mu_j} \times X'_{\mu_j},$$
$$X_{\mu_j} \cap \bigcup_{\sigma' < \sigma, \sigma'_n = j} \mathcal{F}_{\sigma'} \text{ corresponds to } B_{\mu_j} \times \Big(\bigcup_{\sigma'^1 < \sigma^1} \mathcal{F}'_{\sigma'^1} \Big)$$

By induction, we may assume that $\bigcup_{\sigma'^1 < \sigma^1} \mathcal{F}'_{\sigma'^1}$ is open in $X'_{\mu_j} = \mathcal{F}'_{x'}$. This will imply that $B_{\mu_j} \times \bigcup_{\sigma'^1 < \sigma^1} \mathcal{F}'_{\sigma'^1}$ is open in $B_{\mu_j} \times X'_{\mu_j}$, so $X_{\mu_j} \cap \bigcup_{\sigma' < \sigma, \sigma'_n = j} \mathcal{F}_{\sigma'}$ is open in X_{μ_j}.

Note next that $X_{\mu_{j+1}} \cup \cdots \cup X_{\mu_j} = \bigcup_{\sigma'_n = j} \mathcal{F}_{\sigma'} = B - A$ by considering the possibilities for the first subspace in the flag. It follows that $X_{\mu_{j+1}} \cup \cdots X_{\mu_j - 1} = p^{-1}(\mathbb{P}(W_{\mu_j - 1})) \cap (B - A)$ is closed in $B - A$, so X_{μ_j} is open in $B - A$.

This now means that $X_{\mu_j} \cap \bigcup_{\sigma' < \sigma, \sigma'_n = j} \mathcal{F}_{\sigma'}$ is open in X_{μ_j}, which is in turn open in $B - A$; hence it is open in $B - A$, and it is also clearly contained in $B - Z$. We will be done if we can show that it contains F (since this will then be the required open neighbourhood of F in $B - A$). To show that it contains F, we are required to show that F is contained in X_{μ_j}. F is chosen to be in some $\mathcal{F}_\tau$ for some τ with $\tau_n = j$; so the first subspace in the flag lies in W_{μ_j} but not in $W_{\mu_{j+1}}$. We require that the first subspace in the flag F lies in W_{μ_j} but not in $W_{\mu_j - 1}$. To do this, we have freedom in choosing the Jordan basis for x, so we can simply choose the Jordan basis for x so that the first subspace in the flag lies in W_{μ_j} but not in $W_{\mu_j - 1}$.

(b) Continue to denote σ_n by j. Consider the morphism $p : \mathcal{F}_\sigma \to \mathbb{P}(W_{\mu_j}) - \mathbb{P}(W_{\mu_{j+1}})$. We have seen that the fibre of this morphism over any point can be identified with $\mathcal{F}'_{\sigma^1}$. This means that $\mathcal{F}_\sigma$ is a fibre bundle over $\mathbb{P}(W_{\mu_j}) - \mathbb{P}(W_{\mu_{j+1}})$ with fibres isomorphic to $\mathcal{F}'_{\sigma^1}$. We can assume by induction that $\mathcal{F}'_{\sigma^1}$ is an irreducible variety. Since $\mathbb{P}(W_{\mu_j}) - \mathbb{P}(W_{\mu_{j+1}})$ is an open subset of the irreducible variety $\mathbb{P}(W_{\mu_j})$, it is an irreducible variety. Since a fibre bundle over an irreducible variety with irreducible fibres is irreducible, it follows that $\mathcal{F}_\sigma$ is irreducible.

Now assume by induction that $\dim \mathcal{F}'_{\sigma^1} = (\sum_{i\neq j} \frac{1}{2}\mu_i(\mu_i - 1)) + \frac{1}{2}(\mu_j - 1)(\mu_j - 2)$. The base case for this induction is when σ^1 is the standard tableaux with one box (containing 1). In this case, the Springer fibre corresponds to a point (the flag $0 = V_0 \subset V_1 = V$), which has dimension 0, as required. For the induction step, clearly $\dim(\mathbb{P}(W_{\mu_j}) - \mathbb{P}(W_{\mu_{j+1}})) = \mu_j - 1$. Since $\mathcal{F}_\sigma$ is a fibre bundle over $\mathbb{P}(W_{\mu_j}) - \mathbb{P}(W_{\mu_{j+1}})$ with fibres isomorphic to $\mathcal{F}'_{\sigma^1}$ we have:

$$\begin{aligned}
\dim \mathcal{F}_\sigma &= \dim \mathcal{F}'_{\sigma^1} + \dim(\mathbb{P}(W_{\mu_j}) - \mathbb{P}(W_{\mu_{j+1}})) \\
&= (\sum_{i\neq j} \frac{1}{2}\mu_i(\mu_i - 1)) + \frac{1}{2}(\mu_j - 1)(\mu_j - 2) + \mu_j - 1 \\
&= (\sum_{i\neq j} \frac{1}{2}\mu_i(\mu_i - 1)) + \frac{1}{2}(\mu_j - 1)(\mu_j) \\
&= \sum_{i} \frac{1}{2}\mu_i(\mu_i - 1)
\end{aligned}$$

This now proves the required dimension formula for $\mathcal{F}_\sigma$. □

Now we are in a position to prove Theorem 4.3.

Proof. It is clear that $\mathcal{F}_x = \cup\overline{\mathcal{F}_\sigma}$. Since the closure of an irreducible variety is irreducible, (b) of Proposition 4.5 gives us that $\overline{\mathcal{F}_\sigma}$ is irreducible. Since (b) of Proposition 4.5 tells us that the dimensions of the varieties $\overline{\mathcal{F}_\sigma}$ are all equal, it is not possible for any one of these varieties to contain another (since a proper closed subvariety of an irreducible variety has strictly smaller dimension). Finally, it is not possible for $\overline{\mathcal{F}_\sigma} = \overline{\mathcal{F}_{\sigma'}}$ for distinct σ, σ', since then $\mathcal{F}_\sigma$ and $\mathcal{F}_{\sigma'}$ would be disjoint nonempty open subsets (since $\mathcal{F}_\sigma$ is locally closed, hence open in its closure), and it is not possible for an irreducible variety to have disjoint nonempty open subsets. This now completes the proof. □

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