Correction of Quiz 3

1. (4 points) Consider the function $f(x,y,z) = x^3 - 2xy^2 + \frac{1}{z}$.

a. Compute the gradient of f at $P = (2,1,1)$.

$$\frac{\partial f}{\partial x} = 3x^2 - 2y^2,\ \frac{\partial f}{\partial y} = -4xy,\ \frac{\partial f}{\partial z} = -\frac{1}{z^2} \qquad \textit{so} \quad \boxed{\vec{\nabla} f(P) = 10\vec{\imath} - 8\vec{\jmath} - \vec{k}.}$$

b. Give an equation of the plane tangent to the surface with equation $f(x,y,z) = 5$ at $P = (2,1,1)$.

That's the plane through $(2,1,1)$ *normal to* $(10,-8,-1)$ *so it has equation*

$$\boxed{10(x-2) - 8(y-1) - (z-1) = 0.}$$

2. (5 points) Find all the local minima and maxima of $f(x,y) = x^3 + y^3 - 9xy + 27$.
A local extremum should satisfy $\vec{\nabla} f(x,y) = \vec{0}$ *i.e.*

$$\begin{cases} 3x^2 - 9y &= 0 \\ 3y^2 - 9x &= 0 \end{cases}$$

From the first equation we get $y = \frac{x^2}{3}$, *which we plug in the second equation to get* $\frac{x^4}{3} - 9x = 0$. *This last equation gives* $x = 0$ *or* $x^3 = 27$ *so* $x = 0$ *or* 3. *Using the first equation we find two solutions* $(x,y) = (0,0)$ *or* $(x,y) = (3,3)$.

Now, to check whether those values are local minima, maxima, or saddle points (which are neither a local maximum nor a local minimum), we use the criterion of the second partials.

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -9$$

so $D = f_{xx}f_{yy} - fxy^2 = 36xy - 81$. *so* $D < 0$ *at* $(0,0)$ *so we have a saddle point there (it is neither a local maximum nor a local minimum), and* $D > 0$ *at* $(3,3)$ *so* f *has a local extremum there, and it is a local maximum since* $f_{xx}(3,3) < 0$.

$$\boxed{\textit{There is only one local extremum: it is a local maximum located at } (3,3)}$$

Note that the function is not bounded on the plane ($f(x,0)$ goes to $\pm\infty$ as x tends to $+\infty$ or $-\infty$). Therefore it has no global max and no global min.

3. (5 points) Compute the maximum and minimum of $f(x,y) = x^2 + y$ on the ellipse $x^2 + \frac{y^2}{2} = 1$.

We use the Lagrange Method: we solve for x, y, λ in the following system of equations:

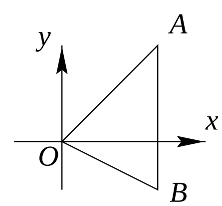
$$\begin{cases} 2x &= 2\lambda x \\ 1 &= \lambda y \\ x^2 + \frac{y^2}{2} &= 1 \end{cases}$$

From the first equation we get $x = 0$ or $\lambda = 1$. If $\lambda = 1$, equation 2 gives $y = 1$, and third equation implies $x^2 = \frac{1}{2}$ so we get two critical points $(x,y) = (\frac{1}{\sqrt{2}}, 1)$ and $(x,y) = (-\frac{1}{\sqrt{2}}, 1)$. If $x = 0$ the third equation gives $y^2 = 2$ so we get two other solutions $(x,y) = (0, \sqrt{2})$ and $(x,y) = (0, -\sqrt{2})$. These four points are the four critical points.

Now since the ellipse is a closed bounded subset of the plane, and since $f(x,y)$ is continuous, we know that it has a maximum and a minimum value. Those values have to be critical points, so we just have to compare the values of f at the critical points to know which is the minimum and which is the maximum. We get that f has its maximum at $(x,y) = (\pm\frac{1}{\sqrt{2}}, 1)$ where its value is 1.5 and its minimum at $(0, -\sqrt{2})$ where its value is $-\sqrt{2}$.

Note: You could also use a parametrization of the ellipse by $x = \cos t$, $y = \sqrt{2}\sin t$ and derive with respect to t using the chain rule to get the critical points.

4. (6 points) Let S be the triangle in the plane with vertices $O = (0,0)$, $A = (2,2)$, $B = (2,-1)$. Compute $\iint_S \sin(\pi x^2)\, dA$. **Note:** the order of integration is important to manage to get the result.



S may be described as

$$S = \{(x,y) \text{ s.t. } 0 \le x \le 2,\ -x/2 \le y \le x.\}$$

Therefore,

$$\iint_S \sin(\pi x^2)\, dA = \int_{x=0}^{2} \int_{y=-x/2}^{x} \sin(\pi x^2)\, dy\, dx = \int_{x=0}^{2} \frac{3}{2} x \sin(\pi x^2)\, dx.$$

Here we recognize $u' sin(u)$ up to a constant where $u = x^2$. Therefore,

$$\iint_S \sin(\pi x^2)\, dA = \frac{3}{2}\left[\frac{-cos(\pi x^2)}{2\pi}\right]_0^2 = 0.$$