Solution of Quiz 4

1. Let $\vec{F}(x,y) = \frac{y}{x^2+y^2}\vec{\imath} - \frac{x}{x^2+y^2}\vec{\jmath}$ and consider $\mathcal{C}$ the curve defined by $x(t) = \cos(t)$, and $y(t) = \sin(t)$ for $-\pi/2 \le t \le \pi/2$. Compute $\int_{\mathcal{C}} \vec{F}.\mathrm{d}\vec{r}$.

$$\vec{F}.\mathrm{d}\vec{r} = \frac{y dx}{x^2+y^2} - \frac{x dy}{x^2+y^2} \text{ so,}$$

$$\int_{\mathcal{C}} \vec{F}.\mathrm{d}\vec{r} = \int_{t=-\pi/2}^{\pi/2} \frac{-\sin^2(t)\mathrm{d}t}{\cos^2(t)+\sin^2(t)} - \frac{\cos^2(t)\mathrm{d}t}{\cos^2(t)+\sin^2(t)} = \int_{t=-\pi/2}^{\pi/2} -dt = -\pi$$

2. Let $\vec{F}(x,y) = e^x y^2 \vec{\imath} + 2ye^x\vec{\jmath}$. Find a function f such that $\vec{\nabla} f = \vec{F}$. Deduce $\int_{\mathcal{C}} \vec{F}.\mathrm{d}\vec{r}$ for any path $\mathcal{C}$ going from $(0,0)$ to $(3,4)$.

One can easily guess and check that $f(x,y) = e^x y^2$ is such that $\vec{\nabla} f = \vec{F}$. If this doesn't seem obvious to you, you may compute f this way: f must satisfy $\frac{\partial f}{\partial x} = e^x y^2$ so $f(x,y) = e^x y^2 + C(y)$ then $\frac{\partial f}{\partial y} = 2ye^x$ has to be equal to $2ye^x + C'(y)$ so $C'(y)$ is zero, and C has to be a constant. Since we just care about finding one such function f, we can take C to be zero, and check that $\vec{\nabla}(e^x y^2) = \vec{F}$. Then

$$\int_{\mathcal{C}} \vec{F}.\mathrm{d}\vec{r} = \int_{\mathcal{C}} \vec{\nabla} f.\mathrm{d}\vec{r} = f(3,4) - f(0,0) = 16e^3$$

3. Let $\mathcal{C}$ be the boundary of the triangle S with vertices $(0,0)$, $(1,0)$, $(1,1)$ oriented counterclockwise. Use Green's theorem to compute $\int_{\mathcal{C}} (2xy + \arctan x)\,\mathrm{d}x + (e^{y^2} + x^2)\,\mathrm{d}y$.

Green's theorem says

$$\int_{\mathcal{C}} M\mathrm{d}x + N\mathrm{d}y = \int_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

Here, we take $M = 2xy + \arctan x$ and $N = e^{y^2} + x^2$ so that $\frac{\partial N}{\partial x} = 2x$ and $\frac{\partial M}{\partial y} = 2x$, so we get

$$\int_{\mathcal{C}} (2xy + \arctan x)\,\mathrm{d}x + (e^{y^2} + x^2)\,\mathrm{d}y = 0.$$

Note: here you see the point of Green's theorem: just try to compute the integral directly!