SMOOTH MANIFOLDS FALL 2022 - MIDTERM SOLUTIONS

Problem 1. Let M be a C^∞ manifold and $i_1 : N_1 \to M$ be an embedding. Show that if $i_2 : N_2 \to M$ is another embedding such that $i_1(N_1) = i_2(N_2)$, then N_1 and N_2 are diffeomorphic.

Proof. By definition an embedding is a homeomorphism onto its image. Therefore, if $N \subset M$ denotes the common image of N_1 and N_2, then $i_2^{-1} : N \to N_2$ is a homeomorphism. Hence, $i_2^{-1} \circ i_1 : N_1 \to N_2$ is a homeomorphism. By invariance of domain, it follows that $\dim(N_1) = \dim(N_2)$. Denote their common dimension by ℓ.

We claim that it is a diffeomorphism. To see this, we need to show that both $i_2^{-1} \circ i_1$ and $i_1^{-1} \circ i_2$ are C^∞. Fix $p_1 \in N_1$, and let $p = i_1(p_1)$ and $p_2 = i_2^{-1}(p)$. Since i_1 is an embedding, we may find an open set $U \subset M$ and a chart $\varphi : U \to \mathbb{R}^\ell \times \mathbb{R}^{n-\ell}$ such that $\varphi \circ i_1({i_1}^{-1}(U)) = \varphi(N \cap U) \subset \mathbb{R}^\ell \times \{0\}$. Then $\varphi \circ i_2({i_2}^{-1}(U)) \subset \mathbb{R}^\ell \times \{0\}$, and $D(\varphi \circ i_2) \subset \mathbb{R}^\ell$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^\ell$ denote the projection onto the first ℓ coordinates. Since i_2 has rank ℓ and φ is a local diffeomorphism, it follows that $D(\pi \circ \varphi \circ i_2)$ is an isomorphism. By the inverse function theorem, its inverse is C^∞. Therefore,

$$i_2^{-1} \circ i_1 = (\pi \circ \varphi \circ i_2)^{-1} \circ (\pi \circ \varphi \circ i_2)$$

is C^∞. By reversing the roles if i_1 and i_2, it follows that the inverse is differentiable as well. □

Problem 2. Consider the function $f(x, y, z) = z^2 - x^2 - 2y^2$. Find the regular values of f. Find the values of $r > 0$ such that the cylinder $x^2 + z^2 = r^2$ intersects $f^{-1}(1)$ transversally. Justify your answers.

Proof. First, note that $Df(x, y, z) = \begin{pmatrix} -2x & -4y & 2z \end{pmatrix}$, so the matrix is full (ie, nonzero) rank unless all of x, y and z are zero. Hence, the regular values of f are $\mathbb{R}^3 \setminus \{0\}$.

For the transversality question, first, let us note that if $f^{-1}(1)$ and the cylinder intersect nontrivially, then $z^2 = r^2 - x^2$. Hence,

$$(r^2 - x^2) - x^2 - 2y^2 = 1 \implies x^2 + y^2 = \frac{1}{2}(r^2 - 1).$$

Therefore, the sets intersect if and only if $r \geq 1$. When $r = 1$, the intersection requires that $x^2 + y^2 = 0$, hence $x = y = 0$, and $z^2 = 1 - 0 = \pm 1$. Hence, the intersection is a finite collection of points. By the transversality theorem, if the intersection were transverse, it would be as 1-manifolds, so when $r = 1$, the intersection is not transverse.

When $r > 1$, we claim that the intersection is tranvserse. Indeed, note that since 1 is regular value of f, if $p \in f^{-1}(1)$, $T_p f^{-1}(1) = \ker Df$. For the same reason, if g denotes the function $x^2 + z^2 - r^2$, the tangent bundle to the cylinder is given by $\ker Dg$. Since these are both planes, their sum spans unless the kernels coincide. This occurs only when $F(p) = (f(p), g(p))$ is *not* of full rank. We know that

$$DF(x, y, z) = \begin{pmatrix} -2x & -4y & 2z \\ 2x & 0 & 2z \end{pmatrix}.$$

This map has full rank when the vectors are linearly independent. Notice that by considering the second column, since each row is a nonzero vector at points of the intersection, the row vectors are linearly independent unless $y = 0$. Since we have assumed that $r > 1$, $x^2 + y^2 > 0$, so if $y = 0$, then

$x \neq 0$. Then since the first column must be opposites and the last column is equal, the rows are linearly independent. That is, DF is of full rank and $f^{-1}(1)$ and the cylinder are transverse. $\square$

Problem 3. Let φ_t, ψ_s, and η_u be fixed-point free flows on a C^∞ manifold M whose generating vectors fields are linearly independent at every point. Assume that η_u commutes with both φ_t and ψ_s, and that there exists a C^∞ function $\sigma : \mathbb{R}^2 \times M \to \mathbb{R}$ such that $\sigma(0,t,p) = 0$ and $\sigma(s,0,p) = 0$ for all $s, t \in \mathbb{R}$ and $p \in M$ and

$$\varphi_t \psi_s(p) = \eta_{\sigma(s,t,p)} \psi_s \varphi_t(p).$$

Find a 3-dimensional foliation in M containing the orbits of φ_t, ψ_s and η_u. Be sure to prove that it is a foliation!

Proof. Let X, Y and Z denote the generating vector fields for φ_t, ψ_s and η_u, respectively. We claim that $E(p) := \operatorname{span}_{\mathbb{R}} \{X(p), Y(p), Z(p)\}$ is an involutive distribution. Then by the Frobenius theorem, it integrates to a foliation $\mathcal{F}$. This distribution $\mathcal{F}$ must contain the orbits of the generating vector fields. To show that

Since η_u commutes with φ_t and ψ_s, it follows that $[X,Z] = [Y,Z] = 0$. So we must compute $[X,Y]$. Recall that $[X,Y] = \frac{d}{dt}\big|_{t=0} (\varphi_t)_* Y$. To compute this, we first compute $(\varphi_t)_* Y$ for a fixed t. If $p \in M$, then $Y(p) = \frac{d}{ds}\big|_{s=0} \varphi_s(p)$. Therefore,

$$(\varphi_t)_* Y = \frac{d}{ds}\bigg|_{s=0} \varphi_t \psi_s \varphi_{-t}(p) = \frac{d}{ds}\bigg|_{s=0} \eta_{\sigma(s,t,p)} \psi_s(p).$$

Now, since the flow η is given by a C^∞ map from $\mathbb{R} \times M$ to M, and s appears in each coordinate, we must differentiate in each and add. It follows that

$$(\varphi_t)_* Y = \frac{\partial \sigma}{\partial s}(0,t,p) Z(p) + Y(p)$$

Now, to get the Lie bracket, we differentiate in t. It follows that $[X,Y] = \frac{\partial^2 \sigma}{\partial s \partial t}(0,0,p) Z(p)$. Since this vector field is subordinate to E, we conclude that E is involutive. $\square$