MATH3210 - SPRING 2022 - SECTION 004

HOMEWORK 8

Problem 1. We say that a partition $\mathcal{P}$ of $[-a, a]$ is *symmetric* if $0 \in \mathcal{P}$ and whenever $x \in \mathcal{P}$, $-x \in \mathcal{P}$. Let $f : [-a, a] \to \mathbb{R}$ be an integrable function. Show that for every $\varepsilon > 0$, there exists a symmetric partition $\mathcal{P}$ such that $U(f, \mathcal{P}) - \varepsilon < \int_{-a}^{a} f\, dx < L(f, \mathcal{P}) + \varepsilon$.

Problem 2. Let $f : [-a, a] \to \mathbb{R}$ be an *even* function. That is, a function such that $f(x) = f(-x)$. Show that if f is integrable, then

$$\int_{-a}^{a} f\, dx = 2 \int_{0}^{a} f\, dx.$$

[*Hint*: Use the previous problem]

Problem 3. Show directly that the function $f(x) = x^2$ is integrable on $[-1, 1]$, and compute $\displaystyle\int_{-1}^{1} f\, dx$. *Do not appeal to the theorem that every continuous function is integrable, every monotone function is integrable or the fundamental theorem of calculus.* You may use the previous problem and the following formula:

$$\sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}$$

Problem 4. Let f be an function on $[a, b]$ such that $|f(x)| \leq B$ for all $x \in \mathbb{R}$.

i) Show that $\left|f(x)^2 - f(y)^2\right| \leq 2B\,|f(x) - f(y)|$ for all $x, y \in [a, b]$.

ii) Show that for any partition $\mathcal{P}$ of $[a, b]$, $U(f^2, \mathcal{P}) - L(f^2, \mathcal{P}) \leq 2B(U(f, \mathcal{P}) - L(f, \mathcal{P}))$.

iii) Show that if f is integrable on $[a, b]$, then so is f^2.