SMOOTH MANIFOLDS FALL 2022 - FINAL

Complete as many problems as you can. The 3 best solutions will be counted towards your score.

Problem 1. Let G and H be connected Lie groups, and $F : G \to H$ be a surjective, C^∞ homomorphism. Show that F is a submersion.

Problem 2. Prove or find a counterexample: for every C^∞ connected manifold M, there exists a vector bundle E such that $E \oplus TM$ is a trivial bundle over M.

Problem 3. Give an example of a 2-dimensional distribution on the Lie group $SL(2,\mathbb{R})$ which is not integrable. Justify your claims.

Problem 4. Let α be a 1-form on a connected 3-manifold M such that $\alpha \wedge d\alpha$ is a volume form. Show that there exists a unique vector field X on M such that $\iota_X \alpha \equiv 1$ and $\iota_X d\alpha \equiv 0$. Furthermore, show that $\alpha \wedge d\alpha$ is invariant under φ_t^X, the flow generated by X.

Problem 5. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be transverse C^∞ foliations on a connected manifold M (ie, foliations such that at every point $p \in M$, $\mathcal{F}_1(p) \pitchfork \mathcal{F}_2(p)$, where $\mathcal{F}_i(p)$ is the leaf of $\mathcal{F}_i$ through p). Show that there exists a unique C^∞ foliation $\mathcal{F}$ such that $\mathcal{F}(p) = \mathcal{F}_1(p) \cap \mathcal{F}_2(p)$.

Problem 6. Let $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, $f : \mathbb{T}^k \to \mathbb{T}^k$ be a C^∞ map, and $U \subset \mathbb{T}^k$ be an open set such that $f^{-1}(x)$ has exactly m elements for every $x \in U$ and $\det(Df(x))\det(Df(y)) \geq 0$ for all $x, y \in \mathbb{T}^k$. Show that $\displaystyle\int_{\mathbb{T}^k} f^*\omega = \pm m$, where $\omega = dx_1 \wedge \cdots \wedge dx_k$ is the standard volume form.