A Capture Problem in Random Walks and Eigenvalues of Spherical Domains

Jesse Ratzkin
University of Connecticut
ratzkin@math.uconn.edu

Andrejs Treibergs
University of Utah
treiberg@math.utah.edu

January 6, 2005

Abstract

We resolve a question of Bramson and Griffeath by showing that the expected capture time of four Brownian predators pursuing one Brownian prey on a line is finite. Our main tool is an eigenvalue estimate for a particular spherical domain, which we obtain by a repeated coning process.

In this paper, we examine the expected capture time of a single Brownian prey pursued by n Brownian predators. All motion is restricted to a line. Bramson and Griffeath [BG] first considered this problem, and estimated the capture time in various circumstances. In particular, they showed that if at time $t = 0$ there are predators on both sides of the prey then the expected capture time is finite. For this reason, we will assume that the initial position of the prey is $x_0(0) = 1$ and the initial positions of the predators are all $x_1(0) = \cdots = x_n(0) = 0$. In this case, [BG] also showed that the expected capture time is infinite for $n = 1, 2, 3$, and conjectured that it is finite for $n \geq 4$ (as indicated by simulations). Li and Shao [LS] (see also [L]) showed that the expected capture time is finite for $n \geq 5$. In our main theorem, we resolve the remaining case by showing that the expected capture time is finite for $n = 4$.

main-thm **Theorem 1** *The expected capture time of a Brownian prey pursued by four Brownian predators, all moving on a line, if finite if and only if there are at least four predators.*

One can reformulate the capture of a Brownian prey pursued by n Brownian predators, as described above, as the exit of a Brownian particle in $\mathbb{R}^{n+1}$ from a specific cone. We denote the posititon of the prey at time t as $x_0(t)$ and the position of the jth predator at time t as $x_j(t)$. By our choice of initial conditions, the initial position of the Brownian particle $x(t) = (x_0(t), x_1(t), \ldots, x_n(t))$ is $x(0) = (1, 0, \ldots, 0)$. The event of capture is then equivalent to the Brownian particle $x(t)$ leaving the cone

$$\mathcal{C}_{n+1} := \{(x_0, x_1, \ldots, x_n) : x_0 \geq x_j, j = 1, \ldots, n\},$$

with $x(0) = (1, 0, \ldots, 0)$, and so we must estimate the expected exit time of a Brownian particle from the cone $\mathcal{C}_{n+1}$, with the starting position $(1, 0, \ldots, 0)$.

Deblassie [DB] developed the theory of estimating exit times for Brownian motion from cones in Euclidean space. Let $C = \{(r, \theta) \mid r \geq 0, \theta \in D \subset S^n\}$ be the cone over a domain $D \subset S^n$. Also let τ_x be the exit time from C of a Brownian particle with starting position x, and let $\mathbb{P}(\tau_x > t)$ be the probability that $\tau_x > t$. DeBlassie showed $\mathbb{P}(\tau_x > t) \sim c(x)t^{-a}$, where

$$2a = \left[\left(\frac{n-2}{2} \right)^2 + \lambda_1(D) \right]^{1/2} - \frac{n-1}{2}.$$

Here $\lambda_1(D)$ is the first Dirichlet eigenvalue of D. In the particular case we are interested in, the expected exit time of a Brownian particle from $\mathcal{C}_{n+1}$ is finite if and only if $a > 1$, which reduces to

$$\lambda_1(\mathcal{D}_n) > 2n + 2. \tag{1}$$

Our method for proving Theorem 1 is to estimate the first eigenvalue of $\mathcal{D}_n$ using a repeated coning proccess and the monotonicity property of eignevalues. The rest of the paper proceeds as follows. In Section 1 we discuss the geometry of $\mathcal{C}_{n+1}$, $\mathcal{D}_n$, and related regions. Section 2 contains the separation of variables background one needs to estimate the expected capture time. For the reader's convenience, we also include the proof that the expected capture time is infinite for $n = 1, 2, 3$ predators in this section. Finally, we prove the relevant eigenvalue estimate in Section 3.

Davar Khoshnevisan first told us of this problem. We would like to thank him and Pedro Mendez for lending their ears and expertise in this project.

1 Geometry of the cone $\mathcal{C}$

The cone $\mathcal{C}_{n+1}$ and its spherical angle $\mathcal{D}_n$ have much symmetry. First observe that $\mathcal{C}_{n+1}$ contains the line spanned by $(1, 1, \ldots, 1)$; this is the line where all the inequalities $x_0 \geq x_j$, $j = 1, \ldots, n$ are equalities. Thus we can split $\mathcal{C}_{n+1}$ as a direct sum

$$\mathcal{C}_{n+1} = \operatorname{span}\{(1, 1, \ldots, 1)\} \oplus \mathcal{V}_n.$$

Notice that $V_j := e_0 - e_j$, $j = 1, \ldots, n$ is orthogonal to $(1, 1, \ldots, 1)$, so $V_1, \ldots, V_n$ provide a basis of $\mathcal{V}_n$. It is convenient to define

$$\mathcal{T}_{n-1} := \mathcal{V}_n \cap S^{n-1},$$

where S^{n-1} is the unit sphere in $\operatorname{span}\{V_j\}$. The domain $\mathcal{D}_n$ is a double cone over $\mathcal{T}_{n-1}$. More precisely, let N be one of the intersection points in $S^n \cap \operatorname{span}\{(1, 1, \ldots, 1)\}$ (there are two such points), and let (r, θ) be polar coordinates in S^n, centered at N. Then

$$\mathcal{D}_n = \{(r, \theta) \mid \theta \in \mathcal{T}_{n-1}, 0 \leq r \leq \pi\}.$$

In later sections, we will use a generalization of this type of spherical cone. In general, let Ω be a domain in the equatorial S^{n-1} of S^n, and let $r_0 \in (0, \pi]$. Then we define the cone

$$\mathcal{D}(\Omega, r_0) := \{(r, \theta) \mid \theta \in \Omega, 0 \leq r \leq r_0\}.$$

We abreviate $\mathcal{D}(\Omega, \pi) = \mathcal{D}(\Omega)$. In this notation, $\mathcal{D}_n = \mathcal{D}(\mathcal{T}_{n-1}) = \mathcal{D}(\mathcal{T}_{n-1}, \pi)$.

The domain $\mathcal{T}_{n-1}$ has symmetry. If we let

$$\mathcal{C}^j_{n+1} = \{(x_0, x_1, \ldots x_n) \mid x_j \geq x_k, j \neq k\},$$

then we see $\mathcal{C}_{n+1} = \mathcal{C}^0_{n+1}$ and the $\mathcal{C}^j_{n+1}$ are pairwise congruent. Thus $\mathcal{T}_{n-1}$ is a face of the regular $(n+1)$-hedral tesselation of the standard S^{n-1} on obtains by connecting the vertices of a regular $(n+1)$-simplex with great circle arcs. In particular, one can compute the diameter of $\mathcal{T}_{n-1}$ as the distance from a vertex to the center of the opposite face, which is

$$\delta(n-1) = \arccos\left(-\sqrt{\frac{n-1}{2n}}\right).$$

Moreover, the spherical angle of $\mathcal{T}_{n-1}$ at a vertex is $\mathcal{T}_{n-2}$, so we can construct a succession of comparison domains for $\mathcal{T}_1, \mathcal{T}_2, \ldots$ starting with $\mathcal{T}_1$ and using the coning proccess described above. To this end, we let

$$\hat{\mathcal{T}}_1 := [0, \frac{2\pi}{3}] = \mathcal{T}_1, \qquad \hat{\mathcal{T}}_n := \mathcal{D}(\hat{\mathcal{T}}_{n-1}, \delta(n)).$$

By induction, $\mathcal{T}_n \subset \hat{\mathcal{T}}_n$, and so $\lambda_1(\mathcal{T}_n) \geq \lambda_1(\hat{\mathcal{T}}_n)$.

2 Separating variables

separate-sec

We discuss two types of separation of variables in this section. The first type is the separation of variables in [DB] to estimate expected exit times of Brownian motion from Euclidean cones, and the second is the separation of variables one performs to estimate eigenvalues of a spherical domain with a conical structure.

2.1 DeBlassie's separations of variables

We first review DeBlassie's [DB] argument. Consider the cone C over a domain $D \subset S^n$:

$$C = \{r\theta \mid r > 0, \theta \in D \subset S^n\}.$$

Let τ_x be the time it takes for a Brownian particle to exit C, with starting position x, and let $u(x,t) = \mathbb{P}(\tau_x > t)$ be the probability that $\tau_x > t$. Then u satisfies the heat equation

$$\begin{array}{ll} u_t = \frac{1}{2}\Delta u & (x,t) \in C \times [0,\infty) \\ u(x,0) = 1 & c \in \bar{C} \\ u(x,t) = 0 & (x,t) \in \partial C \times (0,\infty). \end{array}$$

In polar coordinates (r,θ,t), the PDE becomes

$$2u_t = u_{rr} + \frac{n}{r}u_r + \frac{1}{r^2}\Delta_{S^n} u.$$

Moreover, the solution scales as $u(r,\theta,t) = u(\beta r, \theta, \beta^2 t)$, so we can separate variables and look for a solution of the form $u = R(\xi)U(\theta)$, where $\xi = r^2/2t$. Then we get a positive separation constant $\lambda_j(D)$, and

$$\lambda_j(D) = \frac{\Delta_{S^n} U}{U} = \frac{4\xi^2 \ddot{R} + (4\xi^2 + 2n\xi)\dot{R}}{R}.$$

Letting $R = \xi^a \rho(-\xi)$, this ODE becomes

$$\xi\ddot{\rho} + (a + \frac{n+1}{2} - \xi)\dot{\rho} - a\rho = 0,$$

which has solutions of the form $\rho(\xi) = f(a/2, a + (n+1)/2, \xi)$, where f is the *confluent* hypergeometric function. Using the hypergeometric function f, one obtains (see [?, DB] a formal expansion for u of the form

$$u(x,t) = \sum_{j=1}^{\infty} B_j f(a_j, 2a_j + \frac{n+1}{2}, -|x|^2/2) U_j(\frac{x}{|x|})(\frac{|x|^2}{2t})^{a_j},$$

where U_j is the jth Dirichlet eigenfunction of Δ_{S^n} on D and

$$2a_j = \left[\left(\frac{n-1}{2}\right)^2 + \lambda_j(D)\right]^{1/2} - \frac{n-1}{2}.$$

The leading term in this expansion is $u(x,t) \sim B_1 u_1(x/|x|)(|x|/(2t))^{a_1}$, which yields the conclusion of inequality (1).

2.2 Separating variables on the sphere

Next we separate variables to relate the eigenvalues $\lambda_1(\mathcal{D}_n)$ and $\lambda_1(\mathcal{T}_{n-1})$. First recall that we can write the Laplacian for S^n as

$$\Delta u = u_{rr} + (n-1)\cot r u_r + \csc^2 r \Delta_\theta u, \tag{2}$$ sep-var1

where Δ_θ is the Laplacian on the equatorial S^{n-1}.

Lemma 2 *Let Ω be a nice domain in an equatorial S^{n-1} with first eigenvalue $\lambda = \lambda_1(\Omega)$, and let $\mathcal{D} = \mathcal{D}(\Omega)$ the double cone over Ω. Then the first Dirichlet eigenvalues of $\mathcal{D}$ and Ω are related by*

$$\lambda_1(\mathcal{D}) = \lambda_1(\Omega) - \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} + \lambda_1(\Omega)}. \tag{3}$$

eigen-rel1

In particular, $\lambda_1(\mathcal{D}) > 2n+2$ whenever $\lambda_1(\Omega) > 2n$.

Proof: Set $u(r,\theta) = R(r)T(\theta)$, where $R(0) = 0 = R(\pi)$ and $T(\theta) = 0$ for $\theta \in \partial\Omega$. Then u is an eigenfunction on $\mathcal{D}$ with eigenvalue μ precisely when

$$T\ddot{R} + (n-1)\cot r T\dot{R} + \csc^2 r R\Delta_\theta T = -\mu TR.$$

Separating variables with a positive separation constant λ yields

$$\frac{\sin^2 r\ddot{R} + (n-1)\sin r\cos r\dot{R} + \mu\sin^2 rR}{R} = \lambda = -\frac{\Delta_\theta T}{T}.$$

Choosing T to be the first eigenfunction of Ω, we obtain the ODE

$$\sin^2 r\ddot{R} + (n-1)\sin r\cos r\dot{R} + (\mu\sin^2 r - \lambda)R = 0, \tag{4}$$

sep-var2

which has regular singular points at $r = 0, \pi$. If we try a solution of the form $R = \sin^m r$, for some power m, we find

$$\begin{aligned} 0 &= m(m-1)\sin^m r\cos^2 r - m\sin^{m+2} r + m(n-1)\sin^m r\cos^2 r + (\mu\sin^2 r - \lambda)\sin^m r \\ &= \sin^m r[(m^2 + m(n-2) - \lambda)\cos^2 r + (\mu - m - \lambda)\sin^2 r]. \end{aligned}$$

Both coefficients must vanish, so we have $\mu = m + \lambda$ and $\lambda = m^2 + m(n-2)$. Solving for m, we find

$$m = \frac{2-n}{2} + \sqrt{\frac{(2-n)^2}{4} + \lambda}. \tag{5}$$

rel-eigenval

Next, observe that if $\lambda_1(\Omega) = 2n$ then $\lambda_1(\mathcal{D}) = 2n+2$. Finally, the formula for $\lambda_1(\mathcal{D})$ is monotone increasing in $\lambda_1(\Omega)$, and so $\lambda_1(\mathcal{D}) > 2n+2$ whenever $\lambda_1(\Omega) > 2n$. $\square$

Remark 1 *A second solution to equation (4) has the form $\sin^m r\cos r$, where m is again given by equation (5) but*

$$\mu' = \lambda + 3m + n.$$

This eigenfunction vanishes on $\{\pi/2\} \times \Omega$, so it corresponds to a higher eigenvalue.

At this point, we can prove that the expected capture time for $n = 1, 2, 3$ predators is infinite. To prove the expected capture time is infinite, by inequality (1) we need to show $\lambda_1(\mathcal{D}_n) \leq 2n+2$, or, equivalently, that $\lambda_1(\mathcal{T}_{n-1}) \leq 2n$. In the case of $n = 1$, we have $\mathcal{D}_1 = [-3\pi/4, \pi/4]$, and so $\lambda_1(\mathcal{D}_1) = 1 < 4$. In the case $n = 2$, we have $\mathcal{T}_1 = [0, 2\pi/3]$, and so $\lambda_1(\mathcal{T}_1) = 9/4 < 4$. We cannot compute $\lambda_1(\mathcal{T}_2)$ so easily, but we can find a test function to show that $\lambda_1(\mathcal{T}_2) < 6$. Recall the Rayleigh characterization of the first eigenvalue of a domain Ω:

$$\lambda_1(\Omega) = \inf_{f \in H_0^1(\Omega), f \not\equiv 0} \frac{\int_\Omega |df|^2}{\int_\Omega f^2}.$$

To show that $\lambda_1(\mathcal{T}_2) < 6$, it suffices to find $f_0 \in H_0^1(\mathcal{T}_2)$ so that $\int_{\mathcal{T}_2} |df_0|^2 / \int_{\mathcal{T}_2} f_0^2 < 6$. Let

$$f_0(x) = \sin(\operatorname{dist}(x, \partial\mathcal{T}_2)),$$

and observe that $|df_0|^2 = 1 - f_0^2$ off the set of focal points of $\partial\mathcal{T}_2$, which is a set of measure zero. A computation shows

$$\lambda_1(\mathcal{T}_2) \le \frac{\int_{\mathcal{T}_2} |df_0|^2}{\int_{\mathcal{T}_2} f_0^2} = \frac{2\pi + \sqrt{3}}{\pi - \sqrt{3}} < 6.$$

One can generalize the eigenvalue relationship (3) to spherical cones of the form $\mathcal{D}(\Omega, r_0)$, for $0 < r_0 < \pi$, using the confluent hypergeometric function

$$f(\alpha, \beta, , \gamma, z) = 1 + \frac{\alpha}{\beta}\frac{z}{\gamma} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{z^2}{\gamma(\gamma+1)} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\frac{z^3}{\gamma(\gamma+1)(\gamma+2)} + \dots \qquad (6)$$

confluent

In the next section, we will use the following lemma to relate the eigenvalues of $\hat{\mathcal{T}}_{n-1}$ and $\hat{\mathcal{T}}_n$, which we will then estimate to complete the proof of Theorem 1.

Lemma 3 *Let Ω be a nice domain in a equatorial $S^{n-1} \subset S^n$ with eigenvalue $\lambda = \lambda_1(\Omega)$. Then the first Dirichlet eigenvalues of Ω and $\mathcal{D}(\Omega, r_0)$ are related by*

$$\lambda_1(\mathcal{D}(\Omega, r_0)) = \mu = \mu(n, \lambda, r_0), \qquad (7)$$

eigen-rel2

where μ is the first eigenvalue of the ODE (4) on $[0, r_0]$. If $r_0 \ge \pi/2$ then μ is the unique zero of $f(\alpha_1, \beta_1, \gamma_1, (1/2)(1 - \cos r_0))$ in $(m + \lambda, 3m + \lambda + n)$, where m is defined in equation (5) and

$$\begin{aligned} \alpha_1, \beta_1 &= \frac{7}{2}[1 + \sqrt{(n-2)^2 + 4\lambda} \pm \sqrt{(n-1)^2 + 4\mu}] \\ \gamma_1 &= \frac{2 + \sqrt{(n-2)^2 + 4\lambda}}{2}. \end{aligned}$$

If $r_0 \le \pi/2$ then μ is the unique zero of $f(\alpha_2, \beta_2, \gamma_2, (1/2)(1 - \cos r_0))$ in $(0, n)$, where

$$\alpha_2, \beta_2 = \frac{n - 1 \pm \sqrt{(n-1)^2 + 4\mu}}{2}, \qquad \gamma_2 = \frac{n}{2},$$

with f defined by equation (6).

Proof: We separate variables and look for a solution of the form $R(r) = \sin^m r u(r)$, with $u(r) \ne 0$ on $[0, r_0)$, but $u(r_0) = 0$. Then equation (4) becomes

$$0 = \sin^{m+2} r\ddot{u} + (2m + n - 1)\sin^{m+1} r \cos r\dot{u} + [m(m + n - 2)\cos^2 r + (\mu - m)\sin^2 r - \lambda]\sin^m ru.$$

Now let $u(r) = y(x)$, where $x = (1/2)(1 - \cos r)$, which transforms the ODE above into

$$x(1-x)y'' + (m + \frac{1}{2}n - (2m+n)x)y' - (\lambda + m - \mu)y = 0.$$

The solution to this ODE is the hypergeometric function $y(x) = f(\alpha, \beta, \gamma, x)$, with

$$\begin{aligned} \alpha, \beta &= \frac{2m + n - 1 \pm \sqrt{(2m+n-1)^2 - 4\lambda - 4m + 4\mu}}{2} \\ \gamma &= \frac{2m + n}{2}. \end{aligned}$$

The lemma follows from taking $R(r; n, \lambda, r_0) = \sin^m r f(\alpha, \beta, \gamma, (1/2)(1 - \cos r))$, where we choose μ so that $R(r_0; n, \lambda, r_0) = 0$. $\square$

One can use this Lemma to compute $\lambda_1(\hat{\mathcal{T}}_n)$ iteratively. In this case we start with $\lambda_1(\hat{\mathcal{T}}_1) = 9/4$ and apply equation (7).

3 The eigenvalue estimate

eigen-est-sec

In this section, we complete the proof of Theorem 1. First observe that it suffices to show $\lambda_1(\mathcal{T}_3) > 8$. Given a nice domain $\Omega_2 \subset S^2$ with eigenvalue λ, such that $\mathcal{T}_2 \subset \Omega_2$, one obtains a lower bound for $\lambda_1(\mathcal{T}_3)$ using equation (7) and monotonicity. We define λ_{cr} by

$$8 = \mu(3, \lambda_{cr}, \delta(3)), \qquad \lambda_{cr} \simeq 5.101267527.$$

Constructing a domain Ω_2 with $\lambda_1(\Omega_2) > \lambda_{cr}$ and $\mathcal{T}_2 \subset \Omega_2$ shows that $\lambda_1(\mathcal{T}_3) > 8$ using the argument above.

We now construct a domain $\mathcal{G}_2 \subset S^2$ as a perturbation of $\mathcal{T}_2$, such that $\mathcal{T}_2 \subset \mathcal{G}_2$ and $\lambda_1(\mathcal{G}_2) = 5.102 > \lambda_{cr}$. This completes the proof of Theorem 1. Previous work of Rayleigh [R] and Pólya–Szegö [PS] motivates us to consider this type of domain perturbation. They studied the eigenvalue of a planar domain which has the form $\{(r,\theta) \mid 0 \le r \le c + \epsilon f(\theta)\}$ in polar coordinates, for some small $\epsilon > 0$, giving an expression for the eigenvalue λ_1 in terms of ϵ and f. In our case, we fix λ_1 and find a domain $\mathcal{G}_2$ with λ as its eigenvalue.

Suppose the functions $R(r)$ and $\Theta(\theta)$ satisfy

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0 \quad & 0 \le \theta \le \tfrac{2\pi}{3} \\ \Theta(0) &= 0 \quad & \Theta(2\pi/3) = 0 \\ \sin^2 r \ddot{R} + \sin r \cos r \dot{R} + (\mu \sin^2 r - \lambda) R &= 0 \quad & 0 \le r < \pi \\ R(0) &= 0 \end{aligned}$$

Then $u(r,\theta) = R(r)\Theta(\theta)$ is the first eigenfunction of Δ_{S^2} on its nodal domain $\mathcal{G}_2$. By construction, $\Delta_{S^2} u + \mu u = 0$, so u is an eigenfunction. Also, u does not change sign on its nodal domain, so it must be the first eigenfunction. In polar coordinates, $u(r,0) = 0 = u(r, 2\pi/3)$. Let $m = \sqrt{\lambda}$ and set $R(r) = \sin^m r u(r)$, so that equation (4) becomes

$$\sin^2(r)\ddot{u} + (1+2m)\sin(r)\cos(r)\dot{u} + (\mu - m - \lambda)u = 0.$$

Next we take $\lambda = 9l^2/4$, corresponding to the l^{th} mode of the interval $[0, 2\pi/3]$, and write the solution in terms of the hypergeometric function:

$$u_l(r) = f(3l/2 + .5 \pm \sqrt{1/4 + \mu}, 1 + 3l/2, (1 - \cos r)/2).$$

Finally, we take the $\mu = 5.102$ superposition of the $l = 1, 3$ modes to define

$$\Upsilon := (\sin r)^{3/2} u_1(r) \sin(3\theta/2) - .0003 (\sin r)^{9/2} u_3(r) \sin(9\theta/4)$$

and let $\mathcal{G}_2$ be the nodal domain of Υ. By construction, $\lambda_1(\mathcal{G}_2) = 5.102 > \lambda_{cr}$.

It remains to show that $\mathcal{T}_2 \subset \mathcal{G}_2$. Define r_1 to be the first positive zero of u_1, and observe $r_1 < \delta(2)$. The function Υ is a small perturbation of $\sin^{3/2}(r)u_1(r)$, so its nodal domain $\mathcal{G}_2$ is a small perturbation of $\mathcal{D}_2(\mathcal{T}_1, r_1)$. We convert to a planar domain using stereographic projection, with the south pole cooresponding to $r = 0$ in our polar coordinates. Then in polar coordinates (ρ, θ) in the plane, $\rho = \tan(r/2)$. The radius of the circular outer edge of $\mathcal{T}_2$ satisfies

$$(\beta(\theta)\cos\theta - 1/\sqrt{8})^2 + (\beta(\theta) - \sqrt{3/2})^2 = \frac{3}{2},$$

which we can rewrite as

$$\beta(\theta) = \frac{\sqrt{2}\cos(\theta - \pi/3) + \sqrt{2\cos^2(\theta - \pi/3) + 4}}{2}.$$

We factor out the $\sin^{3/2}(r)\sin(3\theta/2)$ term, so that it remains to show

$$H(r, theta) := u_1(r) - .0003(\sin(r))^3 u_3(r)(4\cos^2(3\theta/2) - 1) > 0$$

for $0 < r < 2\arctan(\beta(\theta))$ and $0 < \theta < 2\pi/3$. One can see that $H > 0$ by plotting its graph. One can also check $H > 0$ using the following algorithm. First observe that one can write H_r in terms of u_1', u_3, u_3'. For the range of r we encounter, which is given by $\sqrt{2} \leq \beta \leq \sqrt{1/2} + \sqrt{3/2}$, the properties of hypergeometric functions imply that u_1' is negative, and bounded away from 0; this term is much larger than the u_3 and u_3' terms in H_r (because of the coefficient .0003), so H_r is negative. Thus, H is a decreasing function in r, so it sufffices to check $H > 0$ on the curve $\beta(\theta) = \rho = \tan(r/2)$.

Our strategy is the following. we first evaluate H at a point θ_0 on the curve, checking $H > 0$ at $(\rho = \beta(\theta_0), \theta_0)$, and bound the derivative H_θ on an interval containing θ_0. Our bound $|H_\theta| \leq M$ gives us a lower bound $H > H_0 - M|\theta - \theta_0|$. Thus $H > 0$ on a possibly smaller neighborhood of θ_0. We then repeat this process with each endpoint of this (smaller) interval. It suffices to check nine points on the curve. $\square$.

References

BG [BG] M. Bramson and D. Griffeath. *Capture problems for coupled random walks.* in Random Walks, Brownian Motion and Interacting Particle Systems, ed. R. Durrett and H. Kester, Birkhäuser , 1991.

DB [DB] R. D. DeBlassie. *Exit times from cones in $\mathbb{R}^n$ of Brownian motion.* Prob. Theory and Rel. Fields. 74:1–29, 1987.

L [L] W. Li. *The first exit time of a Brownian motion from an unbounded convex domain.* Ann. Probability. 31:1078–1096, 2003.

LS [LS] W. Li and Q.-M. Shao. *Capture time of Brownian pursuits.* Prob. Theory and Rel. Fields. 121:30–48, 2001.

PS [PS] G. Pólya & G. Szegö. *Isoperimetric Inequalities in Mathematical physics.* Princeton University Press, 1951.

R [R] Lord Rayleigh (J. W. Strutt) *The Theory of Sound (2nd. ed.)* Dover, 1945 (Originally published by Mcmillan Co., (1894–1896)).