THE SINGULAR VALUE DECOMPOSITION

Let

\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Then

\[ A^T A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

we observe that the eigenvectors of the matrix are

\[ x_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad x_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

\[ \lambda = 3, \quad \lambda = 1, \quad \lambda = 0 \]

we get

\[ \sigma_1 = \sqrt{3}, \quad \sigma_2 = 1, \]

These are called the singular values of \( A \). Let

\[ \sqrt{V_1 = \|x_3\| x_3 = \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix}, \quad \sqrt{V_2 = \|x_1\| x_1 = \begin{pmatrix} -\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \quad \sqrt{V_3 = \|x_0\| x_0 = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} \]

Then as these are eigenvectors of the positive semidefinite matrix \( A^T A \) corresponding to distinct eigenvalues the form an orthonormal basis of \( \mathbb{R}^3 \). We can now give 2 of the matrices of the SVD of \( A \).
Let

\[ \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

the final matrix in the SVD is obtained as follows.

Let

\[ U_1 = \frac{AV_1}{\sigma_1}, \quad U_2 = \frac{AV_2}{\sigma_2} \]

\[ = \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{2}} \\ \frac{\sqrt{2}}{\sqrt{2}} \\ 0 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{\sqrt{2}} \\ \frac{\sqrt{2}}{\sqrt{2}} \end{bmatrix} \]

Note that \( U_1, U_2 \) are orthonormal and of length 1.

We observe that for any vector \( \mathbf{y} \), \( A \mathbf{y} = y_1 \text{col}_1(A) + y_2 \text{col}_2(A) + y_3 \text{col}_3(A) \) is \( A \mathbf{y} \) is in the column space of \( A \). It follows that \( U_1, U_2 \in \text{col}(A) \). But \( \text{col}(A^+) = \text{null}(A^T) \), so we determine 3 more orthonormal vectors \( U_3, U_4, U_5 \) by finding an orthonormal basis for \( \text{null}(A^T) \).

For \( \text{null}(A^T) \)

\[ A^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]
\(y_3, y_4, y_5\) are free variables. Set \(y_3 = 1, y_4 = s, y_5 = t\)

Then \(A^T Y = 0 \iff Y = \begin{pmatrix} -r \\ r \\ r \\ r \\ s \\ t \\ \overline{s} \\ \overline{t} \\ \overline{r} \\ \overline{r} \\ \overline{r} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}

Note the vectors are orthogonal so set

\[
U_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad U_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad U_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\]

Finally we put

\[
U = \begin{bmatrix} \sqrt{r} & 0 & -\sqrt{r_3} & 0 & 0 \\ \sqrt{s} & -\sqrt{s_2} & \sqrt{s_3} & 0 & 0 \\ \sqrt{t} & \sqrt{t_2} & \sqrt{t_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

Then claim

\[
A = U \Sigma V^T
\]

We've \(U + V\) are orthogonal matrices. Thus

Product simplifies by observing that

\[
\Sigma V^T = \begin{bmatrix} \sqrt{r} & \sqrt{s} & \sqrt{t} & 0 & 0 \\ \sqrt{r_2} & -\sqrt{s_2} & \sqrt{t_2} & 0 & 0 \\ \sqrt{r_3} & -\sqrt{s_3} & \sqrt{t_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{r_1} & 0 & 0 & 0 \\ 0 & -\sqrt{s_1} & 0 & 0 \\ 0 & 0 & -\sqrt{t_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

and so \(U_i = \frac{AV_i}{\sigma_i}, i = 1, 2, \ldots\) \(AV_i = \sigma_i U_i, i = 1, 2\)

Then

\[
U \Sigma V^T = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 & U_5 \end{bmatrix}
\]
\[
\begin{align*}
&= \sigma_1 U_1 V_1^T + \sigma_2 U_2 V_2^T \\
&= A V_1 V_1^T + A V_2 V_2^T \\
\end{align*}
\]
So if \( B = U \Sigma V^T \) then
\[
B V_1 = (A V_1 V_1^T + A V_2 V_2^T) V_1 \\
= A V_1 (V_1^T V_1) + A V_2 (V_2^T V_1) \\
But \ x \cdot y = x^T y \ so \ as \ V_1, \& V_2 \ are \ on \ axes \\
= A V_1 \\
\]
Similarly
\[
B V_2 = A V_2 \\
and \quad B V_3 = A V_1 (V_1^T V_3) + A V_2 (V_2^T V_3) = 0 = A V_3 \\
So \ N(B-A) = \mathbb{R}^3 \ as \ it \ contains \ an \ ON \ basis \ of \ \mathbb{R}^3 \\
\Rightarrow \ \dim \operatorname{col}(B-A) = 0 \Rightarrow \ \operatorname{col}(B-A) = \{0\} \ \text{ie} \\
the \ columns \ of \ B = \text{those \ of} \ A