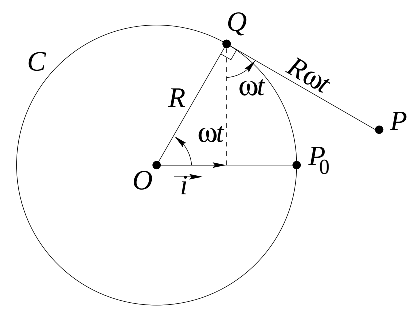
Correction of the problem of First Midterm

Problem II. Unwinding a fixed bobbin.

Consider a fixed bobbin (it cannot turn nor move) around which a thread is wound. The bobbin has the shape of a circle C of radius R. We take the center of the bobbin O to be the origin of the coordinate system (we assume that everything happens in a plane containing the circle C). Let P be the end of the thread. The goal of the problem is to study the movement of P as you unwind the thread, keeping it tight.
We assume that the point P is initially at $P_0(R, 0)$. Since the thread is tight, it is tangent to C at some point Q. We assume that at time t, the angle between $\vec{\imath}$ and $\overrightarrow{OQ}$ is ωt. The length $|PQ|$ of the unwound thread is $R\omega t$.



1 Show that $x = R(\cos\omega t + \omega t \sin\omega t)$ and $y = R(\sin\omega t - \omega t\cos\omega t)$ are the coordinates of P.

$\overrightarrow{OP} = \cos\omega t\vec{\imath} + \sin\omega t\vec{\jmath}$. Moreover, since $\overrightarrow{OP} \perp \overrightarrow{PQ}$, the angle between the vertical direction and $\overrightarrow{PQ}$ is ωt, so that $\overrightarrow{PQ} = R\omega t(\sin\omega t\vec{\imath} - \cos\omega t\vec{\jmath})$. Therefore,

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} = R(\cos\omega t + \omega t\sin\omega t)\vec{\imath} + R(\sin\omega t - \omega t\cos\omega t)\vec{\jmath}.$$

This means that the coordinates of P are $x = R(\cos\omega t + \omega t\sin\omega t)$ and $y = R(\sin\omega t - \omega t\cos\omega t)$.

2 Show that the velocity $\vec{v}$ is $R\omega^2 t(\cos(\omega t)\vec{\imath} + \sin(\omega t)\vec{\jmath})$ and compute the unit tangent vector $\vec{T}$.

$$\begin{aligned}
\vec{r}(t) = \overrightarrow{OP} &= R(\cos\omega t + \omega t\sin\omega t)\vec{\imath} + R(\sin\omega t - \omega t\cos\omega t)\vec{\jmath} \\
\vec{v} = \vec{r}'(t) &= R\Big(-\omega\sin\omega t + (\omega\sin\omega t + \omega^2 t\cos\omega t)\Big)\vec{\imath} + R\Big(\omega\cos\omega t - (\omega\cos\omega t - \omega^2 t\sin\omega t)\Big)\vec{\jmath} \\
&= R\omega^2 t\Big(\cos\omega t\vec{\imath} + \sin\omega t\vec{\jmath}\Big)
\end{aligned}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{R\omega^2 t}R\omega^2 t\Big(\cos\omega t\vec{\imath} + \sin\omega t\vec{\jmath}\Big) = \cos\omega t\vec{\imath} + \sin\omega t\vec{\jmath}$$

3 What is the angle ϕ between $\vec{\imath}$ and $\vec{T}$? Deduce the curvature κ of the curve.

The angle ϕ between $\vec{\imath}$ and $\vec{T}$ is ωt since $\cos\phi = \frac{\vec{\imath}.\vec{T}}{|\vec{\imath}|\,|\vec{T}|} = \frac{\cos\omega t}{1\times 1} = \cos\omega t$.
The easiest way to compute the curvature is by using its formula in terms of ϕ:

$$\begin{aligned}
\kappa &= \frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{\mathrm{d}\phi/dt}{\mathrm{d}s/dt} = \frac{\mathrm{d}\phi/dt}{v} \\
&= \frac{w}{R\omega^2 t} = \frac{1}{R\omega t}
\end{aligned}$$

You could also use the formula of the curvature in terms of $\vec{T}$:

$$\begin{aligned}\kappa &= \frac{|\vec{T}'|}{v} = \frac{1}{R\omega^2 t}\left|\frac{\mathrm{d}}{\mathrm{d}t}(\cos\omega t\vec{\imath} + \sin\omega t\vec{\jmath})\right| \\ &= \frac{1}{R\omega^2 t}\left|(-\omega\sin\omega t\vec{\imath} + \omega\cos\omega t\vec{\jmath})\right| = \frac{1}{R\omega t}\end{aligned}$$

You could also compute the curvature from the formula

$$\kappa = \frac{|x'y'' - x''y'|}{\sqrt{x'^2 + y'^2}^3}.$$

To compute x'' and y'', we just have to derive the coordinates of the velocity:

$$\begin{aligned} x' &= R\omega^2 t\cos\omega t \quad & x'' &= R\omega^2(\cos\omega t - \omega t\sin\omega t) \\ y' &= R\omega^2 t\sin\omega t \quad & y'' &= R\omega^2(\sin\omega t + \omega t\cos\omega t)\end{aligned}$$

$$\begin{aligned} x'y'' - x''y' &= R^2\omega^4 t\Big(\cos\omega t\sin\omega t + \omega t\cos^2\omega t - \cos\omega t\sin\omega t + \omega t\sin^2\omega t\Big) \\ &= R^2\omega^5 t^2 \\ \sqrt{x'^2 + y'^2} &= v = R\omega^2 t \\ \kappa &= \frac{R^2\omega^5 t^2}{(R\omega^2 t)^3} = \frac{1}{R\omega t}\end{aligned}$$

4 What are the tangential and radial accelerations ?

$$\begin{aligned} a_T &= \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}|\vec{v}|}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}R\omega^2 t = R\omega^2 \\ a_N &= \kappa v^2 = \frac{1}{R\omega t}(R\omega^2 t)^2 = R\omega^3 t\end{aligned}$$

5 Compute the length of the curve between $t = 0$ and $t = 2\pi/\omega$.

$$\begin{aligned} L &= \int_0^{2\pi/\omega} |\vec{v}|dt = \int_0^{2\pi/\omega} \sqrt{x'^2 + y'^2}dt \\ &= \int_0^{2\pi/\omega} R\omega^2 t\,dt = R\omega^2\Big[t^2/2\Big]_0^{2\pi/\omega} = R\omega^2\frac{4\pi^2}{2\omega^2} = 4\pi^2 R\end{aligned}$$

6 Show that the line through P perpendicular to the tangent line at P is tangent to the circle C. (*Hint: Show that this line is the line* (PQ))

Since the line (QP) contains P, we just have to prove that $\overrightarrow{QP}$ is normal to the curve, that is $\overrightarrow{QP} \perp \vec{T}$ (or we could have taken $\vec{v}$ instead of $\vec{T}$ since they have same direction). So we just have to check that $\overrightarrow{QP}.\vec{T} = 0$.

$$\overrightarrow{QP}.\vec{T} = R\omega t(\sin\omega t\vec{\imath} - \cos\omega t\vec{\jmath}).(\cos\omega t\vec{\imath} + \sin\omega t\vec{\jmath}) = 0$$

7 **Sketch a graph of the curve for $0 \le t \le 2\pi/\omega$ by plotting the point P for $\omega t = 0, \pi/2, \pi, 3\pi/2, 2\pi$ together with the tangent lines at these points.** ***Hint: you may use the previous question to draw the tangent line.***