MATH3210 - SPRING 2022 - SECTION 001

HOMEWORK 1

Problem 1. In the following, A is a set of real numbers. Negate the following statements:

(1) There exists $x \in A$ such that $x < 0$.
(2) Every $x \in A$ is an integer.
(3) For every $x \in A$, there exists $y \in \mathbb{R}$ such that $yx = 1$.
(4) For every $x, y \in A$, if $x \leq y$, then there exists $z \in A$ such that $x < z < y$.

Problem 2. Recall that a set $R \subset \mathbb{Q}$ is a *Dedekind cut* if it satisfies all 3 of the following properties:

(a) $R \neq \emptyset$.
(b) R is bounded above.
(c) For every $x \in R$ and $y \in \mathbb{Q}$, if $y < x$, then $y \in R$.
(d) For every $x \in R$, there exists $y \in R$ such that $y > x$.

Show that if R and S are Dedekind cuts, then $R + S = \{x + y : x \in R \text{ and } y \in S\}$ is also a Dedekind cut (ie, show properties (a)-(d) for the set $R + S$, assuming (a)-(d) for the sets R and S themselves).

Problem 3. Prove that if $x, y \in \mathbb{R}$ and $x < y$, then there exists some $q \in \mathbb{Q}$ such that $x < q < y$. You may use the following fact: if $a, b \in \mathbb{R}$ and $b - a > 1$, then there exists $m \in \mathbb{Z}$ such that $a < m < b$.

Problem 4. Let $S_1, S_2, \ldots$ be bounded subsets of $\mathbb{R}$, and $x_n = \sup S_n$. Show that if $S = \bigcup_{n \in \mathbb{N}} S_n$ is also bounded above, then

$$\sup S = \sup \{x_n : x \in \mathbb{N}\}.$$