This week we applied the idea of "approximation with increasing accuracies" to functions.

1 Power series

Definition 1. A power series centered at $a$ is a series of the form

$$p(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

where $c_n$’s are called the coefficients. The interval of convergence is the set of $x$ values for which the series converges:

$$I_c(f) = \{ x \mid f(x) \text{ converges} \}.$$  \hspace{1cm} (2)

Theorem 2 (interval of convergence). There are only three possibilities for the interval of convergence of a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$:

(i) $I_c = \{ a \}$,
(ii) $I_c = (-\infty, +\infty),$
(iii) $I_c = (a-R, a+R)$ or $[a-R, a+R)$ or $(a-R, a+R)$ or $[a-R, a+R]$.

$R$ is called the Radius of convergence. $R = 0, +\infty$ for cases (i) & (ii), respectively.

Theorem 3 (Term-by-term differentiation and integration). If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$, then $f(x)$ is differentiable on $(a-R, a+R)$.

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}.$$  \hspace{1cm} (3)

$$\int f(x)dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$  \hspace{1cm} (4)

Furthermore, the radii of convergence of both $f'(x)$ and $\int f(x)dx$ are $R$.

2 Taylor Series

Definition 4. Consider a function $f : (a-R, a+R) \to \mathbb{R}$ with $R > 0$. If $f^{(n)}(x)$ exists at $x = a$, then

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$  \hspace{1cm} (5)

is called the $n$th Taylor polynomial for $f(x)$ at $a$.

$$\lim_{n \to \infty} T_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$  \hspace{1cm} (6)

is called the Taylor series for $f(x)$ at $a$. The remainder of $T_n$ in approximating $f(x)$ is

$$R_n(x) = f(x) - T_n(x).$$  \hspace{1cm} (7)

Lemma 5. $\forall m = 0, 1, 2, \ldots, n, R_n^{(m)}(a) = 0$.

Theorem 6. Let $T_n$ be the $n$th Taylor polynomial for $f(x)$ at $a$.

$$\lim_{n \to \infty} R_n(x) = 0 \iff f(x) = \lim_{n \to \infty} T_n(x).$$  \hspace{1cm} (8)

Theorem 7 (Rolle’s). If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies

(i) $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$,
(ii) $f(a) = f(b)$,
then $\exists x \in (a, b)$ s.t. $f'(x) = 0$.

Theorem 8 (Taylor’s theorem - Lagrangian form). Consider a function $f : \mathbb{R} \to \mathbb{R}$. If $f^{(n+1)}(x)$ exists on the interval $I = (a-d, a+d)$, and $f^{(n)}(x)$ is continuous on $[a-d, a+d]$, then $\forall x \in I$, $\exists y \in I$ s.t.

$$R_n(x) = \frac{f^{(n+1)}(y)}{(n+1)!} (x-a)^{n+1}.$$  \hspace{1cm} (9)

Proof. For any fixed $x \neq a$, $\exists M$ s.t.

$$R_n(x) = f(x) - T_n(x) = \frac{M(x-a)^{n+1}}{(n+1)!}.$$  \hspace{1cm} (10)

Consider function

$$g(t) = R_n(t) - \frac{M(t-a)^{n+1}}{(n+1)!}.$$  \hspace{1cm} (11)

By Lemma 5, $g(a) = 0$ and $g^{(k)}(a) = 0$ for $k = 0, 1, \ldots, n$.

Also, $g(x) = 0$ by (10). By Rolle’s theorem,

$$\exists x_1 \in (a, x) \text{ s.t. } g'(x_1) = 0.$$  \hspace{1cm} (12)

If $x < a$, change $(a, x)$ above to $(x, a)$. Similarly,

$$\exists x_2 \in (a, x_1) \text{ s.t. } g^{(2)}(x_2) = 0.$$  \hspace{1cm} (13)

Repeatedly using Rolle’s theorem,

$$\exists x_{n+1} \in (a, x_n) \text{ s.t. } g^{(n+1)}(x_{n+1}) = 0.$$  \hspace{1cm} (14)

Since $T_n^{(n+1)}(t) = 0$, $g^{(n+1)}(t) = M - f^{(n+1)}(t)$.

(11) completes the proof by showing the existence of $y$ with $y = x_{n+1}$. \hspace{1cm} $\square$

The proof is optional, but I hope you find it elegant.

Theorem 9 (Taylor’s inequality). If $\exists M < \infty$ s.t. $\forall x \in [a-d, a+d], f^{(n+1)}(x) \leq M$, then $\forall x \in [a-d, a+d],$

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$  \hspace{1cm} (15)

Consequently $f(x) = \lim_{n \to \infty} T_n$.

Theorem 10 (The Binomial Series). $\forall k \in \mathbb{R}, |x| < 1$,

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + \sum_{n=1}^{\infty} \frac{n-1}{n!} x^n.$$  \hspace{1cm} (16)