MATH3210 - SPRING 2024 - SECTION 004

HOMEWORK 10

Problem 1. Prove the divergence criteria of the comparison test. That is, show that if $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n \geq b_n \geq 0$ for every n, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Use the following scheme: Let $t_m = \sum_{n=1}^{m} b_n$ be the sequence of partial sums for b_n. Show that if $\sum_{n=1}^{\infty} b_n$ diverges, then t_m must diverge to infinity. Then show that the sequence t_m diverges to ∞, then so does $s_m = \sum_{n=1}^{m} a_n$.

Problem 2. Determine whether the series converges or diverges. Prove that your answer is correct using the following tools only: the term test, the comparison test, the ratio test, and the integral test.

(a) $\displaystyle\sum_{n=1}^{\infty} \cos(n)$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^8}{n!}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+3^n}{5^n}$

Problem 3. Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of positive functions such that $\sum_{n=1}^{\infty} f_n(0)$ converges, and each f_n is L_n-Lipschitz. Show that if $L = \sum_{n=1}^{\infty} L_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly to an L-Lipschitz function.

Problem 4. Let $f(x) = xe^x$, Find a number N such that the Taylor approximation of order N is within .1 of $f(x)$ on the interval $[0,1]$. [*Hint*: First, find and prove a formula for $f^{(k)}(x)$ by induciton, then bound $f^{(k)}(x)$ on $[0,1]$ using this formula by a number depending on k]