Definition 1. The Cartesian product \( X \times Y \) between two sets \( X \) and \( Y \) is the set of all possible ordered pairs with first element from \( X \) and second element from \( Y \):

\[
X \times Y = \{(x, y) \mid x \in X, y \in Y\}. \tag{1}
\]

Definition 2. A function \( f \) from \( X \) to \( Y \), written as \( f : X \to Y \) or \( X \rightarrow Y \), is a subset of the Cartesian product \( X \times Y \) satisfying that for all \( x \in X \), there is exactly one \( y \in Y \) such that \( (x, y) \in X \times Y \). \( X \) and \( Y \) are the domain and range of \( f \), respectively.

Definition 3 (Limit of a scalar function). Consider a function \( f : I \to \mathbb{R} \) with \( I(a, r) = (a-r, a) \cup (a, a+r) \). The limit of \( f(x) \) exists as \( x \) approaches \( a \), written as

\[
\lim_{x \to a} f(x) = L,
\]

iff the following holds:

\[
\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } \forall x \in I(a, \delta), |f(x) - L| < \varepsilon. \tag{3}
\]

Definition 4. A vector function is a function whose range is a set of vectors in \( \mathbb{R}^n \). It is written as \( \mathbb{R}^m \to \mathbb{R}^n \) or \( f : \mathbb{R}^m \to \mathbb{R}^n \) \((m,n \in \mathbb{N}^+)\).

Definition 5. The limit of a vector function \( \mathbf{r} : \mathbb{R} \to \mathbb{R}^3 \),

\[
\lim_{t \to a} \mathbf{r}(t) = \left( \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right).
\]

The derivative and integral of a vector function are also defined component-wise, e.g.,

\[
\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = (f'(t), g'(t), h'(t)). \tag{4}
\]

Definition 6. Given two points \( A = (a_1, a_2, a_3) \) and \( B = (b_1, b_2, b_3) \), \( \mathbf{v} = B - A \) is the vector that starts at \( A \) and ends at \( B \). The \textit{length of vector} \( \mathbf{v} \) equals the distance between \( A \) and \( B \):

\[
|\mathbf{v}| = |AB| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}. \tag{6}
\]

In particular, \( \mathbf{a} = (a_1, a_2, a_3) \) can be regarded as a vector that starts at the origin \( O \) and ends at \( A \).

\[
\mathbf{a} = (a_1, a_2, a_3) \Rightarrow |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}. \tag{7}
\]

Definition 7. \( \mathbf{v} \) is a unit vector iff \(|\mathbf{v}| = 1\). The unit vector in the same direction of \( \mathbf{v} \) is \( \frac{\mathbf{v}}{|\mathbf{v}|} \).

Definition 8. A line is a set of points uniquely determined by a point \( P_0 \) and a direction vector \( \mathbf{v} \):

\[
\{P \mid P(t) = P_0 + tv, \ t \in (\infty, \infty)\}. \tag{8}
\]

Definition 9 (Dot product: algebraic definition). The dot product of two vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \) is a real number:

\[
\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3. \tag{9}
\]

Definition 10. The angle \( \theta \) between two nonzero vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \) satisfies

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}, \quad \theta \in [0, \pi]. \tag{10}
\]

Theorem 11. The algebraic definition of the dot product is equivalent to its geometric definition:

\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos \theta. \tag{11}
\]

Definition 12. The scalar projection of \( \mathbf{b} \) onto \( \mathbf{a} \) is

\[
\text{comp}_\mathbf{a} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}, \tag{12}
\]

and the vector projection of \( \mathbf{b} \) onto \( \mathbf{a} \) is

\[
\text{proj}_\mathbf{a} \mathbf{b} = \mathbf{c} \mathbf{a} = (\text{comp}_\mathbf{a} \mathbf{b}) \frac{\mathbf{a}}{|\mathbf{a}|}. \tag{13}
\]

Definition 13. A plane is a set of points uniquely determined by a point \( P_0 \) and a normal vector \( \mathbf{n} \):

\[
\{P \mid \mathbf{n} \cdot (P - P_0) = 0\}. \tag{14}
\]

Equivalently, the \textit{scalar equation of a plane} is

\[
a x + by + cz + d = 0. \tag{15}
\]

Definition 14. The standard basis vectors in \( \mathbb{R}^3 \) are

\[
i = (1, 0, 0), \quad j = (0, 1, 0), \quad k = (0,0,1). \tag{16}
\]

Definition 15 (Geometric definition of cross product). The cross product of two vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \) is

\[
\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}||\sin \theta|\mathbf{n}, \tag{17}
\]

where \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \), and \( \mathbf{n} \) is the unit vector determined by the right-hand rule from \( \mathbf{a} \) and \( \mathbf{b} \).

Definition 16 (Algebraic definition of cross product).

\[
\mathbf{a} \times \mathbf{b} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{18a}
\]

\[
= \det \begin{vmatrix} a_2 & a_3 & \mathbf{i} - \det \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \mathbf{j} + \det \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ b_2 & b_3 \end{vmatrix} \tag{18b}
\]

\[
= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k \tag{18c}
\]
Theorem 17. The algebraic and geometric definitions of cross product are equivalent.

Definition 18. Two nonzero vectors \( \mathbf{a}, \mathbf{b} \) are perpendicular or orthogonal, written as \( \mathbf{a} \perp \mathbf{b} \), iff \( \mathbf{a} \cdot \mathbf{b} = 0 \).

Definition 19. Two nonzero vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \) are parallel, written as \( \mathbf{a} \parallel \mathbf{b} \), iff \( \exists \lambda \neq 0, \text{ s.t. } \mathbf{a} = \lambda \mathbf{b} \).

Theorem 20. \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \) are parallel iff \( \mathbf{a} \times \mathbf{b} = 0 \).

Definition 21 (scalar triple product). For \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 \),

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 
\end{vmatrix}.
\] (19)

Theorem 22. For \( \mathbf{u}, \mathbf{v} : \mathbb{R} \to \mathbb{R}^3 \), \( c \in \mathbb{R} \), \( f : \mathbb{R} \to \mathbb{R} \),

\[
\frac{d}{dt} [\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t),
\] (20a)

\[
\frac{d}{dt} [(c \mathbf{u})(t)] = c \mathbf{u}'(t),
\] (20b)

\[
\frac{d}{dt} [f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t),
\] (20c)

\[
\frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t),
\] (20d)

\[
\frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t),
\] (20e)

\[
\frac{d}{dt} [\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f).
\] (20f)

Definition 23. A curve is (the image of) a vector function \( \mathbb{R} \to \mathbb{R}^3 \), or \( \mathbf{r}(t) : \mathbb{R} \to \mathbb{R}^3 \). The independent variable \( t \) is its parameterization.

Definition 24. A surface is (the image of) a vector function \( \mathbb{R}^2 \to \mathbb{R}^3 \).

Definition 25. The tangent vector to a curve \( \mathbf{r}(t) \) at a point \( P(t) = O + \mathbf{r}(t) \) is \( \mathbf{r}'(t) \); the corresponding unit tangent vector is

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.
\] (21)

Definition 26. The tangent line to \( \mathbf{r} : \mathbb{R} \to \mathbb{R}^3 \) at \( P(t_0) = O + \mathbf{r}(t_0) \) is the line determined by \( P(t_0) \) and \( \mathbf{T} \):

\[
\{ P \mid P = P(t_0) + t\mathbf{T}, \ t \in \mathbb{R} \}.
\] (22)

Theorem 27. If \( |\mathbf{r}(t)| = c \) where \( c \) is a constant, then \( \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \). Consequently \( \mathbf{r}'(t) \cdot \mathbf{T}(t) = 0 \).

Definition 28. The arc length of a curve \( \mathbf{r} : \mathbb{R} \to \mathbb{R}^3 \) starting from \( P(a) = O + \mathbf{r}(a) \) is a function \( s : \mathbb{R} \to \mathbb{R} \),

\[
s(t) = \int_a^t |\mathbf{r}'(u)| du.
\] (23)

Formula 29. \( \frac{ds}{dt} = |\mathbf{r}'(t)| \).

Definition 30. The curvature of a curve \( \mathbf{r}(t) \) at the point \( P(t) = O + \mathbf{r}(t) \) is

\[
\kappa(t) = \frac{|d\mathbf{T}|}{ds}.
\] (24)

Formula 31.

\[
\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.
\] (25)

Theorem 32. The curvature of a curve \( \mathbf{r}(t) \) at \( P(t) \) is

\[
\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.
\] (26)

Corollary 33. The curvature of a 2D curve \( y = f(x) \) is

\[
\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.
\] (27)

Definition 34. The principal unit normal vector is

\[
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.
\] (28)

and the binormal vector is

\[
\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).
\] (29)

The normal plane of the curve at \( P = O + \mathbf{r}(t) \) is the plane determined by \( \mathbf{N}(t) \) and \( \mathbf{B}(t) \). The osculating plane is that by \( \mathbf{T}(t) \) and \( \mathbf{N}(t) \).

Definition 35. Let \( t \) represent time and \( \mathbf{r}(t) \) the trajectory of a moving particle. Then \( \mathbf{r}'(t) = \mathbf{v} \) is called the velocity of the particle, \( |\mathbf{v}| = v \) the speed of the particle, \( \mathbf{r}''(t) = \mathbf{a} \) the acceleration of the particle.

Theorem 36. The acceleration of a particle following the curve \( \mathbf{r}(t) \) is a vector \( \mathbf{a}(t) \) consists of two parts:

\[
\mathbf{a}(t) = a_T \mathbf{T} + a_N \mathbf{N},
\] (30)

where \( a_T \) is caused by the change of the speed, and \( a_N \) is caused by the change of the velocity direction:

\[
a_T(t) = v' = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|^3},
\] (31)

\[
a_N(t) = \kappa v^2 = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|}.
\] (32)