

Review Sheet(s)

13 Geometry in the plane

The hard part here is the curvature and related stuff.

- parametric representation of a curve, implicit differentiation
- length of a curve, integration of $f(x, y)dx$ on a curve parametrized by $(x(t), y(t))$ (see also 17.2)
- vectors, dot product and perpendicularity. The dot product take two vectors, gives a number.
- vector valued function, position vector \vec{r} ,
- velocity, acceleration, curvature, and tangential and normal components of accelerations. Plenty of points of view on the same thing here. You have to feel comfortable with every approach since one may be very convenient in one situation and not in another (see all the formulas for curvature for instance. Some in terms of \vec{T} , some in terms of ϕ and some in terms of x, y). Remember for instance that the formula $\kappa = \frac{|y''|}{(1+y'^2)^{3/2}}$ works only when your curve is given as a function $y(x)$.

A good way to review is to look at the concepts test. in the chapter review p664. For instance: is the parametrization of a curve unique? Does the dot product satisfies the associative law? Is it true that if $\vec{u} \cdot \vec{v} = 0$ then $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$?

14 Geometry in space

That's mainly a generalization of what was done in the plane.

- Distance and spheres in 3-space
- Planes and lines: remember, *one* linear equation defines a plane *two* are needed to define a line: the symmetrical equations is really as set of two equations (there are 2 = signs). Also, since a line is also a particular kind of curves, it has a parametric equation. You need to know how to find it in chapter 17.2 to compute line integrals. Finding a parametric representation of a line may also be useful in the plane.

- dot product and perpendicularity, angle between planes
- Cross product. Something typically 3-dimensional. The cross product takes two vectors, gives a vector. This vector is zero if and only if the two vectors are colinear. Nice to find equation of line of intersection of two planes, or the line through 2 points or the normal vector of a plane through three points.
- curves/vector functions, tangent lines
- velocity, acceleration, curvature, and tangential and normal acceleration. Unseemingly, there are rather less formulas in 3 space than in the plane because there are less points of view possible. So in some sense, things are simpler. And actually, if you know how to manage in 3-space, you know how to do it in the plane: you may just add a z coordinate always zero. However, the formulas in 3-space are a bit less direct (see for instance the formula for curvature p693: to compute the curvature (a number) you have to take the cross product of speed and acceleration then compute its magnitude, then divide by the cube of the velocity. I think that those formulas are nicer than the one in the plane... more compact in some sense. Anyway remember that if you don't review those concepts, you will probably have hard time during the exam even with the book. There are many formulas, you have to feel very comfortable with their utilization: remember that one formula can be used in plenty of ways (you can for instance solve for \vec{N} in the first formula p 693 like in example 5).
- I assigned homework about the torsion of a curve. I think it is a good exercise to review this.
- Surfaces in 3-space. Cylinders (not necessarily circular), and quadric. How to recognize them.
- spherical and cylindrical coordinates. particularly useful in integration.

Once again, chapter review, and concept tests are a good exercise. For instance, is $3x + y = 1$ the equation of a line? What is $\vec{i} \times \vec{i}$? Does the curvature depend on the speed at which you go on a given curve (take the same curve and change the parametrization so that you go twice faster...)? One more remark. When you do some computation involving vectors. Take care not to transform a vector into a number, replacing $3t\vec{i} + 5t^2\vec{j} - 3\vec{k}$ by

$3t + 5t^2 - 3$, or replacing $\vec{i} + \vec{j} + 2\vec{k}$ by $x + y + 2z$, it does not make any sense. Try not to apply blindly a formula, or you'll get it wrong. You have to understand what it means, what it applies to. An example among others, you have a vector $\langle 1, 2, 3 \rangle$ want to find the line through this vector and through the point $\langle 2, 3, 4 \rangle$, if you come up with $(x-2) + 2(y-2) + 3(z-4)$, you are wrong. The best reason is that this is not the equation of a line but of a plane. Which plane ?

15 Functions of 2 and more variables

That's a really a step here. Mainly, understand what is a function of 2 variable, and what you can do with it and what do you get. The gradient is the main one. It is a VECTOR. there are also partial derivatives. They are numbers. Also directional derivative. Is it a number of or a vector? Answer in the note.¹

- sketch by drawing level lines
- partial derivatives
- limits: remember how to find out that a function is not continuous
- differentiability and gradient
- directional derivative, and link with gradient. Interpretation of gradient as direction of greatest rate of change, its magnitude being the rate of change in this direction. The gradient is perpendicular to level curves (or level surfaces in 3 variables).
- Chain rule, application to implicit functions, and approximations. Tangent planes (take care! several formulas in different contexts. Be sure to use the right one at the right place.) Differentials.
- Maxima and minima. A very important part of analysis. 2 contexts: finding free extrema and constrained extrema. Say you want to find a minimum of a function. If you know that a minimum exists (for instance if you are on a closed bounded domain), then check for all critical points to know which one is the minimum. But if the domain is not bounded, then the function may very well have no maximum and non minimum at all, its values may go from $-\infty$ to $+\infty$! then, knowing that you have a minimum or not requires some work. However, in the case of free extrema, you have a criterion to know if a critical

point is a local minimum or a local maximum or a saddle point (not a local extremum). But this won't tell you if the point is a global minimum. Usually, you are told that there is a minimum, and you have to find where it is, or you can prove that there actually is a minimum somewhere because the region you're looking at is closed and bounded (see th A p.754).

Get a lot of practise with free extrema, using the criterion of second partials, and for lagrange multipliers method.

See Concept test questions p 767. For instance (a) is it true that the gradient of f is perpendicular to the graph $z = f(x, y)$? Try to answer this and justify your answer. (b) If $f(x, y)$, then $|D_{\vec{u}}f(x, y)| \leq 4$ for any direction \vec{u} ? (c) Does the function $\sin(xy)$ attain a maximum on the open disc $x^2 + y^2 < 4$?

I want to give an answer to the first question. But before reading the solution, try to answer yourself. No, I mean it really. Do it otherwise it is not worth anything. Actually, I put the answer as a note at the end so that you are not too tempted. It is very important that you can justify the answer. ²

16 Double integrals

I find it less clever than the previous chapters. But sometimes painful. A tricky part though is to write the limits for the integrals and to change the order of integration. Review this!

- Double, triple integrals, on rectangle or potatoes (x or y simple). How do you find the limits of the iterated integrals when you want to integrate on a set S ? Let's take the example of the tetrahedron bounded by coordinate planes and $x+y/2+z/3 = 1$ (so that it intersects the x, y and z axis at $x = 1, y = 2$, and $z = 3$ respectively): assume you want to integrate first with respect to z , then with respect to y , then with respect to x (so that we want to write it as $\int_{x=..}^{\cdot\cdot} \int_{y=..}^{\cdot\cdot} \int_{z=..}^{\cdot\cdot} \dots dz dy dx$)

1. Find out between which bounds does x vary. Here, between 0 and 1.
2. Now fix an x (here between 0 and 1). Find out the minimum and maximum possible values of y *x being fixed, and whatever z* . here, the minimum value is 0, and the maximum one is [find it out yourself, result in the endnote³]

- Now fix x and y , what are the extreme coordinates possible for z ? well z has to be atleast zero and [... please, play the game, find it yourself! ⁴]
- plug the values you found in the integral:

$$\int_{x=0}^1 \int_{y=0}^{2-2x} \int_{z=0}^{3-3x-3y/2} \dots dz dy dx$$

- check that in the outmost integral limits you really have constants, that in the next one you have functions of one variable, and in the last one, functions of 2 variables.

Practise by changing the order of integration ! (x then y then z for instance). ⁵

- Polar coordinates: the important thing is $dx dy = r dr d\theta$. Also $x = r \cos \theta, y = r \sin \theta, r^2 = x^2 + y^2$. Take care about limits though: you may have to compute equation of a line or something in polar coordinates. If you have the equation in cartesian coordinates, replace x by $r \cos \theta, y$ by $r \sin \theta$ to get it in polar coordinates.
- Mass, center of mass, moment of inertia, radius of gyration
- Surface area. Well, mainly a formula to apply, quite similar to the formula for the length of a curve.
- Cylindrical coordinates: $dx dy dz = r dr d\theta dz$. Spherical coordinates $dx dy dz = \rho^2 \sin(\phi) d\rho d\theta d\phi$. Try to find without looking in the book the values of x, y, z in function of ρ, θ, ϕ . If you know how to do that (I mean, not just remembering the equations but computing them from a picture) then you know what are spherical coordinates. Otherwise...

17 Vector Calculus.

Yet another world. Plenty of ways of taking the derivative of a vector or scalar field: Gradient, Curl, Divergence. It is very important that you remember for each operator (I mean Gradient, Curl, Divergence) what kind of input it has (Vector? or Scalar?) and what is its output (vector or scalar?) I write the curl and gradient as \vec{curl} and $\vec{\nabla}$ with an arrow above to remember that their output is a vector. I don't put any arrow on divergence since its output is a scalar. It doesn't make any sense to take the gradient of a vector

field! the input of the gradient is a scalar field. In the same way it doesn't make any sense to take the curl or divergence of a scalar field ! Also, don't use the definitions to compute the divergence and curl, use the memorizing tools $\text{curl}\vec{F} = \vec{\nabla} \times \vec{F}$ and $\text{div}\vec{F} = \vec{\nabla} \cdot \vec{F}$.

- Integral on a path. 1st take a parametrization of the path (find one if it is not given). Don't confuse $ds, dt, d\vec{r}, dx$ (and dy, dz).

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \text{ corresponds to arclength}$$

$$dx = \frac{dx}{dt} dt \text{ corresponds to the variation of } x$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k} = \left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right) dt$$

corresponds to the variation of the position vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Note that $d\vec{r}$ is a vector. To get a number in the integral, usually take the dot product of \vec{F} with $d\vec{r}$ (like when you compute the work of a force).

- independance of path, curl criterion for independance of path, finding out a function given its gradient. Note: you want to find a function (ie a scalar field) and you are given the gradient which is a vector field. Formula for integration of $\nabla f \cdot d\vec{r}$. May avoid painful calculations. But it only works for conservative vector fields.
- Integral on surfaces: for a surface given by $z = f(x, y)$, $dS = \sqrt{1 + f_x^2 + f_y^2} dx dy$ (see computation of surface area in ch. 16.6!). Definition and shortcut to compute the flux.
- Green theorem, Gauss divergence theorem, and Stokes theorem: How to compute some integrals on some sets only by looking at the boundary. Or the other way around!

Take a look in the chapter review here also!

Good work !

Notes

¹the directional derivative is a number: that's how fast the function grows in some direction.

² Solution of the question (a): The answer is... (I love suspense). Let's think first. You have $f(x, y)$ a function of two variables. Take an example: $f(x, y) = x^2 + y^2$ (so that the surface $z = f(x, y)$ is a ... [here the good student searches the answer] is a diobarap backwards). What is the gradient? I don't need to compute it to say that it is something times \vec{i} plus something times \vec{j} (whatever the function $f(x, y)$). Hey, that's something in the xy -plane, so something horizontal. so if it has to be perpendicular to the surface, the surface should be vertical. Well but this doesn't make sense, for a nice differentiable function f , $z = f(x, y)$ has never a vertical tangent plane! this would mean that there is a direction in which f grows with an infinite derivative, so f wouldn't be differentiable. Actually, you can check that the tangent plane to our paraboloid is never vertical: compute the equation of the tangent plane. If it was vertical, there would be no z in the equation.

OK let's sum up. Why could someone think that it is true that $\vec{\nabla} f$ is perpendicular to this surface? because $\vec{\nabla} f$ is really perpendicular to something, but something else: is perpendicular to the curve $f(x, y) = 0$ (that's a level curve of f and you know that the gradient is perpendicular to level curves). That's the same story when you get confused with the 2 kinds of equations for tangent planes: for the surface $z = f(x, y)$ and for the surface $F(x, y, z) = 0$ (which is a level surface of F). remember that the second case is more general since you can set $F(x, y, z) = z - f(x, y)$.

³ the maximum y is on the plane $x + y/2 + z/3 = 1$ so that $y = 2 - 2x - 2z/3$, and it is maximum when z is minimum (ie $z = 0$) so that $y = 2 - 2x$. A better way to see this may well be to draw a picture. This means, x being fixed, look at the corresponding slice of the tetrahedron (a triangle in a plane parallel to the yz -plane), and ask yourself about the extreme values of y in this slice. You should find 0 and $2 - x$ (note: the equation $x + y/2 + z/3 = 1$ where x is fixed and y and z are free is the equation of the oblique line in your triangle in the corresponding plane.)

⁴the max value of z corresponds to a point on the plane with equation $x + y/2 + z/3 = 1$ so that $z = 3 - 3x - 3y/2$.

⁵answer = $\int_{z=0}^3 \int_{y=0}^{2-2z/3} \int_{x=0}^{1-y/2-z/3} \dots dx dy dz$