preREU - Problem Set 2

Problem 1. Show that if $f : X \to X$ is a dynamical system on a finite set X, and f is one-to-one, then every point of X is periodic.

Problem 2. Let m and n be positive integers, and $\ell = \lfloor m/n \rfloor$. If $q = m - \ell n$, show that $0 < q < n$ whenever m is not divisble by n. What is q when m is divisible by n?

Problem 3. Show that if p is a pre-periodic orbit, then there exists a positive integer n such that $f^k(p)$ is periodic for every $k \geq n$.

Problem 4. Find the flaw in the following argument

"Theorem." Any finite subset of integers consists either of only even numbers or only odd numbers.

"Proof". Let A be a finite subset of integers. We prove this claim by induction on the number of elements in A. For the base case, we consider the case when A has exactly one element. That element is either even or odd, so the base case is proved.

Suppose that we know that every set with k elements, all elements of that are either all even or all odd. Then, consider a set A with $k + 1$ elements. We may write $A = \{x_1, \ldots, x_{k+1}\}$. Define $B = \{x_2, \ldots, x_{k+1}\}$ and $C = \{x_1, \ldots, x_k\}$. Then B and C both have k elements, so all elements of B are either all even or all odd and the same is true by C by the induction hypothesis. Now, $x_2 \in B$ and $x_2 \in C$. If x_2 is even, then every element of B must be even, and the same is true for C. Hence $A = B \cup C$ has only even elements. Similarly if x_2 is odd, because it belongs to both B and C, both B and C consist only of odd elements, and $A = B \cup C$ consists only of odd elements. Hence, we have proven the theorem by induction. □

Problem 5 (Induction Pracitce)**.** Prove the following claims by induction:

(1) $2^n > n^2$ whenever $n \geq 5$
(2) $a^2 - 1$ is divisible by 8 whenever a is an odd integer
(3) Recall that the *Fibonacci sequence* is defined recursively by $f_1 = f_2 = 1$ and $f_{k+2} = f_k + f_{k+1}$ whenever $n \geq 1$.
  (a) Show that $\sum_{k=1}^{n} f_k = f_{n+2} - 1$ for all $n \geq 1$.
  (b) Show that if $\phi = (1 + \sqrt{5})/2$, then $f_n = (\phi^n - (-1/\phi)^n)/\sqrt{5}$ for all $n \geq 1$. [*Hint*: First show (not using induction) that ϕ satisfies $1 + \phi = \phi^2$. Then find a similar formula for $1/\phi$.]

1. The circle as equivalence classes

Recall that the equivalence classes on the circle are denoted by

$$[x] = \{x + k \cdot 2\pi : k \in \mathbb{Z}\},$$

and that the set of equivalence classes is denoted by $\mathbb{R}/2\pi\mathbb{Z}$.

Problem 6. Show that if two equivalence classes intersect, then they are equal.

Problem 7. Show that the product of two equivalence classes is *never* an equivalence class.

Problem 8. We say that if x and y are real numbers, we say that x is *equivalent to* y if $x \in [y]$.

(1) Show that for every x, x is equivalent to itself.
(2) Show that if x is equivalent to y, then y is equivalent to x.
(3) Show that if x is equivalent to y, and y is equivalent to z, then x is equivalent to z.

These three properties are called *reflexivity*, *symmetry* and *transitivity*. Any criterion for comparing two elements of a set (ie, a *relation*) is called an *equivalence* relation if it satisfies all three of these properties.

Problem 9 (Hard)**.** An *endomorphism* of $\mathbb{R}/2\pi\mathbb{Z}$ is a function $F : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ such that $F([x] + [y]) = F([x]) + F([y])$. Show that for each integer m, the function

$$L_m([x]) = [mx]$$

is well-defined and an endomorphism (see a TA for what it means to be "well-defined" if you don't know). For which values of m is the map L_m one-to-one? For which values of m is L_m onto? Try to find some periodic orbits and pre-periodic orbits when $m = 2$.

Problem 10. Let A be a set of equivalence classes (ie, A is a subset of the circle). We say that A is *dense* if for any equivalence class $[x]$ on the circle, there exists a sequence of real numbers y_n such that $[y_n] \in A$ and $\lim_{n\to\infty} y_n = x$. Find an example of an infinite set A which is dense and a set A which is not dense.

2. Rotation Dynamics

Recall that $R_\theta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ is the rotation dynamics defined by $R_\theta([x]) = [x+\theta] = [x]+[\theta]$.

Problem 11. Show that if $\theta = \frac{p}{q} \cdot 2\pi$ (where p and q are integers and p/q is in reduced form), then every point of the circle is periodic of period q for the rotation dynamics R_θ.

Problem 12. Show that if the rotation dynamics R_θ has a periodic orbit of period q, then $\theta = \frac{p}{q} \cdot 2\pi$ for some integer p for which p/q is a reduced fraction.

Problem 13. Consider a rotation by an angle $\theta \in (0, \pi)$, and let I denote the set of equivalence classes starting from $[0]$ and ending at $[\theta]$ in the counterclockwise direction, including 0, but not θ.

(1) Draw a picture to show that $I = \{[x] : 0 \le x < \theta\}$.
(2) Define m to be the smallest integer such that $R_\theta^m([0]) \in I$. Find a formula for m in terms of θ when $\theta = \frac{p}{q} \cdot 2\pi$ for some p and q.
(3) Let $[\alpha] = R_\theta^m([0])$, and choose the number α to represent the equivalence class so that $\alpha \in [0, \theta)$. Find a formula for $\theta - \alpha$ (we'll care about this number soon).