preREU Day 1 Problems

1. Finite dynamical systems

Problem 1. For each natural (ie, whole) number q, find an example of a finite dynamical system where every point is periodic of period q.

Problem 2. Give an example of a finite dynamical system and a point p such that p is preperiodic but not periodic.

Problem 3. Consider a number m between 1 and 10. Let $X = \{0, \dots, 9\}$ and define

$$f(x) = \text{the last digit of } m \cdot x.$$

Classify which points are periodic, and which are pre-periodic for $m = 3$ and $m = 4$. Make a guess for which points are which for a general m (and even better, prove your guess!).

Problem 4 (Proof)**.** Show that every point of a finite dynamical system is preperiodic.

2. Dynamical systems on intervals and real nubmers

Problem 5. Consider the dynamical system f defined on the interval $[0, 4]$ defined by $f(x) = 4x - x^2$.

(1) Sketch the graph of f.
(2) Draw some figures representing the orbits of a few points on X.
(3) Find an expression for $f^2(x)$, $f^3(x)$ and $f^4(x)$, where

$$f^2(x) = f(f(x)) \qquad f^3(x) = f(f(f(x))) \qquad f^4(x) = f(f(f(f(x))))$$

(4) Find all fixed points, and the periodic points of period 2 and 3 on $[0, 4]$.

Problem 6. Let f be a dynamical systems on the real numbers which is diffentiable. Find a formula for $(f^k)'(x)$ [*Hint*: Remember that $f^k(x)$ is not $(f(x))^k$, but the k-fold composition of f at x. Use the chain rule to find a formula for $f'(x)$, $(f^2)'(x)$ and $(f^3)'(x)$, then identify a pattern. If you can, try to prove your pattern by *induction*.]

Problem 7 (Proof)**.** Let f be defined on all real numbers and assume that $0 < f'(x) < 1$ for all x. Show that f has exactly one fixed point. Check this property first for linear functions of the form $f(x) = mx + b$ and find their fixed points. [*Hint*: Define an auxiliary function $g(x) = x - f(x)$. First show that f has a fixed point exactly when $g = 0$. Then show that g has exactly one zero by considering its derivative.]

3. Circle rotations

Problem 8. Find 5 different angles for which the circle rotation with that angle has every point periodic, and find that period.

Problem 9. Classify the circle rotations for which every point of the circle is periodic.

Problem 10 (Open-ended)**.** What is the relationship between the rotation angle and how "sparse" each orbit for a periodic circle rotation? Can you describe the way in which the orbits are built based on the angle?