Instability in a buoyant chemical front driven by curvature

Andrejs Treibergs

May 23, 2006

Abstract

We consider a model of a curvature dependent chemical reaction along a circular interface between two fluids in a Hele-Shaw cell in the presence of gravity. The problem reduces to a nonlocal evolution equation. Global existence is proved for the linearized equation in general and the nonlinear problem for small data. The linearized equation admits polynomial solutions in space with coefficients depending on time whose estimates are used to study fingering of solutions. For a general class of initial data including the polynomials, the solutions of the linearized equation coarsen and ultimately devolve to a single peak. Gevrey-like analytic regularity is shown for the linearized problem. For the nonlinear problem, long time existence is proved by viewing the equation as a perturbation of mean curvature flow and deriving apriori estimates independent of time. Thus the scaled rising expanding circle solution is shown to be strongly asymptotically stable provided that the combustion effects are strong compared to the fluid effects.

We study fingering and stability in a model of the motion of a burning region in fuel filling the narrow gap between two vertical glass plates. The heavier reactant and lighter product are assumed to have the same viscosity but have no interfacial surface tension. The motion is governed by fluid effects, pressure and gravity and combustion effects at the interface through the burning speed of a flame front which depends on its curvature. This reaction has been studied experimentally and numerically. In order to observe buouyancy effects in premixed gas flames, Abid and Romney [1] have experimented with an analogous aqueous autocatalytic reaction in which the density change is relatively small. When placed between two glass plates, the reaction produces a product region which develops fingers along its upper edge as it grows and rises. A numerical study modelling this reaction was carried out by Zhu [18] who assumed that as the shape dependent combustion proceeds producing a lighter product, influenced by fluid effects through Hele-Shaw flow and combustion effects through curvature dependence. Zhu also observed this Rayleigh-Taylor instability in his simulations and derived a dispersion selection mechanism for unstable wave number for linear fronts. We analyze the linear instability globally for circular fronts and derive a selection mechanism for the number of fingers.

Zhu observed computationally that in a parameter regime where the combustion effects dominate the reaction, although the front expands, when scaled to a fixed radius, any small perturbations of initial circle die out and the front tends polynomially to a circle. We prove that the shape is strongly asymptotically stable by showing that the initial value problem for small perturbations of the scaled rising circle solution can be solved in an appropriate space of functions which decay at infinity.

We describe more about our model of the evolution of a compact product region floating in a reactant. The interface is averaged over the width between two glass plates resulting in a one dimensional problem in space. We assume that that the combustion product has less density than the reactant, so it will tend to rise. We assume that the reactant and product are miscible, so that there is no interface surface tension.

But because the gap is small, the fluid senses viscous effects from the bounding plates so the fluid velocity is proportional to the pressure gradient (Darcy's law.) We assume that the combusion is curvature sensitve so that the normal velocity of a flame front is increased if the flame is focussed by negative curvature. The strength of the curvature term determines in large part the stability properties. For the physically reasonable curvature depenedence, the curvature conrtibution is bounded. Even though several modes may initially grow fingers, for a large class of initial data, including all polynomials, the fingers of the solutions will ultimately coalesce into a single maximum.

Acknowledgement. We thank J. Zhu for suggesting the problem and encouraging our study. Part of this work was completed while visiting The University of California, Irvine.

1. Formulation of the problem.

Following Zhu [18], we suppose that there is a moving smooth embedded closed curve in the plane $\Gamma(t) \subset \mathbf{R}^2$ which bounds a compact region of combustion product $\Omega_2(t)$ and an exterior domain of reactant $\Omega_1(t)$. Let ρ_i denote the density of the fluid in the region Ω_i. We are assuming that $\rho_1 > \rho_2$. After averaging over the gap of width ℓ between the plates and assuming no slip boundary conditions on the plate walls, the continuous pressure is related to the fluid velocity in $\mathbf{R}^2$ by Darcy's law and incompressibility

$$-\frac{\ell^2}{12\mu}\nabla(P_i + \rho_i g\langle x, \partial_2\rangle) = \mathbf{w}(x) \qquad \text{for } x \in \Omega_i \tag{1}$$

$$0 = \operatorname{div}\mathbf{w} \tag{2}$$

where μ is the fluid viscosity, ∂_2 represents the vertical unit vector in the plane, $\langle\cdot,\cdot\rangle$ is the usual Euclidean inner product and g is the acceleration due to gravity. Along the front, the normal velocity is assumed to be continuous, $[\langle \mathbf{w}, \mathbf{n}\rangle] = 0$ where $\mathbf{n}$ denotes the outward unit vector field along $\partial\Omega_2$. (Bracket $[P] = \lim_{t\to 0+}((P_2(x - t\mathbf{n}) - P_1(x + t\mathbf{n}))$ denotes the jump across the interface from Ω_1 to Ω_2.) P_i is determined in Ω_i using incompressibility (incomp) for $i = 1, 2$ depending on Γ by solving

$$\begin{array}{rcll} \Delta P_i & = & 0 & \text{inside } \Omega_i \text{ for } i = 1,2 \\ \lim_{x\to\infty} P_1(x) & = & -g\rho_1\langle x, \partial_2\rangle & \end{array} \tag{3}$$

$$\begin{array}{rcll} [P] & = & 0 & \\ \left[\dfrac{\partial P_i}{\partial \mathbf{n}}\right] & = & -g[\rho]\langle \mathbf{n}, \partial_2\rangle & \text{along } \Gamma, \end{array} \tag{4}$$

The jump conditions (4) follow from the continuity of velocity (1).

The fluid velocity and buoyancy contribution to normal velocity in the evolution of Γ is equal for either $i = 1, 2$ and is given by the nonlocal operator

$$\mathcal{N}(\Gamma) := V_h = \langle \mathbf{n}, \mathbf{w}\rangle = -\frac{\ell^2}{12\mu}\left(\frac{\partial P_i}{\partial \mathbf{n}} + g\rho_i\langle \mathbf{n}, \partial_2\rangle\right).$$

Now the fluid front is burning at a rate dependent on curvature. Let κ denote the curvature of the curve Γ. For a circle of radius R we have $\kappa = 1/R$. We assume that the contribution to normal velocity of the interface due to combustion is given by the local second order operator depending on Γ

$$V_c = \mathcal{L}(\Gamma) = \alpha(\kappa)$$

where $\alpha \in C^1(\mathbf{R})$ is strictly decreasing $\alpha'(\kappa) < 0$ and $S_L = \alpha(0)$ is the combustion speed of a linear front. Zhu chose $\alpha(\kappa) = S_L e^{-b\kappa}$ where S_L and b are postitive constants. For this choice, the combustion velocity is outward no matter what the curvature, it increases in velocity as curvature decreases. The linear flame front propogates at a speed S_L, which along with b, is an empirically determined constant. In our study, other forms of the curvature dependence may be more convenient, for example $\alpha = S - b\log\kappa$.

Suppose the curve $\Gamma(t)$ is parameterized $X(t,\theta) : \mathbf{R}^+ \times \mathbf{S}^1 \to \mathbf{R}^2$. We seek a curve whose motion is determined by its normal velocity

$$V = \left\langle \frac{\partial X}{\partial t}, \mathbf{n} \right\rangle = V_h + V_c = \mathcal{N}(\Gamma) + \mathcal{L}(\Gamma) \tag{5}$$

The tangential velocity $\frac{\partial X}{\partial \theta}$ describes only reparameterization of $\Gamma(t)$ and does not affect the set $X(t, \mathbf{S}^1)$.

2. The nonlinear equation.

Let us formulate the nonlinear problem as an integro-differential equation following Zhu [18]. Using the single layer potential, the solution of (3) and (4) is given by

$$P_i(x) = \frac{g[\rho]}{2\pi} \int_\Gamma \log|x-y| \langle \mathbf{n}(y), \partial_y \rangle ds(y) - g\rho_1 \langle x, \partial_2 \rangle.$$

The integral vanishes at infinity [3] since

$$\int_\Gamma \langle \mathbf{n}(y), \partial_y \rangle ds(y) = 0$$

gives the oriented length of the projection of Γ onto the y-axis.

Then the normal derivative for $x \in \Gamma$ and $i = 1, 2$ is given by

$$\frac{\partial P_i}{\partial \mathbf{n}}(x) = \left(\pm \frac{g[\rho]}{2} - g\rho_1 \right) \langle \mathbf{n}(x), \partial_2 \rangle + \frac{g[\rho]}{2\pi} \int_\Gamma \left\langle \mathbf{n}(x), \frac{x-y}{|x-y|^2} \right\rangle \langle \mathbf{n}(y), \partial_2 \rangle \, ds(y)$$

where the integral is in the principal value sense. Hence the fluid contribution to the normal velocity for $x \in \Gamma$ is given (using either $i = 1, 2$)

$$\begin{aligned} \mathcal{N}[\Gamma](x) &= -\frac{\ell^2}{12\mu} \left(\frac{\partial P_i}{\partial \mathbf{n}} + g\rho_i \langle \mathbf{n}, \partial_2 \rangle \right) \\ &= -\frac{\ell^2}{12\mu} \left(\frac{g[\rho] \langle \mathbf{n}(x), \partial_2 \rangle}{2} + \frac{g[\rho]}{2\pi} \int_\Gamma \left\langle \mathbf{n}(x), \frac{x-y}{|x-y|^2} \right\rangle \langle \mathbf{n}(y), \partial_2 \rangle \, ds(y) \right). \end{aligned}$$

3. Rising circle solution of the nonlinear equation.

Although the solution can be immediately written down, we do so slowly to illustrate the method of the appendix. We seek a solution for the problem where $\Gamma(t)$ takes the form

$$X(t,\theta) = (\mathcal{R}(t)\cos\theta, \mathcal{R}(t)\sin\theta + F(t)) \tag{6}$$

where $t \in \mathbf{R}^+$ is the time and $(r,\theta) \in \mathbf{R}^+ \times \mathbf{S}^1$ are polar coordinates of $\mathbf{R}^2$ centered at $(0, F(t))$. Hence the normal vector to Γ is $\mathbf{n} = (\cos\theta, \sin\theta)$ and the vertical coordinate on Γ is $\langle \mathbf{X}, \partial_2 \rangle = r\sin\theta + F$. The normal velocity of Γ is

$$V = \left\langle \mathbf{n}, \frac{\partial X}{\partial t} \right\rangle = \mathcal{R}'(t) + F'(t)\sin\theta \tag{7}$$

On the other hand, by substituting a pressure of the form

$$\begin{aligned} P_1 &= -g\rho_1 (r\sin\theta + F) + \frac{A\sin\theta}{r} \\ P_2 &= B + Cr\sin\theta \end{aligned}$$

into the jump conditions (4), we find

$$\begin{aligned} P_1 &= -g\rho_1 \left(r\sin\theta + F(t)\right) - \frac{g[\rho]\mathcal{R}(t)^2}{2r}\sin\theta, \\ P_2 &= -g\rho_1 \left(r\sin\theta + F(t)\right) - \frac{g[\rho]r}{2}\sin\theta. \end{aligned}$$

It follows that the fluid forces tend merely to translate the circular cell upward with constant velocity ($[\rho] < 0$)

$$V_h = -\frac{g[\rho]\ell^2}{24\mu}\sin\theta.$$

Similarly since $\kappa = 1/\mathcal{R}(t)$ it follows that

$$V_c = \alpha\left(\frac{1}{\mathcal{R}(t)}\right).$$

Because we equate (5) and (7), we deduce that

$$F(t) = F_0 - \frac{g[\rho]\ell^2}{24\mu}t$$

and $\mathcal{R}(t)$ is the solution of the ODE

$$\frac{d\mathcal{R}}{dt} = \alpha\left(\frac{1}{\mathcal{R}(t)}\right)$$

with $F_0 = F(0)$ and $\mathcal{R}(0) = \mathcal{R}_0$ some initial data. It is convenient to introduce

$$\alpha(\kappa) = \beta\left(\frac{1}{\kappa}\right) = \beta(R)$$

where R is the radius of curvature. For the circle, $\dot{\mathcal{R}} = \beta(\mathcal{R})$ and

$$\alpha'(\kappa) = -\beta'\left(\frac{1}{\kappa}\right)\frac{1}{\kappa^2} = -\beta'(\mathcal{R})\mathcal{R}^2.$$

It is possible to eliminate the translation part of the flow. Choosing the parameterization of $\Gamma(t)$ as

$$X(t,\theta) = (R(t,\theta)\cos\theta, R(t,\theta)\sin\theta + F_0 + c_1 t)$$

where $c_1 = -g[\rho]\ell^2/24\mu$ is the vertical velocity of the center of the disk, a parameter of the fluid, yields the equation for the unknown

$$RR_t = \alpha\left(\frac{-RR_{\theta\theta} + 2{R_\theta}^2 + R^2}{\left(R^2 + {R_\theta}^2\right)^{3/2}}\right)\sqrt{R(\theta,t)^2 + R_\theta(\theta,t)^2} + \\ + \frac{c_1}{\pi}\int_{-\pi}^{\pi} \frac{\left\{\begin{array}{c} R(\theta,t)^2 - R(\theta,t)R(\eta,t)\cos(\theta-\eta) \\ -R_\theta(\theta,t)R(\eta,t)\sin(\theta-\eta)\end{array}\right\}\left[\begin{array}{c} R(\eta,t)\sin\eta \\ -R_\theta(\eta,t)\cos\eta\end{array}\right]}{R(\theta,t)^2 - 2R(\theta,t)R(\eta,t)\cos(\theta-\eta) + R(\eta,t)^2}\,d\eta.$$

It is convenient to use the notation $\beta(r) = \alpha(1/r)$ and switch to a new time variable

$$\tau = \int_0^t \frac{dt}{\mathcal{R}(t)}.$$

Then the equation becomes

$$\frac{RR_\tau}{\mathcal{R}(\tau)} = \beta\left(\frac{1}{\kappa}\right) + \frac{c_1}{\pi}\int_{\Gamma_\tau} \frac{n(\theta)\cdot(X(\theta)-X(\eta))}{|X(\theta)-X(\eta)|^2} n_2(\eta)\,ds(\eta)$$

where $n_2(\eta) = n(\eta)\cdot(0,1)$. Note that any function of θ may be added to the quotient without changing the integral because $n_2(\eta)\,ds(\eta) = -dx$ is exact. Also note that the quotient is identically constant if and only if $X(\theta)$ is a circle of constant radius R_1,

$$\frac{n(\theta)\cdot(X(\theta)-X(\eta))}{|X(\theta)-X(\eta)|^2} \equiv \frac{1}{2R_1}.$$

Thus

$$\frac{RR_\tau}{\mathcal{R}(\tau)} = \beta\left(\frac{\left(R^2+R_\theta{}^2\right)^{3/2}}{-RR_{\theta\theta}+2R_\theta{}^2+R^2}\right)\sqrt{R(\theta,\tau)^2+R_\theta(\theta,\tau)^2} -$$

$$-\frac{c_1}{\pi}\int_{-\pi}^{\pi} \frac{\left\{\begin{array}{c} R(\theta,\tau)^2 - R(\theta,\tau)R(\eta,\tau)\cos(\eta-\theta) \\ +R_\theta(\theta,\tau)R(\eta,\tau)\sin(\eta-\theta)\end{array}\right\}\frac{d}{d\eta}\left[R(\eta,\tau)\cos(\eta)\right]\,d\eta}{R(\theta,\tau)^2-2R(\theta,\tau)R(\eta,\tau)\cos(\theta-\eta)+R(\eta,\tau)^2}.$$

Now since we're interested in the relative stability, it is convenient to consider $R(\theta,\tau) = \mathcal{R}(\tau)e^{u(\theta,\tau)}$. Using $\mathcal{R}_t = \beta$ we find

$$e^u u_\tau = \beta\left(\frac{\mathcal{R}e^u\left(1+u_\theta{}^2\right)^{3/2}}{-u_{\theta\theta}+u_\theta{}^2+1}\right)\sqrt{1+u_\theta(\theta,\tau)^2} - \beta(\mathcal{R})e^u -$$

$$-\frac{c_1}{\pi}\int_{-\pi}^{\pi} \frac{\left\{\begin{array}{c} e^{u(\theta,\tau)-u(\eta,\tau)} - \cos(\eta-\theta) \\ +u_\theta(\theta,\tau)\sin(\eta-\theta)\end{array}\right\}}{e^{u(\theta,\tau)-u(\eta,\tau)}-2\cos(\theta-\eta)+e^{u(\eta,\tau)-u(\theta,\tau)}}\frac{d}{d\eta}\left[e^{u(\eta,\tau)}\cos(\eta)\right]\,d\eta.$$

Subtracting 1/2 from the quotient yields

$$\begin{aligned}\frac{\partial}{\partial\tau}u(\theta,\tau) \quad &= \quad \beta\left(\frac{\mathcal{R}e^u\left(1+u_\theta{}^2\right)^{3/2}}{-u_{\theta\theta}+u_\theta{}^2+1}\right)\frac{\sqrt{1+u_\theta(\theta,\tau)^2}}{e^{u(\theta,\tau)}} - \beta(\mathcal{R}) + \\ &\quad + \frac{c_1}{2\pi}\int_{-\pi}^{\pi}\mathcal{K}(\eta,\theta,\tau)\frac{d}{d\eta}\left[e^{u(\eta,\tau)-u(\theta,\tau)}\cos(\eta)\right]\,d\eta. \end{aligned} \tag{8}$$

where

$$\mathcal{K}(\eta,\theta,\tau) = \frac{\sinh\left[u(\eta,\tau)-u(\theta,\tau)\right] - u_\theta(\theta,\tau)\sin(\eta-\theta)}{\cosh\left(u(\eta,\tau)-u(\theta,\tau)\right)-\cos(\eta-\theta)}. \tag{9}$$

Linearizing about the rising circle solution given by $\hat{R} = \mathcal{R}(t)+\epsilon u$ is done in Appendix A. The linearized equation (100) reduces to

$$\frac{\partial u}{\partial t} = \beta'(\mathcal{R}(t))\left(\frac{\partial^2 u}{\partial\theta^2}+u\right) + \frac{c_1}{\mathcal{R}(t)}\mathcal{M}u. \tag{10}$$

The operator

$$\begin{aligned}\mathcal{M}u &= \frac{\partial}{\partial\theta}\left(\tilde{u}\sin\theta + (u-u_0)\cos\theta\right)\\ &= \sum_{k=1}^{\infty} kv_{k+1}\cos k\theta - ku_{k+1}\sin k\theta.\end{aligned}$$

where $\tilde{u} = \mathcal{H}u$ denotes the Hilbert transform.

Consider solving equation (100) spectrally, thus regard $u_k(t)$ and $v_k(t)$ as functions of t. Averaging over $\mathbf{S}^1$ equation (100) becomes

$$\dot{u}_0 = -\frac{\alpha'(1/\mathcal{R})}{\mathcal{R}^2}u_0 = \beta'(\mathcal{R})u_0$$

so that

$$u_0 = c\beta(\mathcal{R}(t)).$$

In the case $\alpha = S_L e^{-b\kappa}$, integration yields $u_0(t) = u_0(0)\exp(b/\mathcal{R}_0 - b/\mathcal{R}(t))$. Observe that $\sin\theta$ and $\cos\theta$ are in the null space of $\mathcal{V}$.

4. Solvability of the linearized equation.

If u has Hölder continuous derivatives ($u \in C^{k+\alpha}$) then so does $\tilde{u}$, ($\tilde{u} \in C^{k+\alpha}$.) The argument in [19] yields an estimate for the norm $|\cdot|_{k+\alpha} = \|\cdot\|_{C^{k+\alpha}}$. Similarly using the formula, for H^k, the functions of $L^2(\mathbf{S^1})$ whose k-th and lower derivatives are in L^2 there are estimates in the norm $\|\cdot\|_k = \|\cdot\|_{H^k}$. Thus there are finite constants such that

$$|\tilde{u}|_{k+\alpha} \le c_2(k,\alpha)|u|_{k+\alpha} \quad \text{for all} \quad u \in C^{k+\alpha}, \tag{11}$$

$$\|\tilde{u}\|_k \le \|u-u_0\|_k \quad \text{for all} \quad u \in H^k. \tag{12}$$

Using the inequalities (11), (12), we find that

$$|\mathcal{M}u|_{k-1+\alpha} \le c_2(k,\alpha)|u-u_0|_{k+\alpha} \quad \text{for all} \quad u \in C^{k+\alpha}, \tag{13}$$

$$\|\mathcal{M}u\|_{k-1} \le \|u-u_0-u_1\sin\theta - v_1\cos\theta\|_k \quad \text{for all} \quad u \in H^k. \tag{14}$$

We begin by computing the adjoint equation. Suppose we expand

$$z(\theta) = a_0 + \sum_1^\infty a_k\cos k\theta + b_k\sin k\theta.$$

Then in the L^2 the inner product

$$\int_{-\pi}^{\pi}(\mathcal{M}u)z = \pi\sum_{k=1}^{\infty} ka_kv_{k+1} - kb_ku_{k+1}$$

so that

$$\left|\int_{-\pi}^{\pi}(\mathcal{M}u)z\right| \le \|u_\theta\|\|z\|.$$

By reinterpreting the sum,

$$\begin{aligned}\int_{-\pi}^{\pi}(\mathcal{M}u)z &= \int_{-\pi}^{\pi}u\mathcal{M}^*z\\ \left|\int_{-\pi}^{\pi}u\mathcal{M}^*z\right| &\le \|u-u_0-u_1\cos\theta - v_1\sin\theta\|\,\|z_\theta\|\end{aligned}$$

where the adjoint operator is given by

$$\mathcal{M}^* z = \sum_2^\infty (k-1)a_{k-1}\sin k\theta - (k-1)b_{k-1}\cos k\theta \tag{15}$$
$$= \tilde{z}_\theta \sin\theta - z_\theta\cos\theta. \tag{16}$$

Or this can be seen directly as follows. Let Z be the holomorphic function on the disk which we assume has boundary values $z + \tilde{z}i$ on the unit circle. By the Cauchy Riemann equations in polar coordinates one has $Z' = e^{-i\theta}(\tilde{z}_\theta - iz_\theta)$. By Cauchy's formula,

$$\begin{aligned} 0 &= \int_{\mathbf{S}^1} UZ'(z)\,dz \\ &= \int_{-\pi}^{\pi} (u+\tilde{u}i)(\tilde{z}_\theta - z_\theta i)\,d\theta \\ &= \int_0^{2\pi} (u\tilde{z}_\theta + \tilde{u}z_\theta) + i\left(\tilde{u}\tilde{z}_\theta - uz_\theta\right)\,d\theta \end{aligned}$$

from which it follows that

$$\int u\mathcal{M}^* z = \int \frac{d}{d\theta}\Big((u\sin\theta)^\sim + u\cos\theta\Big)z.$$

Now we can see that $\tilde{u}\sin\theta - u_0\cos\theta - u_1/2$ is the conjugate of $u\sin\theta$ by checking that

$$F(\theta) = u\sin\theta + i\left(\tilde{u}\sin\theta - u_0\cos\theta + \frac{u_1}{2}\right)$$

is the restriction of a holomorphic function. For example, one checks that

$$\int e^{-in\theta}F(e^{i\theta})\,d\theta = 0$$

for all integers $n > 0$.

An energy estimate can be deduced as follows. We consider the inhomogeneous equation 10

$$\frac{\partial u}{\partial t} = \beta'(\mathcal{R}(t))\left(\frac{\partial^2 u}{\partial\theta^2} + u + \frac{\beta(\mathcal{R}(t))f(\theta,t)}{S_0}\right) + \frac{c_1}{\mathcal{R}(t)}\mathcal{M}u. \tag{17}$$

Changing time in this equation to $T = \log(\beta(\mathcal{R}(t))/S_0)$ where $S_0 = \beta(R_0)$ yields

$$u_T = u_{\theta\theta} + u + g(T)\mathcal{M}u + f(\theta,T)e^T$$

where $g(T) = c_1/(\beta'(R(t))R(t))$. As usual, we assume $\beta \in C^1([R_0,\infty))$ such that $\beta' > 0$. This means that if β is bounded and $\beta \nearrow S_1$ as $R \to \infty$, then $0 \le T < T^* := \log(S_1/S_0)$. Also,since β is bounded we have $g \to \infty$ as $T \to T^*$. Otherwise, for some $\delta > 0$, $\beta'(R) \ge \delta/R$ which yields a contradiction after integrating from R_0 to R.

The average a_0 of u satisfies

$$a_0 = \left(a_0(0) + \int_0^T f_0(x)\,dx\right)e^T$$

where

$$f_0 = \fint_{\mathbf{S}^1} f(\theta,T)\,d\theta$$

is the average of f. Write $f = f_0 + f_1$. Thus changing dependent variables to

$$u(\theta, T) = a_0(T) + e^T v(\theta, T)$$

gives the equation

$$v_T = v_{\theta\theta} + g(T)\mathcal{M}v + f_1(\theta, T) \tag{18}$$

where $\int v\, d\theta = \int f_1\, d\theta = 0$. Now consider the v-energy

$$E_v(T) := \frac{1}{2}\int_\gamma v_\theta^2 + v^2\, d\theta$$

for a solution $v \in C^1([0, T_1), H^2(\gamma))$ of (18). We compute

$$\begin{aligned}
\frac{dE_v}{dT} &= \int v_\theta v_{\theta T} + v v_T\, d\theta \\
&= \int (v_{\theta\theta} + g\mathcal{M}v + f_1)(-v_{\theta\theta} + v)\, d\theta \\
&\leq -\|v_{\theta\theta}\|^2 - \|v_\theta\|^2 + g\|v_{\theta\theta}\|\|v_\theta\| + g\|v_\theta\|\|v\| + \|f_1\|\|v_{\theta\theta}\| + \|f_1\|\|v\| \\
&\leq \left(\frac{g^2}{2} + \frac{1}{2}\right)(\|v_\theta\|^2 + \|v\|^2) + \|f_1\|^2 \\
&= (g^2 + 1)E_v + \|f_1\|^2.
\end{aligned}$$

By an approximation we prove an energy estimate.

Lemma 1. *Assume $\beta \in C^1([R_0, \infty))$ such that $0 < S_0 = \beta(R_0)$ and $\beta'(R) > 0$. Assume $f \in C_{\mathrm{loc}}([0, \infty), L^2(\mathbf{S}^1))$. Choose $t_2 \in [0, \infty)$. Let u be a weak solution of (10) in $L^2((0, t_2), H^1)$. Then the u-energy given by*

$$E(t) := \frac{1}{2}\int u_\theta(\theta, t)^2 + (u(\theta, t) - a_0(t))^2\, d\theta$$

satisfies for some $c > 0$ depending only on β,

$$E(t) \leq c e^{\zeta(t)}\left(E(0) + \int_0^t \|f_1\|^2(s) e^{-\zeta(s)}\, ds\right) \tag{19}$$

where

$$\zeta(t) = c_1^2 \int_{R_0}^{R(t)} \frac{dR}{\beta(R)\beta'(R)R^2}.$$

Hence a solution to the initial value problem is unique.

We state the existence theorems for the equation (100.) It is simpler to treat (18). The proofs will be slight modifications of the standard ones. Viewing (18) as an ODE in Banach space , we set $F(T, u) := g(T)\mathcal{M}u(\cdot, T) + f_1(\theta, T)$ and consider

$$v_T = Av + F(T, v) \tag{20}$$

$$v(0) = \phi \tag{21}$$

where $A = \partial_{\theta\theta}$. This is a perturbation of the heat equation on the circle. We assume that $\beta \in C^{k+1+\alpha}$ for $k \geq 0$ and

$$\beta(0) = S_0 > 0, \qquad \beta' > 0. \tag{22}$$

$T = \log(\beta(R)/S_0) \in C^{k+1+\alpha}_{loc}([0,\infty))$ implies $g(T) = c_1/(\beta'(R(t))R(t)) \in C^{k+\alpha}_{loc}([0,T^*))$. T^* is the time where $g(T) \to \infty$ as $T \nearrow T^*$. If β is unbounded then $T^* = \infty$. Otherwise $T^* = \log(S_1/S_0) < \infty$ where $\beta \to S_1$ as $R \to \infty$. We choose $0 \le T_2 < T^*$. Then $|g|_{k+\alpha,[0,T_2]}$ is bounded.

Assuming $\beta \in C^{1+\alpha}_{loc}([0,T^*))$ and $f \in C^{\alpha}_{\rm loc}([0,T^*),H^j)$ and using (14) there is a constant dependng on K so that

$$\begin{aligned} \|F(t,u)-F(s,v)\|_j &\le C_K\left(|s-t|^\alpha + \|u-v\|_{j+1}\right) \\ \text{for all} \quad & 0\le s,t,\le T_2 \quad \text{and} \quad \|u\|_{j+1}+\|v\|_{j+1}\le K. \end{aligned} \tag{23}$$

The following existence theorem is a consequence. The fractional space $(H^k)_\alpha$ is the Hilbert space completion of C^∞ in the norm $\|\cdot\|_k + \|(\partial_{\theta\theta})^\alpha\cdot\|_k$.

Theorem 2. *Suppose $\beta \in C^{1+\alpha}([0,T^*))$ satisfies (22), $\phi \in H^1$, $j \ge 0$ is any integer and $f \in C^\alpha_{\rm loc}([0,T^*),H^j)$. For every $T_2 \in [0,T^*)$ there is a unique solution $u \in C([0,T_2],H^1)\cap C^1((0,T_2],H^j)$ to (20), (21) . If in addition, $\phi \in (H^k)_\alpha$ which is the fractional space continuously embedded in H^{k+1} for $1/2<\alpha<1$, then the solution satisfies $u \in C([0,T_2],(H^k)_\alpha)\cap C^1((0,T_2],H^j)$.*

Proof. The local existence for $k=j=0$ is ([13], p. 316) where we consider the heat equation on the circle instead of the interval and since (23). The global existence follows from the energy inequality (19). The higher regularity for $k=j$ follows from ([13], p. 318) because (23) implies the inequality

$$\|F(t,u)-F(s,v)\|_k \le C_K\left(|s-t|^\alpha + \|u-v\|_{(H^k)_\alpha}\right)$$

for all $\|u\|_{(H^k)_\alpha}, \|v\|_{(H^k)_\alpha} \le K$. By restarting the evolution from $0<t_1<t$ we see that the regularity of the initial condition is improved. Thus, by a bootstrapping argument, for $\phi \in H^k$ we see that the solution is $u \in C^1((0,T_1),H^j)$ for any j. □

See also ([4], p. 316.) Because of the Sobolev embedding in one dimension, $H^k \subset C^{k-1/2}$ and the solution is classical for $k \ge 3$.

Let us recall the definitions of the parabolic Hölder spaces $C^{\ell,\ell/2}(\bar Q)$ where $Q = I\times \mathbf{S^1}$, I is an interval in $\mathbf{R}$ and for nonintegral $\ell>0$ following [12]. It consists of functions $w \in C(\bar Q)$ with continuous derivatives of the form $\partial_t^r\partial_\theta^s w$ for $2r+s<\ell$ in $\bar Q$ with finite norm $|w|_{\ell,\ell/2}$. Let $\alpha\in(0,1)$. Let $\langle w\rangle^{(\alpha)}_{x,Q} := \sup\{|u(x,t)-u(y,t)|\cdot |x-y|^{-\alpha} : (x,t),(y,t)\in\bar Q, x\ne y\}$ and $\langle w\rangle^{(\alpha)}_{t,Q} := \sup\{|u(x,t)-u(x,s)|\cdot|t-s|^{-\alpha} : (x,s),(x,s)\in \bar Q, s\ne t\}$. Then the norm

$$\begin{aligned} |w|_{\ell,\ell/2} \;:=\; & \sum_{j=0}^{\lfloor\ell\rfloor}\sum_{2r+s=j} |\partial_t^r\partial_\theta^s w|_{C^0(\bar Q)} + \\ & + \sum_{2r+s=\lfloor\ell\rfloor}\langle\partial_t^r\partial_\theta^s w\rangle^{(\ell-\lfloor\ell\rfloor)}_{x,Q} + \sum_{0<\ell-2r-s<2}\langle\partial_t^r\partial_\theta^s w\rangle^{\left(\frac{\ell-2r-s}{2}\right)}_{t,Q}. \end{aligned}$$

In particular, for $0<\alpha<1$,

$$\begin{aligned} |w|_{1+\alpha,(1+\alpha)/2} &= |w|_0 + |w_\theta|_0 + \langle w_\theta\rangle^{(\alpha)}_{x,Q} + \langle w\rangle^{((1+\alpha)/2)}_{t,Q} + \langle w_\theta\rangle^{(\alpha/2)}_{t,Q}, \\ |w|_{\alpha,\alpha/2} &= |w|_0 + \langle w\rangle^{(\alpha)}_{x,Q} + \langle w\rangle^{(\alpha/2)}_{t,Q}. \end{aligned}$$

We can deduce further regularity of solutions from the following crude estimate.

Lemma 3. *Let $1<\nu<\ell<2$. Then there is a constant $c_3<\infty$ depending on ℓ and ν so that for any $u\in C^{\ell,\ell/2}(Q)$ then $\mathcal{M}u \in C^{\ell-\nu,(\ell-\nu)/2}(Q)$ and there holds*

$$|\mathcal{M}u|_{\ell-\nu,(\ell-\nu)/2,(Q)} \le c_3\,|u|_{\ell,\ell/2,(Q)}. \tag{24}$$

Proof. It suffices to find time estimates for $\tilde{u}$ and $\tilde{u}_\theta$. In case $1 < \ell < 2$ we consider the Hilbert transform and write constants independent of u. Let $\eta = \ell - 1$.

$$\begin{aligned}
&|\tilde{u}(x,t+h) - \tilde{u}(x,t)| \\
&= \left| \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{\varepsilon}^{2\pi-\varepsilon} [u(x-y,t+h) - u(x-y,t)] \cot\left(\frac{y}{2}\right) dy \right| \\
&= \left| \frac{1}{2\pi} \int_0^{2\pi} [u(x-y,t+h) - u(x,t+h) - u(x-y,t) + u(x,t)] \cot\left(\frac{y}{2}\right) dy \right| \\
&\le c\langle u\rangle_x h^\eta + \frac{1}{2\pi} \int_h^{2\pi-h} \begin{pmatrix} |u(x-y,t+h) - u(x-y,t)| \\ + |u(x,t+h) - u(x,t)| \end{pmatrix} \cot\left(\frac{y}{2}\right) dy \\
&\le c\langle u\rangle_x h^\eta + c\langle u\rangle_t h^{\eta/2} \int_h^{2\pi-h} \cot\left(\frac{y}{2}\right) dy \\
&= c\langle u\rangle_x h^\eta + c\langle u\rangle_t h^{\eta/2} \log\left(\frac{1}{h}\right) \\
&\le c\langle u\rangle_x h^\eta + c\langle u\rangle_t h^{(\nu-1)/2}.
\end{aligned}$$

For $u \in C^{\ell,\ell/2}$, the derivative is estimated similarly since $\tilde{u}$ satisfies ([5], p.313)

$$\tilde{u}_\theta(\theta,T) = \frac{1}{2\pi} PV\!\int_{\mathbf{S}^1} u_\theta(\theta-\sigma,T) \cot\left(\frac{\sigma}{2}\right) d\sigma.$$

The lemma follows from inequality (11). □

Theorem 4. *Suppose $\beta \in C^{1+\alpha}([R_0,\infty))$ satisfies (22), $f \in C^{\alpha,\alpha/2}_{\text{loc}}([0,T^*) \times \mathbf{S}^1)$ and $\phi \in C^{2+\alpha}$. Then the initial value problem (20), (21) has a unique solution $u \in C^{2+\alpha,1+\alpha/2}([0,T_2] \times \mathbf{S^1})$ for all $T_2 \in [0,T^*)$ and any $\alpha \in (0,1)$. The solution can be bounded by an expression depending on the data*

$$|u|_{2+\alpha,1+\alpha/2} \le c_4(\alpha,\beta,|f|_{\alpha,\alpha/2},|\phi|_{2+\alpha},T_2).$$

where the Hölder norms are taken in $\bar{Q} = [0,T_2] \times \mathbf{S^1}$.

Proof. We use the continuity method (Leray Schauder fixed point theorem) to prove that the initial value problem (20), (21) can be solved in $X = C^{1+\alpha,(1+\alpha)/2}(\bar{Q})$. We consider the mapping $\mathcal{T} : X \to X$ given by taking $u \in X$ to the solution v of the initial value problem

$$\begin{aligned}
v_T &= v_{\theta\theta} + g(T)\mathcal{M}v(x,T) + f_1(\theta,T) \\
v(x,0) &= \phi(x).
\end{aligned}$$

By the conditions on β and by the lemma for any $0 < \gamma < \alpha$, $u \in X$ implies that $g(T)\mathcal{M}u(x,T) \in C^{\gamma,\gamma/2}$. The embeddings $C^{\ell',\ell'/2} \subset C^{\ell,\ell/2}$ are continuous, for nonintegers $0 < \ell < \ell'$ and there are constants $c_5(\ell,\ell') < \infty$ so that

$$|u|_{\ell,\ell/2} \le c_5(\ell,\ell')|u|_{\ell',\ell'/2}. \tag{25}$$

Thus, by the existence theorem for the heat equation ([12], p. 320), there is a unique solution $v \in C^{\gamma+2,1+\gamma/2}(\bar{Q})$ which satisfies the inequality

$$|v|_{2+\gamma,1+\gamma/2} \le c_6\left(|g(T)\mathcal{M}u(x,T)|_{\gamma,\gamma/2} + |f|_{\gamma,\gamma/2} + |\phi|_{2+\gamma}\right) \tag{26}$$

for some constant independent of u and ϕ. Hence the operator $\mathcal{T} : X \to X$ is compact. The conditions for the Leray-Schauder fixed point theorem are satisfied if we can show that there is a constant independent of $u \in X$ and $s \in [0,1]$ such that $|u|_{1+\alpha,(1+\alpha)/2} \le c$ whenever u is a solution of $u = s\mathcal{T}u$ ([13], p. 260.) We make a little bootstrap argument. In particular for some $1+\alpha < \ell' < 2$ we have the fixed point $u \in C^{2+\gamma,1+\gamma/2} \subset C^{\ell',\ell'/2}$ by (26) and so $\mathcal{M}u \in C^{\alpha,\alpha/2}$ by the lemma and

$$|u|_{2+\alpha,1+\alpha/2} \le c_7 \left(|g(T)\mathcal{M}u(x,T)|_{\alpha,\alpha/2} + |f|_{\alpha,\alpha/2} + |\phi|_{2+\alpha}\right). \tag{27}$$

The bound on the fixed point follows from the interpolation inequality ([9], p124.) There is a constant $c_8 < \infty$ so for every $\varepsilon \in (0,1)$ and every $u \in C^{2+\alpha,1+\alpha/2}(\bar{Q})$,

$$|u|_{1+\alpha,(1+\alpha)/2} \le \varepsilon |u|_{2+\alpha,1+\alpha/2} + c_8 \varepsilon^{-\frac{2}{\alpha}} |u|_0. \tag{28}$$

Combining inequalities (24), (26), (25), (27), (28) we get for fixed points u,

$$|u|_{1+\alpha,\frac{1+\alpha}{2}} \le \varepsilon c_7 \left(s c_3^2 c_5 c_6 |g|_{\frac{\alpha}{2}} |u|_{1+\alpha,\frac{1+\alpha}{2}} + |f|_{\alpha,\frac{\alpha}{2}} + |\phi|_{2+\alpha} \right) + c_8 \varepsilon^{-\frac{2}{\alpha}} |u|_0.$$

Thus by choosing ε so $\varepsilon c_7 (c_3^2 c_5 c_6 |g|_{\alpha/2} + 1) \le 1/2$ we get that there is a constant $c_9 < \infty$ depending on α so that

$$|u|_{1+\alpha,\frac{1+\alpha}{2}} \le c_9 \left(|f|_{\alpha,\alpha/2} + |\phi|_{2+\alpha} + |g|_{\alpha/2}^{2/\alpha} |u|_0 \right).$$

The desired apriori estimate is (27) using inequalities (24), (26), (25), (28) and the energy inequality for fixed points since $\int u(y,T)\,dy = 0$ and

$$|u|_0^2 \le \max\{\pi E_v(T) : T \in [0,T_2]\} \le c(T_2, f, g, \|\phi\|_1^2).$$

□

Local existence also holds for analytic initial data. (see theorems 6. and 12.)

5. Polynomial solutions of the linearized equation: separable case.

Remarkably, like the standard heat equation $v_T = v_{\theta\theta}$, equation (10) possesses solutions which are polynomials of fixed degree in θ whose coefficients depend on time. We derive these solutions in this section. They will allow a detailed study of the asymptotic behavior.

We first analyze the case when $g(T) = c_1/b > 0$ is constant. Then

$$\beta(R) = S_0 + b \log\left(\frac{R(t)}{R_0}\right)$$

where $b > 0$ and $S_0 > 0$. Near the circle $r = R_0$ this is as good a first order approximation of curvature dependence as any. However, it does tend to blow up as the curve flattens out $\kappa \searrow 0$ and we may expect strong curvature effects away from the circle.

Then we may rewrite (100) as

$$R\frac{\partial u}{\partial t} = b\left(\frac{\partial^2 u}{\partial \theta^2} + u\right) + c_1 \mathcal{M}u.$$

Separating variables $u(t,\theta) = x(t)y(\theta)$ gives two equations

$$\begin{aligned} \frac{dx}{dt} + \frac{b\lambda}{R(t)}x &= 0 \\ \frac{d^2 y}{d\theta^2} + \frac{c_1}{b}\mathcal{M}y + (1+\lambda)y &= 0 \end{aligned} \tag{29}$$

where λ is a real separation constant. The first equation may be solved by

$$x(t) = x_0 \exp\left(-b\lambda \int_0^t \frac{dt}{R(t)}\right) = x_0 \left(1 + \frac{b}{S_0} \log \frac{R(t)}{R_0}\right)^{-\lambda} \tag{30}$$

Depending on the choice of stability criterion, we can now identify which modes are transient. For example, if transience means $x(t)$ is decreasing, then the transient modes have $\lambda > 0$, and thus solutions are unstable whenever (29) has eigensolutions for $\lambda \le 0$. Let us call this type *arithmetic transience.* If however, we are interested in the relative size change, then the stable modes must be decreasing relative to the growth of the cell. For example then $d \log x/dt < d \log R/dt$ for all t would be a *relative transience* criterion. Typically for our problems, the solutions of the linearized equations grow, but at a slower rate than the radius. If this is to hold initially only we may call the $d \log x/dt(t_0) \ge d \log R/dt(t_0)$ the *instantaneous relative fingering* criterion. With this particular curvature dependence, this becomes $b\lambda > -S_0 - b \log R_0$ which is weaker in this case.

The second equation is a generalized Riemann-Hilbert-Poincaré boundary value problem, of the type studied by Vekua. He devised the first solution method depending on singular representation of a holomorphic function and stated the alternative. Sherman, and Gakhov [5] reformulated the problem as a Fredholm integral equation. This theory tells us that there is a disrete spectrum for λ which tends to infinity and that each eigenspace is finite dimensional ([5]). In this particular instance, however, we find the solution analytically in closed form.

We begin by reformulating (29) following [5]. An analytic function $\Phi = Y + i\tilde{Y}$ is sought in the unit disk whose boundary values are $y + i\tilde{y}$. If we parameterize the circle $f(\theta) = e^{i\theta}$, then

$$\begin{aligned} \Phi_\theta &= \Phi' f_\theta &&= ie^{i\theta}\Phi' \\ \Phi_{\theta\theta} &= \Phi''(f_\theta)^2 + \Phi' f_{\theta\theta} &&= -e^{2i\theta}\Phi'' - e^{i\theta}\Phi' \end{aligned}$$

Thus (29) becomes the generalized Riemann-Hilbert-Poincaré boundary values for a holomorphic function Φ on the disk with boundary $\gamma = \mathbf{S}^1$. Since the average of the conjugate is fixed at zero, we render it as $\tilde{y} - \tilde{y}_0$ where $\tilde{y}_0$ is the average over the circle.

$$0 = \Re e \left(-z^2\Phi'' + \left(\frac{c_1 i}{b} - z\right)\Phi' + (1+\lambda)\Phi - \frac{c_1 i}{bz}(\Phi - \Phi(0))\right). \tag{31}$$

But by the Cauchy formula, we have

$$\Phi(0) = \frac{1}{2\pi i}\int_\gamma \Phi(\zeta)\,\frac{d\zeta}{\zeta}.$$

thus this problem is exactly of the type discussed by Vekua [5], [14].

We observe that although (31) holds on the boundary of the disk, the function inside the parenthesis is holomorphic on the whole disk, so by the maximum principle, we seek an analytic function satisfying on the unit disk

$$-z^2\Phi'' - z\Phi' + (1+\lambda)\Phi + \frac{c_1 i}{b}\left(\Phi' - \frac{\Phi - \Phi(0)}{z}\right) = 0 \tag{32}$$

Although this equation does not have a regular singular point, we can still use the series solution method. Thus we assume that

$$\Phi(z) = \sum_{k=0}^{\infty} a_k\, z^k. \tag{33}$$

Making the substitution into the equation and equating powers gives

$$\begin{aligned} (1+\lambda)a_0 &= 0; \\ b\lambda a_1 + c_1 i a_2 &= 0; \\ b(1+\lambda-k^2)a_k + c_1 i k a_{k+1} &= 0, \qquad \text{for } k \ge 2. \end{aligned}$$

Thus $\lim |a_{k+1}/a_k| = \infty$ and Φ converges only at $z = 0$ unless $1 + \lambda = k^2$ for some integer k. Thus we have found eigenvalues $\lambda_k = k^2 - 1$ for k a nonnegative integer. In that case the solutions are polynomials given by the expressions

$$\begin{aligned} \Psi_0 &= 1 \\ \Psi_k &= \sum_{j=1}^{k} \frac{(k+j-1)!}{(k-j)!(j-1)!} \left(\frac{ibz}{c_1}\right)^j \qquad \text{for } k \geq 1. \end{aligned} \tag{34}$$

We conlude by remarking that using the criterion $\lambda \leq 0$ there are only two nontransient modes corresponding to $\lambda = -1, 0$ for which the linearized solutions are exactly $y = 1$, $y = \cos\theta$ and $y = \sin\theta$, corresponding to dilation and translations.

6. Polynomial solutions of the linearized equation: independent of curvature case.

Assume that u extends to a one parameter family of holomorphic functions on the disk $\Phi = U + i\tilde{U}$ such that $U = u$ on the circle. Then the equation (10) on γ may be expressed

$$0 = \Re e \left(-\Phi_t - \beta'(z^2\Phi'' + z\Phi' - \Phi) + \frac{c_1 i}{\mathcal{R}} \left(\Phi' - \frac{\Phi - \Phi(0)}{z} \right) \right),$$

where Φ' and Φ'' on the bounary are the limits of $(\Phi')^+$ and $(\Phi'')^+$ from the interior, which are assumed to exist and be Hölder continuous. Thus the argument is a holomorphic function on the entire disk. Hence

$$\Phi_t = -\beta'(z^2\Phi'' + z\Phi' - \Phi) + \frac{c_1 i}{\mathcal{R}} \left(\Phi' - \frac{\Phi - \Phi(0)}{z} \right). \tag{35}$$

Assuming that we write the solution as a power series, we have a representation

$$\Phi(z,t) = \sum_{k=0}^{\infty} a_k(t) z^k. \tag{36}$$

Inserting into the differential equation and equating terms gives

$$\dot{a}_k = -\beta'(\mathcal{R})(k^2 - 1)a_k + \frac{ic_1}{\mathcal{R}} k a_{k+1} \qquad \text{for } k \geq 0. \tag{37}$$

Observe that if $\Phi_n(z,0)$ is a polynomial of degree n then the solution is also polynomial of degree n for all t. Indeed, one can solve for the coefficients by integrating the homogeneous first order system. For example, since $d\mathcal{R}(t)/dt = \beta(\mathcal{R}(t))$ we have $a_0 = c_0\beta(\mathcal{R}(t))$ for some constant c_0.

In the simplest burning models, combustion is assumed to proceed at a constant normal velocity $\beta = S_1$, independent of curvature. This leads to a first order real equation involving Hilbert transform or a first order complex partial differential equation. The model vorticity equation is of this type and is analyzed similarly [2]. Thus $\mathcal{R}(t) = R_0 + S_1 t$. To analyze this case, it is convenient to change the time variable to the new time

$$\tau = \int_0^t \frac{dt}{\mathcal{R}(t)}. \tag{38}$$

If $\beta = S_1$ is constant,

$$\tau = \begin{cases} \frac{1}{S_1} \log\left(1 + \frac{S_1}{R_0} t\right), & \text{if } S_1 > 0; \\ \frac{t}{R_0}, & \text{if } S_1 = 0. \end{cases} \tag{39}$$

In this case, we substitute $\Phi(z,t) = a_0 + z\Upsilon(z,t)$ into (35) and find a first order PDE. Hence the general solution is

$$\Phi(z,t) = a_0 + zF(z + ic_1\tau), \tag{40}$$

where F is an arbitrary holomorphic function in the neighborhhod of the unit disk and a_0 is constant. This shows that the problem is sensitive to initial data, since an arbitrarily small perturbation on the circle $F = \delta^2(i + \delta i - z)^{-1}$ will cause blowup in arbitrarily small time δ.

Corollary 5. *The analytic solutions of (35) with constant $\beta = S_1 \geq 0$ blow up in finite time if and only if the set Z defined by*

$$Z := \{z : |\Re e z| \leq 1,\ |z| > 1,\ \Im m z < 0 \text{ and } \Phi_0 \text{ is singular at } z\}$$

is nonempty. In time variable (39), $\Phi(z,\tau)$ becomes infinite when $\tau \nearrow \tau_3$ where

$$c_1\tau_3 = \inf\left\{-\Im m z - \sqrt{1 - (\Re e z)^2} : z \in Z\right\}.$$

The solution also admits solutions periodic in the τ variable, such as $F = e^z$ which causes the bubble to shimmy as it rises. However, by Theorem 6., the problem is well posed for short time.

Taking $F = z^{n-1}$ we find a family of solutions for (10) with constant $\beta = S_1$ is

$$\Psi_n(z,t) = \begin{cases} 1, & \text{if } n = 0; \\ z\,(z + ic_1\tau)^{n-1}, & \text{if } n > 0. \end{cases} \tag{41}$$

In this case, as time proceeds, the weighting of the lower modes dominates. Also the instabilities produce fingers on the upper side of the circle. This is easy to see for $\beta = S_1$ constant because in that case, by equation (41), we see that the norm satisfies

$$\begin{aligned} |\Psi_n(\theta,\tau)| &= |e^{i\theta}(e^{i\theta} + ic_1\tau)^{n-1}| \\ &= (1 + 2c_1\tau\sin\theta + c_1^2\tau^2)^{\frac{n-1}{2}}. \end{aligned}$$

Thus, for some angles, the initial growth is relatively faster than the radius. There the norm at θ satisfies the infinitessimal fingering criterion

$$c_1(n-1)\sin\theta = \left.\frac{d\log|\Psi_n(\theta,\tau)|}{d\tau}\right|_{\tau=0} \geq \left.\frac{d\log\mathcal{R}}{d\tau}\right|_{\tau=0} = S_1.$$

Thus *all* modes above a certain critical number

$$n_c = \frac{S_1}{c_1} + 1$$

develop unstable fingers initially. If there is curvature dependence, $\beta'(R_0) > 0$, then we shall see that the situation is radically different.

In case $F = e^{-i\mu z}$ and μ real we get solutions $\Xi_0 = 1$ for $\mu = 0$ and for $\mu \neq 0$,

$$\Xi_\mu(z,t) = ze^{c_1\mu\tau - i\mu z}.$$

Which of these solutions are relatively infinitessimally unstable? We require

$$\mu c_1 = \left.\frac{d\log|\Xi_\mu|_0}{d\tau}\right|_{\tau=0} \geq \left.\frac{d\log\mathcal{R}(\tau)}{d\tau}\right|_{\tau=0} = S_0$$

or $c_1\mu \geq S_0$. How many fingers actually grow? We interpret that to mean for some fixed fraction $\gamma \in (0,1)$ to determine the number of relative maxima of $\Re e\,\Xi_\mu$ of norm $e^{c_1\mu\sin(\theta)}$ above $\gamma|\Xi_\mu|_0 = \gamma e^{c_1\mu}$. This is the number of relative maxima in the interval $\{\theta : \sin\theta \geq 1 + (\log\gamma)/c_1\mu\} \subset [0,\pi]$. Hence for $c_1\mu \geq -\log\gamma$, the image of this interval by the imaginary part of $\log \Xi_\mu$ or by $\theta - c_1\mu\cos\theta$ has length less than $2c_1\mu + \pi$ so there are at most $c_1\mu/\pi + 2 \geq S_1/\pi + 3/2$ unstable fingers. Using the precise interval gives $(-2c_1\mu\log\gamma - \log^2\gamma)^{1/2}/\pi + \sin^{-1}(1 + (\log\gamma)/c_1\mu)/\pi + 3/2 \sim c\sqrt{c - 1\mu}$ unstable fingers.

Since we found global solutions with polynomial initial data, we can prove a short time existence theorem for analytic initial data. Since Ψ_n grow polynomially in τ, as we have seen, there is no long time statement.

Theorem 6. *Suppose $f(\theta)$ is a real analytic function on $\mathbf{S^1}$. Then there exists some time t_1 depending on S_1, c_1, f and R_0 and an analytic solution $u(\theta,t)$ to (10) with $\beta = S_1 \geq 0$ constant such that $u(\theta,0) = f(\theta)$.*

Proof. f is analytic if and only if there are constants K and $a > 1$ depending on f such that its Fourier series $f(\theta) = \sum_{j=-\infty}^{\infty} f_j e^{ij\theta}$ satisfies $|f_j| \leq K a^{-|j|}$ for all j ([7], p.26). If $\beta_0 \neq 0$ then the solution is a superposition of Ψ_j given by

$$u(\theta,t) = f_0 + 2\Re e\left\{e^{i\theta}\sum_{j=1}^{\infty} f_j\left(e^{i\theta} + \frac{ic_1}{S_1}\log\left(1 + \frac{S_1 t}{R_0}\right)\right)^{j-1}\right\}.$$

The sum converges uniformly in $0 \leq t \leq t_1$ provided that

$$1 + \frac{c_1}{S_1}\log\left(1 + \frac{S_1 t_1}{R_0}\right) < a.$$

For the $\beta \equiv 0$ case, $\mathcal{R}(t) = R_0$ and $\tau = t/R_0$. This time

$$u(\theta,t) = f_0 + 2\Re e\left\{e^{i\theta}\sum_{j=1}^{\infty} f_j\left(e^{i\theta} + \frac{ic_1 t}{R_0}\right)^{j-1}\right\},$$

which converges uniformly in $0 \leq t \leq t_1$ provided $t_1 < (a-1)R_0/c_1$. □

For example, if $\beta = 0$ and we take as initial condition

$$f(\theta) = \frac{1 - 2\cos\theta}{5 - 4\cos\theta} = \Re e\left(\frac{-z}{2-z}\right)\Bigg|_{z=e^{i\theta}} = \Re e\left(-\sum_{k=1}^{\infty}\frac{z^k}{2^k}\right)\Bigg|_{z=e^{i\theta}}$$

then the solution of the equation is

$$\begin{aligned}\Re e\left(-\frac{z}{2}\sum_{k=0}^{\infty}\frac{(z + c_1 i\tau)^k}{2^k}\right)\Bigg|_{z=e^{i\theta}} &= \Re e\left(\frac{-z}{2 - z - c_1 i\tau}\right)\Bigg|_{z=e^{i\theta}} \\ &= \frac{1 - 2\cos\theta + c_1\tau\sin\theta}{5 - 4\cos\theta + 2c_1\tau\sin\theta + c_1^2\tau^2}.\end{aligned}$$

Note that $f(\theta) = \Re e(1 + 2e^{i\theta})^{-1}$ also but the power series for $(1+2z)^{-1}$ doesn't converge on the circle.

7. Polynomial solutions of the linearized equation: general case.

If the curvature dependence is physical then $\alpha(\kappa) \sim 1/S_1 - c\kappa$ near $\kappa = 0$ for some $S_1, c > 0$ and a condition such as (47) holds. Our basic model satisfies $\beta(\mathcal{R}) = S_1\exp(-b/\mathcal{R})$. We may assume S_1 is any constant, but we have in mind $\beta(\mathcal{R}) \nearrow S_1$ as $\mathcal{R} \to \infty$. Taking the time variable as (38) we obtain

$$\frac{d}{d\tau}\left(\beta^{k^2-1}a_k\right) = ic_1 k\beta^{k^2-1}a_{k+1} \qquad \text{for } k \geq 0. \tag{42}$$

Hence, for polynomial initial data, the solutions stay polynomial. Let

$$\Phi_n(z,\tau) = \sum_{p=0}^{n-1} a_{n,p}(\tau) c_1^p i^p z^{n-p} \tag{43}$$

be a solution whose initial data is $\Phi_n(z,0) = z^n$. Then for all $k = 1, \ldots, n-1$,

$$a_{n,0} = \frac{S_0^{n^2-1}}{\beta^{n^2-1}(\tau)}, \tag{44}$$

$$a_{n,k} = \tag{45}$$

$$\frac{\lfloor n-1 \rfloor_k \, S_0^{n^2-1}}{\beta^{(n-k)^2-1}(\tau)} \int_0^{\tau} \int_0^{\tau_{k-1}} \cdots \int_0^{\tau_2} \frac{d\tau_1 \, d\tau_2 \, \cdots \, d\tau_{k-1} \, d\tau_k}{\beta^{2n-2k+1}(\tau_k) \beta^{2n-2k+3}(\tau_{k-1}) \cdots \beta^{2n-1}(\tau_1)}.$$

For example, in the separable case where β is not bounded, $\beta(\tau) = S_0 + b \log \mathcal{R}/R_0$, then $\tau = b^{-1} \log(1 + bS_0^{-1} \log(\mathcal{R}R_0^{-1}))$ so $\beta = S_0 e^{b\tau}$. Thus $a_{n,k} \sim c_{n,k} e^{-b[(n-k)^2-1]\tau}$ as $\tau \to \infty$ which is consistent with (30). Consider the case $\alpha(\kappa) = S_1 \exp(-b\kappa)$. Then $\beta(\tau) = S_1 \exp(-be^{-\tau})$ so β is bounded. If $0 < S_0 \le \beta \le S_1$ for $0 \le \tau$ then we may estimate the iterated integrals to get

$$\binom{n-1}{k} \frac{S_0^{n^2-1} \tau^k}{S_1^{n^2-1}} \le |a_{n,k}(\tau)| \le \binom{n-1}{k} \tau^k. \tag{46}$$

This can be improved.

Lemma 7. *Suppose $\beta \in C^1([R_0,\infty))$ satisfies $\beta' > 0$ and*

$$\frac{S_0 S_1}{S_0 + (S_1 - S_0)e^{-k_1\tau}} \le \beta(\tau) \le S_1 \tag{47}$$

for all $0 \le \tau$ where k_1 and $S_0 = \beta(R_0)$ are positive constants. Let $s = S_0/S_1$, $T > 0$ and

$$k_2 = -\frac{\log(s + [1-s]e^{-k_1 T})}{T}. \tag{48}$$

Then for $\tau \in [0,T]$, and $p = 1, \ldots, n-1$,

$$\beta(\tau)^{(n-p)^2-1} |a_{n,p}| \le M_{n,p} = S_0^{(n-p)^2-1} \binom{n-1}{p} \left(\frac{1 - e^{-(2n-2p+1)k_2\tau}}{(2n-2p+1)k_2} \right)^p. \tag{49}$$

For $\tau \in [0,T]$, and for $1 \le N \le n$ and $p = n-N+1, \ldots, n-1$,

$$|a_{n,p}| \le \frac{S_0^{(n-p)^2-1}}{\beta(\tau)^{(n-p)^2-1}} \binom{n-1}{p} e^{(N-n+p)(N-n+p+1)k_2 T} \left(\frac{1 - e^{-(2N+1)k_2\tau}}{(2N+1)k_2} \right)^p, \tag{50}$$

and for all $\tau \ge 0$,

$$|a_{n,p}| \le \binom{n-1}{p} \tau^p \left(s^{n^2-1} + \frac{2p+2}{1+k_1\tau} \right). \tag{51}$$

Proof. First,

$$\frac{S_0}{\beta(\tau)} \leq s + [1-s]\, e^{-k_1\tau} \tag{52}$$

thus, for any $T > 0$ and all $\tau \in [0, T]$,

$$\frac{S_0}{\beta(\tau)} \leq e^{-k_2\tau}.$$

where k_2 is chosen so that $y(T) = z(T)$ where $y(\tau) = s + [1-s]e^{-k_1\tau}$ and $z(\tau) = e^{-k_2\tau}$. The inequality follows from the maximum principle since $y'' - k_1^2 y \geq z'' - k_1^2 z$ because by the arithmetic-geometric mean inequality, $y(T) \geq e^{-(1-s)k_1 T}$ so $k_2 \leq k_1(1-s)$.

The estimate (50) follows iteratively using the differential recursion. For convenience, we let

$$A_{n,k}(\tau) := \beta(\tau)^{(n-k)^2-1} a_{n,k}(\tau).$$

This satisfies the recursion

$$\frac{d}{d\tau} A_{n,k} = \frac{(n-k)A_{n,k-1}}{\beta(\tau)^{2n-2k+1}}. \tag{53}$$

For our initial data, this means $A_{n,0}(z,\tau) = S_0^{n^2-1} = M_{n,0}$. We majorize the terms inductively as long as $n - p \geq N$ by

$$\begin{aligned}
A_{n,1} &= (n-1)S_0^{(n-1)^2-1} \int_0^\tau \frac{S_0^{2n-1}}{\beta(\tau)^{2n-1}}\, d\tau \\
&\ll (n-1)S_0^{(n-1)^2-1} \int_0^\tau e^{-(2n-1)k_2\tau}\, d\tau \\
&= (n-1)S_0^{(n-1)^2-1} \frac{\left(1 - e^{-(2n-1)k_2\tau}\right)}{(2n-1)k_2} \\
&= M_{n,1} \\
&\cdots \\
A_{n,p} &= (n-p) \int_0^\tau \frac{A_{n,p-1}}{\beta(\tau)^{2n-2p+1}}\, d\tau \\
&= S_0^{n^2-1} \lfloor n-1 \rfloor_p \int_0^\tau \int_0^{\tau_{n-p}} \cdots \int_0^{\tau_2} \frac{d\tau_{n-p} \cdots d\tau_1}{\beta(\tau_{n-p})^{2n-2p+1} \cdots \beta(\tau)^{2n-1}} \\
&\ll (n-p)S_0^{2p-2n-1} \int_0^\tau \frac{S_0^{2n-2p+1}}{\beta(\tau)^{2n-2p+1}} M_{n,p-1}\, d\tau \\
&\leq (n-p)\binom{n-1}{p-1} S_0^{(n-p)^2-1} \int_0^\tau e^{-(2n-2p+1)k_2\tau} \left(\frac{1 - e^{-(2n-2p+3)k_2\tau}}{(2n-2p+3)k_2}\right)^{p-1} d\tau \\
&\leq \binom{n-1}{p} S_0^{(n-p)^2-1} \left(\frac{1 - e^{-(2n-2p+1)k_2\tau}}{(2n-2p+1)k_2}\right)^{p}.
\end{aligned}$$

The last inequality can be checked by comparing derivatives of both sides and using that $(1 - e^{-ax})/a$ is a decreasing function of a. If $n-p=N-1$ then the last inequlity becomes

$$\begin{aligned}
A_{n,n-N+1} \leq (n-p)\binom{n-1}{p-1} S_0^{(n-p)^2-1} \int_0^\tau e^{-(2N-1)k_2\tau} \left(\frac{1-e^{-(2N+1)k_2\tau}}{(2N+1)k_2}\right)^{p-1} d\tau \\
\leq \quad (n-p)\binom{n-1}{p-1} S_0^{(n-p)^2-1} e^{2k_2T} \int_0^\tau e^{-(2N+1)k_2\tau} \left(\frac{1-e^{-(2N+1)k_2\tau}}{(2N+1)k_2}\right)^{p-1} d\tau \\
= \quad \binom{n-1}{p} S_0^{(n-p)^2-1} e^{2k_2T} \left(\frac{1-e^{-(2N+1)k_2\tau}}{(2N+1)k_2}\right)^{p}.
\end{aligned}$$

Thus for $n-p<N-1$,

$$A_{n,p} \quad \leq \quad \binom{n-1}{p} S_0^{(n-p)^2-1} e^{(2+4+\cdots+2(N-n+p))k_2T} \left(\frac{1-e^{-(2N+1)k_2\tau}}{(2N+1)k_2}\right)^{p},$$

which completes inequality (50)

To get (51), we observe from Jensen's inequality on (52),

$$\frac{S_0^q}{\beta^q} \leq s^q + [1-s^q]e^{-k_1\tau} \tag{54}$$

yields an estimate on $a_{n,0}$. Put $N=(2n-2p+1)+(2n-2p+3)+\cdots+(2n-1)=2np-p^2$. Rearranging the order of integration and noting that $\beta(\tau_p) \leq \beta(\tau_j)$ yields

$$\begin{aligned}
I \quad &= \quad p! \int_0^\tau \int_{\tau_1}^\tau \cdots \int_{\tau_{p-1}}^\tau \frac{S_0^N \, d\tau_p \cdots d\tau_1}{\beta(\tau_p)^{2n-2p+1} \cdots \beta(\tau_1)^{2n-1}} \qquad (55) \\
&\leq \quad p! \int_0^\tau \int_{\tau_1}^\tau \cdots \int_{\tau_{p-1}}^\tau \frac{S_0^N \, d\tau_p \cdots d\tau_1}{\beta(\tau_p)^N} \\
&\leq \quad p! \int_0^\tau \int_{\tau_1}^\tau \cdots \int_{\tau_{p-1}}^\tau \left(s^N + [1-s^N]e^{-k_1\tau_p}\right) d\tau_p \cdots d\tau_1 \\
&= \quad s^N\tau^p + [1-s^N]p \int_0^\tau (\tau-\tau_p)^{p-1} e^{-k_1\tau_p} \, d\tau_p \\
&= \quad s^N\tau^p + [1-s^N]\left(\tau^p - k_1 \int_0^\tau (\tau-\tau_p)^p e^{-k_1\tau_p} \, d\tau_p\right).
\end{aligned}$$

By Jensen's inequality,

$$\begin{aligned}
k_1 \int (\tau-\tau_p)^p e^{-k_1\tau} \, d\tau_p \quad &\geq \quad \frac{1-e^{-k_1\tau}}{k_1^p} \left(\frac{e^{-k_1\tau}-1+k_1\tau}{1-e^{-k_1\tau}}\right)^p \\
&\geq \quad \left(1-e^{-k_1\tau}\right)\tau^p \left(1-\frac{1}{2e^{-k_1\tau}+k_1\tau}\right)^p \\
&\geq \quad \tau^p\left(1-\frac{2p+1}{1+k_1\tau}\right)
\end{aligned}$$

and the (51) follows using (54). □

Another estimate is obtained as follows. First we use piecewise constant bounds on β to simplify the integrations.

A third estimate is obtained as follows. We assume piecewise constant bounds on β.

Lemma 8. *Let $\beta \in C^1$ satisfy $\beta' > 0$, $\beta(0) = S_0 > 0$ and $\lim_{x\to\infty} \beta(x) = S_1$. Let the constants $\xi > 0$ and $\hat{s} > \check{s} > 0$ such that*

$$\frac{S_0}{\beta(\tau)} \leq \begin{cases} \hat{s}, & \text{if } \tau < \xi \\ \check{s}, & \text{if } \tau \geq \xi \end{cases} \tag{56}$$

Suppose Φ_n satisfies (42) and $\Phi_n(z, 0) = z^n$. Then

$$a_{n,p} \leq \begin{cases} \binom{n-1}{p} \hat{s}^{n^2-1} \tau^p, & \text{if } \tau < \xi; \\ \binom{n-1}{p} \sum_{j=0}^{p} \binom{p}{j} \hat{s}^{2n(p-j)-(p-j)^2} \check{s}^{(n-p+j)^2-1} \xi^{p-j} (\tau - \xi)^j, & \text{if } \tau \geq \xi; \end{cases} \tag{57}$$

Proof. Again, using the recursions (37) for $A_{n,k} = (\beta/S_0)^{(n-k)^2-1} a_{n,k}$, namely

$$\frac{d}{d\tau} A_{n,p} = (n-p) \left(\beta/S_0\right)^{2n-2p+1} A_{n,p-1},$$

we get upper bounds by integrating the equations in which (56) replaces the power. Hence

$$\begin{aligned} A_{n,0} &\leq 1, \\ A_{n,p} &\leq \begin{cases} \dfrac{\lfloor n-1 \rfloor_p}{p!} \hat{s}^{2np-p^2} \tau^p, & \text{if } \tau < \xi; \\ \lfloor n-1 \rfloor_p \displaystyle\sum_{j=0}^{p} \frac{\hat{s}^{2n(p-j)-(p-j)^2} \check{s}^{j^2+2nj-2pj}}{j!(p-j)!} \xi^{p-j} (\tau - \xi)^j, & \text{if } \tau \geq \xi; \end{cases} \end{aligned}$$

and (57) follows using (56). □

Lemma 9. *Let β, $\xi > 0$ and $\hat{s} > \check{s}$ be as in Lemma 8. Assume (56) and $\hat{s} = S_0/\beta(\xi)$. Suppose Φ_n satisfies the forced version of (42), namely*

$$\frac{d}{d\tau} \left(\beta^{(n-p)^2-1} a_{n,p} \right) = \beta^{(n-p)^2-1} \left((n-p) a_{n,p-1} + f_{n,p} \right) \tag{58}$$

with initial data $\Psi_n(0) = 0$ and where $f_{n,n} = f(\tau)$ and $f_{n,j} = 0$ for $j \neq n$. Then

$$|a_{n,0}| \leq I_1(\tau) \tag{59}$$

$$|a_{n,p}| \leq \begin{cases} \lfloor n-1 \rfloor_p \left(\dfrac{\hat{s}\beta}{S_0} \right)^{2np-p^2} I_{p+1}(\tau), & \text{if } \tau < \xi; \\ \lfloor n-1 \rfloor_p \left(\left(\dfrac{\check{s}\beta}{S_0} \right)^{2np-p^2} I_{p+1}(\tau) + \right. & \text{if } \tau \geq \xi. \\ \left. + \displaystyle\sum_{j=1}^{p} \left[\left(\frac{\check{s}}{\hat{s}} \right)^{(n-j)^2-1} - \left(\frac{\check{s}}{\hat{s}} \right)^{n^2-1} \right] \frac{I_{j+1}(\xi)(\tau-\xi)^{p-j}}{(p-j)!} \right), & \end{cases} \tag{60}$$

where

$$I_1(\tau) = \int_0^\tau |f(\sigma)| \, d\sigma \quad \text{and} \quad I_{k+1}(\tau) := \int_0^\tau I_k(\sigma) \, d\sigma \quad \text{for all} \quad k \geq 1.$$

Proof. Use the recursions (58) for $A_{n,k} = (\beta/S_0)^{(n-k)^2-1} a_{n,k}$ and $F_{n,k} = g f_{n,k}$ where $g = (\beta/S_0)^{n^2-1}$. Arguing as in Lemma 8. we get supersolutions for $p = 1 \dots n-1$,

$$
\begin{aligned}
A_{n,0} &\leq J_1(\tau) \\
A_{n,p} &\leq \begin{cases} \lfloor n-1 \rfloor_p \, \hat{s}^{2np-p^2} J_{p+1}(\tau), & \text{if } \tau < \xi; \\ \lfloor n-1 \rfloor_p \Big(\check{s}^{2np-p^2} J_{p+1}(\tau) + & \text{if } \tau \geq \xi. \\ + \displaystyle\sum_{j=1}^{p} \check{s}^{(2n-j-p)(p-j)} \left[\hat{s}^{2nj-j^2} - \check{s}^{2nj-j^2} \right] \frac{J_{j+1}(\xi)(\tau-\xi)^{p-j}}{(p-j)!} \Big), & \end{cases}
\end{aligned}
$$

where $J_1 = \int_0^\tau F(\sigma)\, d\sigma$ and $J_{k+1} = \int_0^\tau J_k(\sigma)\, d\sigma$. (60) follows from $J_p \leq g I_p$ which hold since g is increasing and $J_p(\xi) \leq \hat{s}^{1-n^2} I_p(\xi)$. □

Lemma 10. *Suppose $\beta \in C^1([0,\infty))$ is increasing and satisfies $S_0 = \beta(0) > 0$ and $\xi, \check{s} \in (0,1)$, so that $S_0/\beta \leq \check{s}$ if $\tau \geq \xi$. Then*

$$
\sum_{n=1}^{\infty} (\tau + n) \; a_{n,n-1} \left(c_1 + \frac{1}{\tau} \right)^{n-1} \leq C(\beta)(F + F') \left(\frac{(\tau-\xi)(1+\tau)}{\tau - \xi\tau - \xi} \right)
$$

as $\tau \to \infty$ where $F(z) = \sum_{j=1}^{\infty} \check{s}^{j^2-1} z^j$.

Proof. Let $\hat{s} = 1$. Thus if $\tau > c_1\tau + \xi$,

$$
a_{n,n-1} \leq \sum_{j=0}^{n-1} \binom{n-1}{j} \check{s}^{j^2+2j} \xi^{n-1-j} (\tau - \xi)^j .
$$

Hence the first sum

$$
\begin{aligned}
\sum_{n=1}^{\infty} a_{n,n-1} \left(1 + \frac{1}{\tau} \right)^{n-1} &\leq \sum_{n=1}^{\infty} \left(c_1 + \frac{1}{\tau} \right)^{n-1} \sum_{j=0}^{n-1} \binom{n-1}{j} \check{s}^{j^2+2j} \xi^{n-1-j} (\tau-\xi)^j \\
&= \sum_{j=0}^{\infty} \check{s}^{j^2+2j} (\tau-\xi)^j \sum_{k=1}^{\infty} \binom{j+k-1}{j} \xi^{k-1} \left(c_1 + \frac{1}{\tau} \right)^{k+j-1} \\
&= \tau \sum_{j=0}^{\infty} \check{s}^{j^2+2j} \frac{(\tau-\xi)^j (1+c_1\tau)^j}{(\tau - c_1\xi\tau - \xi)^{j+1}} \\
&= \frac{\tau}{(\tau-\xi)(1+c_1\tau)} F\left(\frac{(\tau-\xi)(1+c_1\tau)}{\tau - c_1\xi\tau - \xi} \right)
\end{aligned}
$$

which is finite for $(1-\xi)\tau > \xi$. Similarly for the second sum,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n \, a_{n,n-1} \left(c_1 + \frac{1}{\tau} \right)^{n-1} \\
&\leq \sum_{n=1}^{\infty} n \left(c_1 + \frac{1}{\tau} \right)^{n-1} \sum_{j=0}^{n-1} \binom{n-1}{j} \check{s}^{j^2+2j} \xi^{n-1-j} (\tau-\xi)^j
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \check{s}^{j^2+2j}(\tau-\xi)^j \sum_{k=1}^{\infty} \binom{j+k-1}{j}(j+k)\xi^{k-1}\left(c_1+\frac{1}{\tau}\right)^{k+j-1} \\
&= \sum_{j=0}^{\infty} \check{s}^{j^2+2j}(j+1)(\tau-\xi)^j \sum_{k=1}^{\infty} \binom{j+1+k-1}{j+1}\xi^{k-1}\left(c_1+\frac{1}{\tau}\right)^{k+j-1} \\
&= \frac{\tau^2}{(\tau-c_1\tau\xi-\xi)^2}\sum_{j=0}^{\infty} \check{s}^{j^2+2j}\frac{(j+1)(\tau-\xi)^j(1+c_1\tau)^j}{(\tau-c_1\xi\tau-\xi)^j} \\
&= \frac{\tau^2}{(\tau-c_1\tau\xi-\xi)^2}F'\left(\frac{(\tau-\xi)(1+c_1\tau)}{\tau-c_1\xi\tau-\xi}\right).
\end{aligned}
$$

□

This estimate implies that the solutions of the linearized equations are relatively asymptotically stable.

Corollary 11. *Suppose $\beta \in C^1([0,\infty))$ satisfies $S_0 = \beta(0) > 0$, $\beta' > 0$ and $\beta \le S_1 < \infty$. Let Φ be a solution of the buoyant combustion equation (35) linearized about the rising circle. Let $\mathcal{R}(\tau)$ be the radius of the rising circle. Then $\lim_{\tau\to\infty}\Phi/\mathcal{R} = 0$. In particular, for any $\alpha > 0$,*

$$
|\Phi(\tau,\bullet)|_0 = \mathbf{o}\left(\exp\left(\tau^{\alpha}\right)\right) \tag{61}
$$

as $\tau \to \infty$.

Proof. First observe that the radius grows exponentially in τ. Since

$$
S_0 \le \frac{d\log\mathcal{R}}{d\tau} = \beta(\mathcal{R}) \le S_1,
$$

we get $R_0\exp(S_0\tau) \le \mathcal{R}(\tau) \le R_0\exp(S_1\tau)$. Hence, assuming (61), $\Phi/\mathcal{R} \to 0$ uniformly. Using notation from the previous lemma,

$$
\begin{aligned}
|\Phi| &= \left|\sum_{n=1}^{\infty} C_n\,\Phi_n\right| \\
&\le \sup_{n\in N}|C_n|\sum_{n=1}^{\infty} a_{n,n-1}\left(c_1+\frac{1}{\tau}\right)^{n-1} \\
&\le \frac{\tau\sup_{n\in N}|C_n|}{(\tau-\xi)(1+c_1\tau)}F\left(\frac{(\tau-\xi)(1+c_1\tau)}{\tau-c_1\xi\tau-\xi}\right).
\end{aligned}
$$

However, $F(z)$ is an entire fuction of order zero which implies (61). This holds since for any $\alpha > 0$, $\check{s}^{n^2-1} \le n^{-n/\alpha}$ for n large. Hence order does not exceed α [15]. □

The estimate in Lemma 7. provides a short time existence theorem for analytic solutions for general β case analogous to the β constant case. For initial data $a_k(0) = 0$ for $k \ne n$ and $a_n(0) = 1$ majorizing the solution using (46),

$$
\Phi_n(z,\tau) = \sum_{k=0}^{n-1} a_{n,k} i^k z^{n-k} \ll \sum_{k=1}^{n}\binom{n-1}{k-1} z^{n-k}\tau^k = z\left(z+\tau\right)^{n-1}.
$$

As in Theorem 6., by analyticity, the Fourier coefficients $|f_j| \le Ka^{|j|}$ for all j for some positive constants $a < 1$ and K. Majorizing to establish convergence,

$$
\begin{aligned}
u(\theta,t) &= \frac{f_0}{S_0}\beta(R(t)) + 2\Re e\left\{\sum_{n=1}^{\infty} f_n\Phi_n\left(e^{i\theta},\tau(t)\right)\right\} \\
&\ll \frac{|f_0|S_1}{S_0} + 2a\sum_{n=1}^{\infty} Ka^{n-1}\left(1+\tau\right)^{n-1}
\end{aligned}
$$

which converges uniformly on $0 \le \tau \le \tau_2$ provided that $a(1+\tau_2) < 1$.

But using lemma 7., one gets long time existence of an analytic solution.

Theorem 12. *Suppose $f(\theta) \in L^2(\mathbf{S^1})$ so that $f = \sum_{n=0}^{\infty} C_n z^n$. Suppose $\beta \in C^1([R_0,\infty))$ satisfies (47) and $\beta(0) = S_0$ for some positive constants k_1, S_1, S_0. Then there exists a solution $u(\cdot,t) \in C^1((0,\infty), C^\omega(\mathbf{S}^1))$ to (10) of the form*

$$u = \sum_{n=0}^{\infty} C_n \Phi_n(z,\tau)$$

where Φ_n are the solutions with initial data $\Phi_n(z,0) = z^n$. The series converges since the there holds for all $n, \tau \ge 0$ and $0 < \eta < 1$,

$$|\Phi_n| \le \left(\frac{\beta(\tau)}{S_0}\right)^{\frac{1}{2\eta^2 k_2^2}} \left(\eta + (1-\eta) e^{-\frac{\tau}{\eta}}\right)^{n-1} \tag{62}$$

where k_2 is defined in (48)

Proof. We select constants $N, T, V > 0$ for now and esimate Φ_n from Lemma 7.. We split the sum into two parts, the first may be zero if $n < N$. Using $(1-e^{-a\tau})/a$ is decreasing in a and $V(n-p-1) \le (n-p)^2 - 1 + (V/2-1)^2$, we may majorize using Lemma 7.

$$\begin{aligned}
\Phi_n \ll & \sum_{\substack{p=0 \\ p \le n-N}}^{n} \frac{M_{n,p} z^{n-p}}{\beta^{(n-p)^2+1}} + \sum_{\substack{p=0 \\ p > n-N}}^{n} \frac{M_{n,p} z^{n-p}}{\beta^{(n-p)^2+1}} \\
\le & \frac{\beta(\tau)^{(V/2-1)^2}}{S_0^{(V/2-1)^2}} z \left(\sum_{N \le n-p} \binom{n-1}{p} \frac{S_0^{V(n-p-1)} z^{n-p-1}}{\beta(\tau)^{V(n-p-1)}} \left(\frac{1-e^{-(2N+1)k_2\tau}}{(2N+1)k_2}\right)^p \right. \\
+ & \left. \sum_{N > n-p} \binom{n-1}{p} \frac{S_0^{V(n-p-1)} z^{n-p-1} e^{(N-n+p)(N-n+p+1)k_2 T}}{\beta(\tau)^{V(n-p-1)}} \left(\frac{1-e^{-(2N+1)k_2\tau}}{(2N+1)k_2}\right)^p \right) \\
\le & \frac{\beta(\tau)^{(V/2-1)^2}}{S_0^{(V/2-1)^2}} e^{N(N-1)k_2 T} z \left(e^{-Vk_2\tau} z + \frac{1-e^{-(2N+1)k_2\tau}}{(2N+1)k_2}\right)^{n-1}.
\end{aligned}$$

Since the Fourier expansion $f = \sum_0^\infty C_n z^n$ has bounded coefficients, we get convergence of $\sum C_n \Psi_n$ to an analytic solution provided that the series converges uniformly. This happens if $g < 1$ in $[0,T]$ for appropriate choices of N, V depending on β and T but not on n, p, where

$$g(\tau) = e^{-Vk_2\tau} + \frac{1-e^{-(2N+1)k_2\tau}}{(2N+1)k_2}.$$

But $g'(0) < 0$ provided $1 < Vk_2$. Also $g < 1$ as $\tau \to \infty$ if $(2N+1)k_2 > 1$. These conditions imply $g < 1$ on $(0,\infty)$.

For concreteness, choose $(2N-1)k_2 = Vk_2 \le 1/\eta < (2N+1)k_2$. Then $g \le \eta + (1-\eta)e^{-\tau/\eta}$. Since $S_0 e^{k_2 T} \le \beta$ we obtain the inequality (62) □

We show that as time proceeds, every mode grows relative to higher modes.

Lemma 13. *Suppose $\beta \in C^1([R_0,\infty))$ satisfies $\beta' > 0$, $\beta(R_0) = S_0$ and (47) for positive constants k_1, S_0, S_1, R_1. Let $\Phi_n(\tau,z) = \sum_{k=0}^{n-1} a_{n,k} c_1^k i^k z^{n-k}$ satisfy (35) with initial condition $\Phi_n(z,0) = z^n$. Then the* *coefficients satisfy the following inequalities for* $0 \le p < n < N$,

$$(n-p-1)S^{2n-2p-1}\tau \ \ge \ \frac{a_{n,p+1}}{a_{n,p}} \ \ge \ \frac{n-p-1}{p+1}\tau, \tag{63}$$

$$\binom{n-1}{p}\frac{a_{n,n-1}}{a_{n,p}} \ \ge \ \tau^{n-1-p}, \tag{64}$$

$$\frac{a_{n+1,p}}{a_{n,p}} \ \ge \ \frac{ns^{2n+1}}{n-p}. \tag{65}$$

$$\min\left\{\frac{1}{2k_2(\tau)}, \tau s^{2n+1} + \frac{2[1-s^{2n+1}]\tau}{2+k_1\tau}\right\} \ \ge \ \frac{a_{n+1,n}}{a_{n,n-1}} \ \ge \ s^{2n+1}\tau \tag{66}$$

$$\left(s^{2n+1} + \frac{2[1-s^{2n+1}]}{2+k_1\tau}\right)^{N-n}\tau^{N-n} \ \ge \ \frac{a_{N,N-1}}{a_{n,n-1}} \ \ge \ s^{N^2-n^2}\tau^{N-n} \tag{67}$$

$$a_{n,n-1}\left(c_1 + \frac{1}{\tau}\right)^{n-1} \ \ge \ |\Phi_n| \tag{68}$$

Proof. First we observe that since $\beta^{(n-p)^2-1}a_{n,p}$ is increasing,

$$\begin{aligned} \beta^{(n-p-1)^2-1}a_{n,p+1} &= (n-p-1)\int_0^\tau \beta^{-2n+2p+1}\beta^{(n-p)^2-1}a_{n,p} \\ &\le (n-p-1)S_0^{-2n+2p+1}\tau\beta^{(n-p)^2-1}a_{n,p} \end{aligned}$$

and the first inequality of (63) follows.

Let $R_{n,p} = a_{n,p+1}/a_{n,p}$. Dividing the relation (42) by $\beta^{(n-p)^2-1}a_{n,p}$ yields

$$((n-p)^2-1)\frac{\beta_\tau}{\beta} + \frac{(a_{n,p})_\tau}{a_{n,p}} = \frac{n-p}{R_{n,p-1}}.$$

Subtracting gives equations for $R_{n,p}$ when $0 < p < n$,

$$\frac{(R_{n,0})_\tau}{R_{n,0}} = \frac{n-1}{R_{n,0}} + (2n-1)\frac{\beta_\tau}{\beta} \tag{69}$$

$$\frac{(R_{n,p})_\tau}{R_{n,p}} + \frac{n-p}{R_{n,p-1}} = \frac{n-p-1}{R_{n,p}} + (2n-2p-1)\frac{\beta_\tau}{\beta}. \tag{70}$$

Using the representation (44), $R_{n,p}(0) = 0$ and $\beta' > 0$, one finds $R_{n,0} \ge (n-1)t$ and then checks the right side of (63) by induction.

The inequalities (64) then follow from (63). Inequality (65) follows directly from (44) using $S_0 \ge s\beta$. Combining (64) and (65) with $p = n-1$ and repeating yields the second inequality of (67). The first follows from (55) which implies

$$\begin{aligned} a_{n+1,n} &= n\int_0^\tau \left(\frac{S_0}{\beta(\tau_0)}\right)^{2n+1} y_n(\tau_0)\,d\tau_0 \\ &\le ny_n(0)\int_0^\tau \left(\frac{S_0}{\beta(\tau_0)}\right)^{2n+1} d\tau_0 \\ &\le na_{n,n-1}\min\left\{\tau, s^{2n+1}\tau + [1-s^{2n+1}]\frac{1-e^{k_1\tau}}{k_1}, \frac{1-e^{(2n+1)k_2\tau}}{(2n+1)k_2}\right\} \end{aligned}$$

since

$$y_n = (n-1)! \int_{\tau_0}^{\tau} \int_{\tau_1}^{\tau} \cdots \int_{\tau_{n-1}}^{\tau} \frac{S_0^{n^2-1} \, d\tau_n \cdots d\tau_1}{\beta(\tau_n)^3 \cdots \beta(\tau_1)^{2n-1}}$$

is a decreasing function of τ_0. To improve we must integrate from infinity. To this end, consider the differential equation satisfied by $y_n(\sigma)$ for $\sigma \in [0, \tau]$.

$$\frac{dy_{n+1}}{d\sigma} = -n \left(\frac{S_0}{\beta} \right)^{2n+1} y_n.$$

$y_1 \equiv 1$, $y_n(0) = a_{n,n-1}(\tau)$ and $y_n(\tau) = 0$ for all $n \geq 1$. Now let $q_n = y_{n+1}/y_n$. We have $q_1 \equiv y_2$ so $q_n(\tau) = 0$ if $n \geq 2$. Also As before we find the differential equation for q_n if $n > 1$,

$$\begin{aligned} \frac{(q_1)_\sigma}{q_1} &= -\frac{1}{q_1} \left(\frac{S_0}{\beta} \right)^3 \\ \frac{(q_n)_\sigma}{q_n} &= -\frac{n}{q_n} \left(\frac{S_0}{\beta} \right)^{2n+1} + \frac{n-1}{q_{n-1}} \left(\frac{S_0}{\beta} \right)^{2n-1} \end{aligned}$$

Since difference quotients increase with σ, we see that

$$\begin{aligned} q_1 = \int_{\sigma}^{\tau} \left(\frac{S_0}{\beta(\tau_0)} \right)^3 d\sigma &\leq \int_\sigma^\tau s^3 + [1 - s^3] e^{-k_1 \sigma} \, d\sigma \\ &= (\tau - \sigma) \left(s^3 + \frac{[1 - s^3](e^{-k_1\sigma} - e^{-k_1\tau})}{k_1(\tau - \sigma)} \right) \leq (\tau - \sigma) \left(s^3 + \frac{2[1 - s^3]}{2 + k_1 \tau} \right) \end{aligned}$$

By induction we check that

$$\grave{q}_n(\sigma) = (\tau - \sigma) \left(s^{2n+1} + \frac{2[1 - s^{2n+1}]}{2 + k_1 \tau} \right)$$

is a supersolution. To see this, using Jensen's inequality,

$$\begin{aligned} \frac{(\grave{q}_n)_\sigma}{\grave{q}_n} &= -\frac{1}{\tau - \sigma} \\ &\leq \frac{-1}{\tau - \sigma} \left(\frac{1}{s^{2n-1} + \frac{2[1-s^{2n-1}]}{2+k_1\tau}} \right) \left(s^{2n-1} + [1 - s^{2n-1}] e^{-k_1 \tau} \right) \\ &\leq \frac{-1}{\tau - \sigma} \left(\frac{n}{\left(s^{2n+1} + \frac{2[1-s^{2n+1}]}{2+k_1\tau} \right)^{\frac{2n-1}{2n+1}}} - \frac{n-1}{s^{2n-1} + \frac{2[1-s^{2n-1}]}{2+k_1\tau}} \right) \left(\frac{S_0}{\beta} \right)^{2n-1} \\ &\leq \frac{-1}{\tau - \sigma} \left(\frac{n \left(\frac{S_0}{\beta} \right)^2}{s^{2n+1} + \frac{2[1-s^{2n+1}]}{2+k_1\tau}} - \frac{n-1}{s^{2n-1} + \frac{2[1-s^{2n-1}]}{2+k_1\tau}} \right) \left(\frac{S_0}{\beta} \right)^{2n-1} \\ &= -\frac{n}{\grave{q}_n} \left(\frac{S_0}{\beta} \right)^{2n+1} + \frac{n-1}{\grave{q}_{n-1}} \left(\frac{S_0}{\beta} \right)^{2n-1}. \end{aligned}$$

Substituting $\sigma = 0$ completes (67).

Finally, using (64),

$$|\Phi_n| \le \left|\sum_{p=0}^{n-1} a_{n,p} c_1^p i^p z^{n-p}\right| \le \sum_{p=0}^{n-1} a_{n,p} c_1^p \le a_{n,n-1} \sum_{p=0}^{n-1} \binom{n-1}{p} c_1^p \tau^{1-n+p}$$

and (68) follows. □

Polynomial solutions satisfy their own energy estimates. The consequences are strong enough to yield global analytic existence.

Lemma 14. *Suppose $\beta \in C^1([0,\infty))$ are there are positive constants S_0, S_1, k_1, T so that (47) for all $\tau \ge 0$. Then for all $n > p \ge 0$, and $k_1\tau \ge 2\log S - \log\varepsilon$,*

$$a_{n,p} \le \frac{\tau^p}{\lfloor n+1 \rfloor_p \log(S-\varepsilon)^p}.$$

Proof. Let the polynomial energy

$$\mathcal{E}_n = \sum_{p=0}^{n-1} \frac{k_2^p(T)}{(n-p+1)!} \left(\frac{e^{-k_2(T)\tau}\beta(\tau)}{S_0}\right)^{(n-p)^2-1} a_{n,p}$$

where k_2 is given by (48) for $\tau \in [0,T]$. Since we have the recursion (53) for $\tau \in [0,T]$, differentiating yields $d\mathcal{E}_n/d\tau \le 0$. Thus $\mathcal{E}_n \le \mathcal{E}_n(0) = 1/(n+1)!$.

Since the coefficients are postitve, each term is bounded above by

$$a_{n,p} \le \frac{1}{\lfloor n+1 \rfloor_p k_2^p} \left(\frac{S_0 e^{k_2\tau}}{\beta}\right)^{(n-p)^2-1}.$$

The lemma follows since for $k_1 T \ge 2\log S - \log\varepsilon$, there holds

$$\begin{aligned} k_2 T &= -\log(s(1+[S-1]e^{-k_1 T})) \\ &\ge \log\left(\frac{S}{1+[S-1]s^2\varepsilon}\right) \ge \log\left(S(1-[1-s]s\varepsilon)\right). \end{aligned}$$

□

If for some $k_4 > 0$ there holds $\beta(\tau) \le S_0 e^{k_4\tau}$ then a lower bound for energy $\tilde{\mathcal{E}}_n$ with k_4 in place of k_2 follows in the same way $\tilde{\mathcal{E}}_n \ge 1/(n+1)!$.

It follows from Lemma 7. that the polynomial solutions (43) of the heat equation have the following time dependent majorizations.

$$\begin{aligned} |\Psi_n(z,t)| &\le \{1+c_1\tau\}^{n-1}\left(\frac{S_0^{n^2-1}}{S_1^{n^2-1}} + \frac{4n+3}{1+k_1\tau}\right), \\ \Re\left(i^{-n}\Psi_n(i,t)\right) &\ge \frac{S_0^{n^2-1}}{S_1^{n^2-1}}\{1+c_1\tau\}^{n-1}. \end{aligned}$$

Observing that $\max_\theta |\Psi_n(e^{i\theta},\tau)| = i^{-n}\Psi_n(i,\tau)$, computing the instantaneous relative fingering criterion

$$\begin{aligned} 1 &= \left.\frac{d\log R}{d\tau}\right|_{\tau=0} < \left.\frac{d\log\left(i^{-n}\Psi_n(i,\tau)\right)}{d\tau}\right|_{\tau=0} \\ &= -(n^2-1)\beta'(R_0)R_0 + \max\{n-1,0\}c_1 \end{aligned}$$

we see that Ψ_0 and Ψ_1 are relatively stable but there are fingering modes (integer solutions $n > 1$) whenever c_1 is relatively large, namely, it is sufficient that $3\beta'(R_0)R_0 + 1 \le c_1$. Hence, in this case the unstable modes are $2 \le n$ and

$$n < 1 + \frac{c_1 + \sqrt{8\beta'(R_0)^2 R_0^2 - 4\beta'(R_0)R_0 + c_1^2}}{2\beta'(R_0)R_0}.$$

In the case $\beta = S_1 e^{-b/R}$ means that Φ_n is is instantaneously fingering if $(n+1)bS_1 + R_0 S_1 < c_1 R_0 e^{-b/R_0}$. Thus there is a critical number

$$n_c = \frac{c_1 R_0 e^{-b/R_0}}{bS_1} - \frac{R_0}{b} - 1$$

above which the modes are instantaneously relatively nonincreasing. This analysis agrees with the linearized stability analysis of Zhu [18].

We expect to sharpen sufficient conditions for instability since $\beta \approx S_0$ at initial times.

Lemma 15. *Assume that $\beta(R)$ satisfies the conditions of Lemma 7. and*

$$\beta(R_0 e^{\tau}) \le S_0 e^{k_3 \tau}$$

for some $k_3 > 0$. Then for $k = n = 0$ or $1 \le k \le n$,

$$a_{n,k} \ge \binom{n-1}{k} \tau^k \left(1 - \frac{k_3(6n(n-k) - 6 + k(2k+1))\tau}{6}\right). \tag{71}$$

Proof. Let Δ be the simplex $0 \le \tau_1 \le \cdots \le \tau_k \le \tau$. By Jensen's inequality,

$$\begin{aligned}
a_{k,n}(\tau) &\ge \binom{n-1}{k} k! e^{-k_3((n-k)^2-1)\tau} \int_{\Delta} e^{-k_3 \sum_{j=1}^{k}(2n-2j+1)\tau_j} \, d\tau_{n-k} \cdots d\tau_1 \\
&\ge \binom{n-1}{k} \tau^k e^{-k_3((n-k)^2-1)\tau} \exp\left(-k_3 \sum_{j=1}^{k} \frac{(2n-2j+1)k!}{\tau^k} \int_{\Delta} \tau_j \, d\tau_{n-k} \cdots d\tau_1 \right) \\
&= \binom{n-1}{k} \tau^k e^{-k_3((n-k)^2-1)\tau} \exp\left(\frac{-k_3 k(6n-4k+1)\tau}{6} \right)
\end{aligned}$$

and the Lemma follows. □

We have the estimate for (43) using (46)

$$\begin{aligned}
|\Re e \Phi_n(\theta,\tau)| &= \left| \Re e \sum_{k=0}^{n-1} a_{n,k} i^k e^{i\theta(n-k)} \right| \\
&\le \left| \Re e \sum_{k=0}^{n-1} \binom{n-1}{k} (i\tau)^k e^{i(n-k)\theta} \right| + \left| \sum_{k=0}^{n-1} i^k \left[a_{n,k}(\tau) - \binom{n-1}{k} \tau^k \right] e^{i(n-k)\theta} \right|, \\
&\le \left| \Re e e^{i\theta} \left(e^{i\theta} + i\tau\right)^{n-1} \right| + \frac{k_3 \tau}{6} \sum_{k=0}^{n-1} \binom{n-1}{k} \tau^k (6n(n-k) - 6 + k(2k+1)), \\
&\le \left| \Re e e^{i\theta} \left(e^{i\theta} + i\tau\right)^{n-1} \right| + 2k_3 n^2 \tau (1+\tau)^{n-1}.
\end{aligned}$$

Since $\check{r} e^{i\check{\theta}} = e^{i\theta} + i\varepsilon$ implies $\sin^2(\theta - \check{\theta}) \le \varepsilon^2$ and $\check{r}^2 = 1 + 2\varepsilon \sin\theta + \varepsilon^2$. Thus we get $(\theta - \check{\theta})^2 \le \varepsilon^2 \cos^2\theta (1 + \varepsilon \sin\theta)^{-2}$ and $|(e^{i\theta} + \varepsilon)^p - \check{r}^p e^{ip\theta}| \le 2\varepsilon$ for all θ and $\varepsilon \le 1/2$. Then if $2\tau \le 1$,

$$|\Re e \Phi_n(\theta,\tau)| \le |\cos n\theta| \left|1 + \tau^2 + \tau \sin\theta\right|^{n-1} + n^2 2^n \tau (1 + k_3)$$

Here we derive a simple way to see that solutions have the property that the norm increases whenever $\sin\theta > 0$. Thus, expanding the norm

$$
\begin{aligned}
|\Phi_n|^2 &= \sum_{k=0}^{n-1}\sum_{\ell=0}^{n-1} i^k \bar{i}^\ell a_{n,k}(\tau) a_{n,\ell}(\tau) z^{n-k}\bar{z}^{n-\ell} \\
&= \sum_{k=0}^{n-1} a_{n,k}(\tau)^2 + 2\sum_{j=1}^{n-1}\sum_{k=0}^{n-1-j} a_{n,k}(\tau)\, a_{n,j+k}(\tau) \cos\left(\left(\theta - \tfrac{\pi}{2}\right) j\right).
\end{aligned}
$$

This time, consider the expansion of Ψ_n in terms of τ. By Lemmas 7. and 15. we have

$$\left| a_{n,k}(\tau) - \binom{n-1}{k}\tau^k \right| = \mathbf{O}(\tau^{k+1}).$$

near $\tau = 0$. Thus

$$
\begin{aligned}
|\Phi_n|^2 &= 1 + \left(2a_{n,0}(0) \left.\frac{d\,a_{n,0}}{d\,\tau}\right|_{\tau=0} + 2a_{n,0}(0)a_{n,1}(0)\sin(\theta) \right)\tau + \cdots \\
&= 1 + \left(\frac{-2(n^2-1)\beta'(R_0)R_0}{c_1} + 2(n-1)\sin\theta \right)\tau + \cdots
\end{aligned}
$$

Again, this shows that the n-th mode is infinitessimally stable if

$$(n+1)\beta'(R_0)R_0 \ge c_1,$$

and infinitessimally relatively stable if

$$(n-1)c_1 \le \beta(R_0) + (n^2-1)\beta'(R_0)R_0$$

which is sharper than the previous criterion. Again, in the diffusive case $\beta' > 0$ all sufficiently high modes are stable, and there are more stable modes whenever the combustion effect is stronger than the fluid effect.

8. Gevrey like energy estimates for the linearized equation.

The instantaneous analyticity of solutions of the linearized equation may be shown using Gevrey type energy estimates. We define a norm which is L^2 at the initial time, and which shows that for positive time, the coefficients have strong enough decay so that the solution is analytic in a neighborhood of the the unit circle. The estimates have no convergence problem for polynomial initial data since the sums remains finite. Thus the validity for initial data in L^2 or L^1 may be established using approximation. We assume that the solution and right dside is expanded by

$$\Phi = \sum_{k=0}^{\infty} a_k(\tau)\, i^{-k}\, z^k, \qquad F = \sum_{k=0}^{\infty} F_k(\tau)\, i^{-k}\, z^k. \tag{72}$$

Then the solution of the inhomogeneous linearized equation satisfies (53)

$$\frac{d}{d\tau}\left[\left(\frac{\beta}{S_0}\right)^{k^2-1} a_k\right] = \left(\frac{\beta}{S_0}\right)^{k^2-1} \left[k a_{k+1} + \mathcal{R}F_k\right]. \qquad \text{for } k \ge 0. \tag{73}$$

The zeroth coefficient is given by

$$a_0(\tau) = \beta(\tau)\left[\frac{C_0}{S_0} + \int_0^\tau \frac{\mathcal{R}(\sigma)F_0(\sigma)}{\beta(\sigma)}\,d\sigma\right],$$

thus this need not be estimated.

Lemma 16. *Suppose $\beta \in C^1([0,\infty))$ are there are constants $0 < S_0 < S_1$ so that $\beta(0) = S_0$, $\beta' > 0$ and $\lim_{\tau\to\infty} \beta(\tau) = S_1$. Let $s = S_0/S_1$. Choose $1 < M < \infty$ and $T > 0$ so that $2M - 1 \geq e^{2MT}$. If the initial data*

$$\Phi(0, z) = \sum_{n=0}^{\infty} C_k\, i^{-k}\, z^k$$

is in L^2 then for $0 \leq \tau \leq T$, the energy

$$\mathcal{E}_1(\tau) = \sum_{k=1}^{\infty} f_k(\tau) \left| \left(\frac{\beta}{S_0}\right)^{k^2-1} a_k \right|^2 \leq \|\Phi(0)\|_0 + \int_0^\tau \mathcal{R}(\sigma)^2 \mathcal{E}_3[F](\sigma)\, d\sigma \tag{74}$$

where

$$f_k = \exp\left(-2Mk \int_0^\tau \left(\frac{S_0}{\beta}\right)^{2k} d\tau\right)$$

and $F(\cdot, \tau)$ is assumed to have bounded (integrable)

$$\mathcal{E}_3[F] = \sum_{k=1}^{\infty} f_k(\tau) \left| \left(\frac{\beta}{S_0}\right)^{k^2+k-1} F_k \right|^2.$$

Proof. We first prove the result for polynomial initial data $\sum_{k=1}^{N} C_k i^{-k} z^k$ and polynomial right side $\sum_{k=1}^{N} F_k i^{-k} z^k$ whose solution stays polynomial.

Observe that $\mathcal{E}_1(\Phi) \leq \mathcal{E}_2(\Xi)$ where Ξ satisfies the system (73) but with initial data $\sum_{k=0}^{N} |C_k| i^{-k} z^k$ and right side $\sum_{k=0}^{N} |F_k| i^{-k} z^k$. This follow inductively from the fact that a_N is constant and that the solutions of $dy/d\tau = g(\tau)$ with $y(0) = C$ have $|y(\tau)|$ bounded by the solution of $dz/d\tau = |g(\tau)|$ and $z(0) = |C|$ because $|y(\tau)| = |C + \int_0^\tau g| \leq |C| + \int_0^\tau |g|$.

For convenience, let $Q_k = (\beta/S_0)^{k^2-1}|a_k|$ and $P_k = (\beta/S_0)^{k^2-1}\mathcal{R}|F_k|$. Differentiating, we find for $\epsilon > 0$, $\eta_k > 0$,

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_1(\Xi) &= \sum_{k=1}^{\infty} (f_k)_\tau {Q_k}^2 + 2 f_k Q_k (Q_k)_\tau \\
&= \sum_{k=1}^{\infty} (f_k)_\tau {Q_k}^2 + 2k f_k Q_k \left(\frac{S_0}{\beta}\right)^{2k+1} Q_{k+1} + 2 f_k Q_k P_k \\
&\leq \sum_{k=1}^{\infty} (f_k)_\tau {Q_k}^2 + k f_k \left[\epsilon Q_k^2 + \frac{1}{\epsilon} Q_{k+1}^2\right] \left(\frac{S_0}{\beta}\right)^{2k+1} + \eta_k f_k Q_k^2 + \frac{f_k P_k^2}{\eta_k} \\
&\leq \sum_{k=1}^{\infty} (f_k)_\tau {Q_k}^2 + k\epsilon f_k Q_k^2 \left(\frac{S_0}{\beta}\right)^{2k+1} \\
&\quad + \frac{k-1}{\epsilon} f_{k-1} \left(\frac{S_0}{\beta}\right)^{2k-1} Q_k^2 + \eta_k f_k Q_k^2 + \frac{f_k P_k^2}{\eta_k}
\end{aligned}$$

Taking $\epsilon = \beta/S_0$, $\eta_k = (S_0/\beta)^{2k}$, and using the definition of f_k,

$$\frac{d}{dt}\mathcal{E}_1(\Xi) \leq \sum_{k=1}^{\infty} [(1 - 2M)k f_k + (k-1) f_{k-1} + f_k] \left(\frac{S_0}{\beta}\right)^{2k} Q_k^2.$$

Now note that

$$(2k-2)\int_0^\tau \left(\frac{S_0}{\beta}\right)^{2k-2} \geq 2k\int_0^\tau \left(\frac{S_0}{\beta}\right)^{2k-2} - 2\tau \geq 2k\int_0^\tau \left(\frac{S_0}{\beta}\right)^{2k} - 2\tau$$

so that

$$f_{k-1} \leq f_k e^{2M\tau}.$$

If we restrict to the interval $\tau \in [0,T]$ so that $e^{2M\tau} \leq 2M-1$ then

$$(1-2M)kf_k + (k-1)f_{k-1} + f_k \leq (1-2M)kf_k + (k-1)f_k e^{2M\tau} + f_k < 0,$$

so

$$\frac{d}{d\tau}\mathcal{E}_1[\Xi] \leq \mathcal{R}(t)^2\mathcal{E}_3[F](\tau)$$

and the Lemma follows.

The energy estimate (74) says that Φ_N, solutions for truncated data form a Cauchy sequence since the partial sums are Cauchy in Φ in L^2. In particular, $\mathcal{E}_1$ is uniformly bounded for the truncations of the solution, hence bounded for Φ. □

Note that $f_k(T) \geq \exp(-2MkT)$ so that Lemma 16. implies that the hypothesis of Lemma 17. is fulfilled.

Lemma 17. *Suppose β, F, M, S_0 and T are as in Lemma 16. Let $1 > s_1 = S_0/\beta(T)$. Suppose for some solution (72) of the linearized equation the solution satisfies*

$$\mathcal{E}_2 = \sum_{n=1}^{\infty} {s_1}^{2+2ck-2k^2}|a_k(T)|^2 < \infty$$

where $c = -MT/\ln s_1$. Then for $T \leq \tau < \infty$, the energy satisfies

$$\mathcal{E}_2(\Phi) = \sum_{k=1}^{\infty} {s_1}^{2ck}\left|\left(\frac{\beta}{S_0}\right)^{k^2-1} a_k\right|^2 \leq E_2 e^{c_2(\tau-\tau_1)} + \int_{\tau_1}^{\tau} e^{c_2(\tau-\sigma)}\mathcal{R}(\sigma)^2\mathcal{E}_2[F](\sigma)\,d\sigma$$

where $c_2 = 1 - s_1 e^{MT-1}/\ln s_1$.

Proof. Arguing as in the previous lemma, where $f_k = s_1^{2ck}$ we find that

$$\begin{aligned}
\frac{d\mathcal{E}_2(\Xi)}{dt} &\leq \\
&\leq \sum_{k=1}^{\infty}\left[ks_1^{-c}f_k\left(\frac{S_0}{\beta}\right)^{2k+1} + (k-1)s_1^c f_{k-1}\left(\frac{S_0}{\beta}\right)^{2k-1} + f_k\right]Q_k^2 + f_k P_k^2 \\
&\leq \sum_{k=1}^{\infty}\left[ks_1^{2ck-c}s_1^{2k+1} + (k-1)s_1^{2c(k-1)+c}s_1^{2k-1} + s_1^{2ck}\right]Q_k^2 + s_1^{2ck}P_k^2 \\
&\leq \sum_{k=1}^{\infty}\left[ks_1^{2k+1} + (k-1)s_1^{2(k-1)+1} + s_1^c\right]s_1^{2ck-c}Q_k^2 + s_1^{2ck}P_k^2 \\
&\leq (1+2s_1^{-c}M_2)\mathcal{E}_2[\Xi] + \mathcal{R}^2\mathcal{E}_2[F],
\end{aligned}$$

since the function $x \mapsto xs_1^{2x+1}$ is bounded by $M_2 = -s_1/2e\ln s_1$. The conlusion follows as before. □

9. Coalescence of Fingers in the linearized equation.

It is a feature in this equation that the evolution proceeds to move energy from high frequency modes to low frequency since the lower modes have faster growth in time. Speaking picturesquely about the linearization, it means that an initial n-fingered perturbation of the circle eventually collapses into a perturbation whose fingers are concentrated near i on the circle, and then these fingers join together to form a single bump. This finger collapsing must be evidence that the linearization of the circle fails to portray the stability of the perturbed figure, once the nonlinear fingering process has progresses away from the circle.

As an application of lemma 13., we show that solutions to (35) eventually coarsen, that is they become one fingered. This global property of the linearized solution shows that

Lemma 18. *Let $\Psi_k = e^{i\theta_0} z(z+i\tau)^{n-1}$ be a solution of (35) with $\beta = S_1 > 0$ constant. Then, when $\tau > 3n$, the solution is* single peaked. *This means that $u = \Re e\Psi_n$ has a single minimum at θ_1 and a single maximum at θ_2 such that $\theta_1 < \theta_2 < \theta_1 + 2\pi$ and u is strictly increasing on (θ_1, θ_2) and strictly decreasing on $(\theta_2, \theta_1 + 2\pi)$.*

Proof. Let $X : \theta \mapsto e^{i\theta}(e^{i\theta} + i\tau)^{n-1}$. Note that for $\tau > 1$ the minimum and maximum distances of $X(\mathbf{S}^2)$ to the origin are $d_1 = (\tau-1)^{n-1}$ and $d_2 = (\tau+1)^{n-1}$. We show that for τ sufficiently large, X is a diffeomorphism of $\mathbf{S}^1$ to a convex curve in the plane enclosing the origin. Then $\Re e(e^{i\theta_0} X(\theta))$ is single peaked.

The curvature is computed as usual: the tangent vector

$$\mathbf{t} = \frac{X_\theta}{|X_\theta|}, \qquad \mathbf{n} = i\mathbf{t}, \qquad \frac{d\mathbf{t}}{ds} = \frac{\mathbf{t}_\theta}{|X_\theta|} = \kappa \mathbf{n}$$

so that since curvature is real,

$$\kappa = \frac{\mathbf{t}_\theta}{i\mathbf{t}|X_\theta|} = \frac{1}{iX_\theta}\left(\frac{X_{\theta\theta}}{|X_\theta|} - \frac{X_\theta(|X_\theta|)_\theta}{|X_\theta|^2}\right) = \frac{\Im m\left(\overline{X_\theta} X_{\theta\theta}\right)}{|X_\theta|^3}.$$

For this X and $n \geq 2$ we get

$$\kappa = \frac{1}{M}\left(1 + \frac{(n-2)(1+\sin(\theta)\tau)}{1+2\sin(\theta)\tau+\tau^2} + \frac{n^2+n\sin(\theta)\tau}{n^2+2n\sin(\theta)\tau+\tau^2}\right)$$

where

$$M = \left(1+2\sin(\theta)\tau+\tau^2\right)^{\frac{n-2}{2}} \left[n^2+2n\sin(\theta)\tau+\tau^2\right]^{\frac{1}{2}}.$$

Thus $\kappa > 0$ if $\tau \geq 3n$. □

The same method shows that fingers coalesce for arbitrary β and polynomial initial data. The first step is to prove that the norm of a solution with polynomial initial data is uniformly positive for large time.

Lemma 19. *Let $\beta \in C^1([0,\infty))$ satisfy (47) for some constants k_1, S_0, S_1. Let $\Xi(z,\tau)$ be a solution of (35) with initial data $f = \sum_{n=0}^N C_n z^n$ so that $C_N \neq 0$. Then for all $\varepsilon \in (0,1)$, there is a sufficiently large time $\tau > c(S_0, S_1, \varepsilon, f)$ such that the solution is positive and satisfies*

$$|\Xi(z,\tau) - \Xi(0,\tau)| \geq \varepsilon |C_N| a_{N,N-1}. \tag{75}$$

Moreover, the solution is single peaked.

Proof. When f is polynomial of degree $N = 1$ then $\Xi = C_0\beta(\tau)/S_0 + C_1 e^{i\theta}$ and $\Xi(\mathbf{S}^1, \tau)$ and is a circle. Put $\Phi_0 = \beta(\tau)/S_0$. For convenience, write real coefficients for the polynomial basis functions (43) for $n \geq 1$ as

$$\Phi_n = \sum_{k=0}^{n-1} a_{n,k} i^k z^{n-k} = \sum_{k=0}^{n-1} a_{n,k}\, e^{i\left(\frac{\pi}{2}k + (n-k)\theta\right)}. \tag{76}$$

Now for $N \geq 2$, set $X = \sum_{n=1}^{N} C_n \Phi_n$. First, using Lemma 13. we have

$$\begin{aligned}
\left|\Phi_n - a_{n,n-1} e^{i(\frac{\pi}{2}(n-1)+\theta)}\right| &= \left|\sum_{q=0}^{n-2} a_{n,p} e^{-i(\frac{\pi}{2}p+(n-p)\theta)}\right| \\
&\leq +a_{n,n-1} \sum_{p=0}^{n-2} \binom{n-1}{p} t^{1-n+p} \\
&= a_{n,n-1} \frac{(1+\tau)^{n-1} - \tau^{n-1}}{\tau^{n-1}} \\
&\leq a_{n,n-1} \frac{n(1+\tau)^{n-2}}{\tau^{n-1}}
\end{aligned}$$

which is less than $en\tau^{-1} a_{n,n-1}$ if $\tau \geq n$. Second, by Lemma 13., if $\tau \geq N$,

$$\begin{aligned}
\left|\sum_{1}^{N} C_n \Phi_n\right| &\geq |C_N| a_{N,N-1} \left(1 - \frac{eN}{\tau}\right) - \sum_{n=0}^{N-1} |C_n| a_{n,n-1} \left(1 + \frac{en}{\tau}\right) \\
&\geq |C_N| a_{N,N-1} \left(1 - \frac{eN}{\tau}\right) - 4a_{N,N-1} \sum_{n=0}^{N-1} |C_n| S^{N^2-n^2} \tau^{n-N} \\
&\geq a_{N,N-1} \left(|C_N| - \frac{|C_\bullet|_0 \left(eN + 8S^{N^2}\right)}{\tau}\right)
\end{aligned}$$

and (75) complete.

To prove single peakedness we argue as in Lemma 18.

$$\begin{aligned}
\left|\overline{\frac{d\Phi_m}{d\theta}} + i a_{m,m-1} e^{-i(\frac{\pi}{2}(m-1)+\theta)}\right| &= \left|\sum_{q=0}^{m-2} a_{m,q}(m-q) e^{-i(\frac{\pi}{2}q+(m-q)\theta)}\right| \\
&\leq a_{m,m-1} \sum_{p=0}^{m-2} \binom{n-1}{p} (n-p) t^{1-n+p} \\
&= a_{m,m-1} \frac{(1+\tau)^{m-2}(m+\tau) - \tau^{m-1}}{\tau^{m-1}} \\
&\leq a_{m,m-1} \frac{2m(1+\tau)^{m-2}}{\tau^{m-1}}
\end{aligned}$$

which is less than $2emt^{-1} a_{m,m-1}$ if $t \geq m$. Similarly,

$$\left|\frac{d^2\Phi_n}{d\theta^2} + a_{n,n-1} e^{i(\frac{\pi}{2}(n-1)+\theta)}\right| = \left|\sum_{p=0}^{n-2} a_{n,p}(n-p)^2 e^{i(\frac{\pi}{2}p+(n-p)\theta)}\right|$$

$$\begin{aligned}
&\leq a_{n,n-1} \sum_{p=0}^{n-2} \binom{n-1}{p} (n-p)^2 t^{1-n+p} \\
&= a_{n,n-1} \frac{(1+\tau)^{n-3}(n^2 + (3n-1)\tau + \tau^2) - \tau^{n-1}}{\tau^{n-1}} \\
&\leq a_{n,n-1} \left[\frac{n^2}{\tau^2} + \frac{4n}{\tau}\right] \left(1 + \frac{1}{\tau}\right)^{n-3}
\end{aligned}$$

which is less than $5en\tau^{-1}a_{n,n-1}$ if $t \geq n$. It follows that

$$
\begin{aligned}
&\Im m\left(\overline{X_\theta}X_{\theta\theta}\right) = \\
&\quad \sum_{m=1}^{N}\sum_{\ell=1}^{N}\overline{C_m}C_\ell\left(-ia_{m,m-1}e^{-i(\frac{\pi}{2}(m-1)+\theta)}+\right. \\
&\quad \left.\left[\frac{d\bar{\Phi}_m}{d\theta}+ia_{m,m-1}e^{-i(\frac{\pi}{2}(m-1)+\theta)}\right]\right)\times \\
&\quad \times\left(ia_{m,m-1}e^{i(\frac{\pi}{2}(m-1)+\theta)}+\left[\frac{d^2\Phi_m}{d\theta^2}+ia_{m,m-1}e^{-i(\frac{\pi}{2}(m-1)+\theta)}\right]\right) \\
\geq\ & \sum_{m=1}^{N}|C_m|^2a_{m,m-1}^2-2\sum_{m=1}^{N-1}\sum_{\ell=m+1}^{N}|C_m||C_\ell|a_{m,m-1}a_{\ell,\ell-1}- \\
&-2\sum_{m=1}^{N}\sum_{\ell=1}^{N}|C_m||C_\ell|a_{m,m-1}a_{\ell,\ell-1}\left\{\frac{2m(1+\tau)^{m-2}}{\tau^{m-1}}+\right. \\
&\left.+\left[\frac{\ell^2}{\tau^2}+\frac{4\ell}{\tau}\right]\left(1+\frac{1}{\tau}\right)^{\ell-3}+\frac{2m(1+\tau)^{m-2}}{\tau^{m-1}}\left[\frac{\ell^2}{\tau^2}+\frac{4\ell}{\tau}\right]\left(1+\frac{1}{\tau}\right)^{\ell-3}\right\} \\
\geq\ & |C_N|^2s^{2N^2-2}\tau^{2N-2}-\left(\sum_{m=1}^{N}|C_m|\right)^2\left(2N^3+9N^2+6N\right)(1+\tau)^{2N-3}
\end{aligned}
$$

which is positive for large τ. □

Single peakedness can also be proved for nonploynomial initial data, provided that it satisfies a nondegeneracy condition satisfied by polynomials.

Lemma 20. *Let $\beta \in C^1([0,\infty))$ satisfy (47) for some constants k_1, S_0, S_1. Let $\Xi(z,\tau)$ be a solution of (35). Assume that the initial data is nonpolynomial $f = \sum_{n=0}^{\infty} C_n z^n$ and satisfies the nondegeneracy condition*

$$\limsup_{\tau\to\infty}\epsilon(\tau) < \frac{1}{6}$$

where

$$\epsilon(\tau) = \frac{\sum_{n=1}^{\infty}|C_n|a_{n,n-1}(\tau)(n\tau+n^2)\left(1+\frac{1}{\tau}\right)^{n-1}}{\tau^2\left|\sum_{n=1}^{\infty}C_n i^{n-1}a_{n,n-1}(\tau)\right|}. \tag{77}$$

Then, for a sufficiently large time $\tau > c(C_\bullet, S_0, S_1, \chi, f)$, the solution is single peaked.

Proof. As before, write the polynomial basis functions (43) for $n \geq k \geq 1$ as Φ_n. Set $X = \sum_{n=0}^{\infty} C_n\Phi_n$. Let

$$F(\tau) = \left|\sum_{k=1}^{\infty}C_n i^{n-1}a_{n,n-1}\right|$$

This time, as in Lemma (19.),

$$
\begin{aligned}
|X_\theta| &\geq \left|\sum_{k=1}^{\infty}C_n i^{n-1}a_{n,n-1}\right| - 2\sum_{k=1}^{\infty}\frac{n|C_n|}{\tau}a_{n,n-1}\left(1+\frac{1}{\tau}\right)^{n-1} \geq F(\tau)(1-2\epsilon), \\
|X_\theta| &\leq F(\tau)(1+2\epsilon),
\end{aligned}
$$

$$|X_{\theta\theta}-iX_\theta| \leq \sum_{k=1}^{\infty}|C_n|a_{n,n-1}\left[\frac{n^2}{\tau^2}+\frac{6n}{\tau}\right]\left(1+\frac{1}{\tau}\right)^{n-1} \leq 6\epsilon F(\tau).$$

We find that curvature is a positive multiple of

$$\begin{aligned}\Im m\left(X_{\theta\theta}\overline{X_\theta}\right) &= \Im m\left(iX_\theta\overline{X_\theta} + (X_{\theta\theta} - iX_\theta)\overline{X_\theta}\right)\\ &\geq |X_\theta|^2 - |(X_{\theta\theta} - iX_\theta)||X_\theta|\\ &\geq (F-2\epsilon)^2 - 6\epsilon F(F+2\epsilon).\end{aligned}$$

Since $F(\tau) \to \infty$ as $\tau \to \infty$, this quantity is positive for large τ such that $(1-6\epsilon)F > 1$. □

The constant can be improved by better estimates of the derivatives. The condition holds for initial data whose coefficients decay sufficiently fast and which do not alternate. For example, we may assume that the arguments of the $C_n i^{n-1}$ are all the same and the coefficients are monotone in the sense that for a constant $\alpha \in (0,1)$,

$$|C_{n+2}|(n+2)^2 + |C_{n+1}|(n+1) \leq \alpha^n |C_n|$$

for large n. Then, for a sufficiently large time the solution is single peaked. Since any finite number of terms of ϵ are negligible for large τ, we may suppose that $\rho > 0$ is arbitrarily small to be chosen, and

$$s^{2n+1} + \frac{2}{2+k_1\tau} \leq \rho$$

for $n \geq N$. Then, using Lemma 13.,

$$\begin{aligned}&\sum_{k=N}^{\infty} |C_n| a_{n,n-1}\left[\frac{n^2}{\tau^2} + \frac{6n}{\tau}\right]\left(1+\frac{1}{\tau}\right)^{n-1}\\ &\leq \sum_{k=N}^{\infty} |C_n|\left[n^2\rho^2 a_{n-2,n-3} + n\rho a_{n-1,n-2}\right]\left(1+\frac{1}{\tau}\right)^{n-1}\\ &\leq \sum_{k=N}^{\infty}\left\{|C_{n+2}|(n+2)^2\rho^2\left(1+\frac{1}{\tau}\right)^{n+1} + |C_{n+1}|(n+1)\rho\left(1+\frac{1}{\tau}\right)^{n}\right\} a_{n,n-1}\\ &\leq \sum_{k=N}^{\infty}\left\{\rho^2\left(1+\frac{1}{\tau}\right)^2 + \rho\left(1+\frac{1}{\tau}\right)\right\}\left(1+\frac{1}{\tau}\right)^n \alpha^n |C_n| a_{n,n-1}\\ &\leq \frac{8}{49}\left|\sum_{k=N}^{\infty} C_n i^{n-1} a_{n,n-1}\right|\end{aligned}$$

provided $\alpha < (1-\alpha)\tau$ and $7\alpha\rho < 1$. An example of data satisfying such a recursion formula is $C_n i^{n-1} = s^{n^2}$ where $s \in (0,1)$.

Consider the function

$$F(z,\tau) = \sum_{n=1}^{\infty} \frac{c_n i^{n-1} a_{n,n-1}(\tau)}{\tau^{n-1}} z^{n-1}$$

where $K = 1/\log(S/2)$. By the estimate $a_{n,n-1} \leq 2K^{n-1}\tau^{n-1}/(n+1)!$ from Lemma (7.?), we see that F is an entire function. Moreover, it is majorized by $e^{K\tau}$ so it has order one. Let $\nu(r)$ be the central index of F. That is, it is the index of the maximal term,

$$\frac{|c_k| a_{k,k-1}}{\tau^{k-1}} r^{k-1} \leq \mu = \frac{|c_\nu| a_{\nu,\nu-1}}{\tau^{\nu-1}} r^{\nu-1} > \frac{|c_\ell| a_{\ell,\ell-1}}{\tau^{\ell-1}} r^{\ell-1}$$

for all $k < \nu$ and $\ell > \nu$. Let n be the central index corresponding to $r = \alpha\tau$ where $\alpha \in (0,1)$. Then

$$|c_k| a_{k,k-1} \leq |c_n| a_{n,n-1} \alpha^{n-k}$$

for all k. In particular,

$$\begin{aligned}\left|\sum_{p=1}^{n} c_p a_{p,p-1}\right| &\geq |c_n| a_{n,n-1} - \sum_{p=0}^{n-1} |c_p| a_{p,p-1} \\ &\geq |c_n| a_{n,n-1} - |c_n| a_{n,n-1} \sum_{p=0}^{n-1} \alpha^{n-p} \\ &\geq |c_n| a_{n,n-1} \frac{1-2\alpha+\alpha^{n+1}}{1-\alpha} \\ &\geq \frac{1}{2}|c_n| a_{n,n-1} \qquad \text{if } \alpha = \frac{1}{3}\end{aligned}$$

10. Estimates for the integral operator.

The operator is (8)

$$\mathcal{N}[u](\theta,\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{K}(\theta,\eta,\tau)\frac{d}{d\eta}\left[e^{u(\eta,\tau)-u(\theta,\tau)}\cos(\eta)\right] d\eta$$

where the kernel is given by (9).

Lemma 21. *For any $0 < \varepsilon < \alpha$ there is a constant $c_2(\epsilon, j, k)$ so that for $j,k = 0,1,2,\ldots$ and $\partial_\theta^j \partial_\tau^k u \in C^{1+\alpha,(1+\alpha)/2}$ then*

$$|\partial_\theta^j \partial_\tau^k \mathcal{N}[u]|_{\epsilon,(1+\epsilon)/2} \leq c_2 e^{2\omega} P_j(|\partial_\theta^j \partial_\tau^k u|_{1+\alpha,(1+\alpha)/2}). \tag{78}$$

where

$$\omega = \sup_{\theta,\eta\in\mathbf{S}^1} |u(\theta,\tau) - u(\eta,\tau)| \leq |u_\theta|_0$$

is the oscillation of u and $P_j(x) = x(1+x)^{j+k+3}$.

Proof. We regroup the kernel

$$\mathcal{K} = \frac{\mathcal{AB} + u_\theta(\theta,\tau) + \mathcal{C}}{\mathcal{A}+1}$$

where

$$\mathcal{A}(\theta,\eta,\tau) = \frac{\mathrm{ch}(u(\eta,\tau)-u(\theta,\tau))-1}{1-\cos(\eta-\theta)}, \tag{79}$$

$$\mathcal{B}(\theta,\eta,\tau) = \frac{\mathrm{sh}(u(\eta,\tau)-u(\theta,\tau))-u(\eta,\tau)+u(\theta,\tau)}{\mathrm{ch}(u(\eta,\tau)-u(\theta,\tau))-1}, \tag{80}$$

$$\begin{aligned}\mathcal{C}(\theta,\eta,\tau) &= \frac{u(\eta,\tau)-u(\theta,\tau)\cos(\eta-\theta)-u_\theta(\theta,\tau)\sin(\eta-\theta)}{1-\cos(\eta-\theta)} \\ &= \frac{\int_\eta^\theta (u(s,\tau)+u_{\theta\theta}(s,\tau))\sin(s-\eta)\,ds}{1-\cos(\eta-\theta)}.\end{aligned} \tag{81}$$

$$\mathcal{D}(\theta,\eta,\tau) = e^{u(\theta,\tau)-u(\theta,\tau)}\left[u_\eta(\eta,\tau)\cos(\eta)-\sin(\eta)\right] \tag{82}$$

The operator satisfies the following estimate.

The functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ and u_θ cause no trouble. $\mathcal{A}$ is essentially the difference quotient and $\mathcal{B}$ is a bounded analytic function with bounded dericative of $u(\eta,\tau)-u(\theta,\tau)$. For $u \in C^{1+\alpha}$ we have that the first order Taylor polynomial approximates

$$|u(\eta) - u_\theta(\theta)\sin(\eta-\theta) - u(\theta)\cos(\eta-\theta)| \leq c|u|_{1+\alpha}|\eta-\theta|^{1+\alpha}.$$

To avoid the singularity at $\theta - \eta = \pi$, choose $\chi(s) \geq 0$ to be a smooth 2π-periodic even cutoff function such that on $[-\pi, \pi]$, $\chi(s) = 1$ if $|s| \leq 1$, $\chi(s) = 0$ if $|s| \leq 2$ and $|\chi|_3 \leq 100$. Then we may decompose $\mathcal{K}$ as follows. Let

$$f = \frac{(u(\eta) - u_\theta(\theta)\sin(\eta-\theta) - u(\theta)\cos(\eta-\theta))\,\chi(\theta-\eta)}{\sin(\eta)}.$$

Then $f \in C^\alpha$. It follows that the nonlinear operator may be decomposed

$$\mathcal{N}[u](\theta,\tau) = N(\theta,\eta,\theta,\tau)$$

where

$$\begin{aligned} N(\theta,\eta,\zeta,\tau) &= \\ &= \int_{-\pi}^{\pi} \frac{\mathcal{A}(\zeta,\eta,\tau)\mathcal{B}(\zeta,\eta,\tau) + u_\theta(\zeta,\tau) + \mathcal{C}(\zeta,\eta,\tau)(1-\chi(\theta-\eta))}{2\pi(\mathcal{A}(\zeta,\eta,\tau)+1)}\mathcal{D}(\eta)\,d\eta + \\ &+ \frac{1}{2\pi} PV \int_{-\pi}^{\pi} \frac{f(\zeta,\eta,\tau)\mathcal{D}(\eta)}{\mathcal{A}(\zeta,\eta,\tau)+1}\cot\left(\frac{\theta-\eta}{2}\right)\,d\eta \\ = &\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} K_1(\theta,\eta,\tau)\,d\eta + \mathcal{H}\left[K_2(\cdot;\zeta,\tau)\right](\theta) \end{aligned}$$

and $\mathcal{H}$ designates the Hilbert transform. For $u \in C^{1+\alpha,(1+\alpha)/2}$ K_1 is Hölder continuous in θ, so first integral has the same regularity. Since $\mathcal{D}$ is exact, the same argument as Lemma (24) works but no regularity is lost for ζ and τ since $|\mathcal{D}|$ is integrable. The Hilbert transform preserves Hölder continuity with respect to η by Privalov's theorem, but loses regularity in ζ and τ according to Lemma (24). The dependence is as follows. $\mathcal{A}$ is like a squard difference quotient so behaves like u_θ^2. f and $\mathcal{C}$ is a linear combination of u's and u_θ's. $\mathcal{C}$ is an average. Writing c for constants independent of u we find for $\partial_\theta^j u \in C^{1+\alpha,(1+\alpha)/2}$ that in $\phi = \theta$ or ζ spatial derivatives, for $m = 1, 2$ and $j = 0, 1, 2$,

$$|\partial_\phi^j K_m|_{\alpha,\alpha/2} \leq c e^{2\omega}(|u|_{1+\alpha,(1+\alpha)/2} + |\partial_\phi^j u|_{1+\alpha,(1+\alpha)/2})(1 + |u|_{j+1,\alpha/2}^{j+3}).$$

The result follows from Lemma (24).

The integrals can be differentiated with respext to ζ or τ provided that the resulting derivative has some extra θ Hölder regularity to maintain integrable dominating functions. □

11. Apriori estimates for the nonlinear equation.

To study the nonlinear equation we regard the nonlocal term as given forcing function

$$\frac{\partial}{\partial\tau}u(\theta,\tau) = \frac{1}{c_1}\alpha\left(\frac{-u_{\theta\theta} + {u_\theta}^2 + 1}{\mathcal{R}e^u\left(1 + {u_\theta}^2\right)^{3/2}}\right)\frac{\sqrt{1 + u_\theta(\theta,\tau)^2}}{e^{u(\theta,\tau)}} - \frac{1}{c_1}\alpha\left(\frac{1}{\mathcal{R}}\right) + f(\theta,\tau) \tag{83}$$

This equation is parabolic since we assume $0 < \alpha \in C^1$ and $\alpha' < 0$. Note that when the curvature is positive, we may replace $\alpha(x) = \beta(1/x)$ as usual. The β notation suffers no loss of generality since our arguments such as the application of the maximum principle occurs at points where curvature is positive. Be begin by showing solutions satisfy an apriori C^0 estimate.

Lemma 22. *Suppose there are positive constants $S_0 < S_1$ and $s = S_0/S_1$, $S = S_1/S_0$ so that $\beta \in C^1(Q)$ where $Q = \mathbf{S}^1 \times [0, \infty)$ satisfies the usual hypotheses:*

$$\beta' > 0, \qquad \beta(0) = S_0, \qquad s < \frac{S_0}{\beta(\mathcal{R}(\tau))} \le s + (1-s)e^{-k_1\tau} \tag{84}$$

Suppose f satisfies the inequality

$$|f| \le c_2 \tag{85}$$

for positive c_2. Then a solution $u \in C^2(Q)$ has the estimate

$$\min\left\{e^{-|u_0|_0}, \frac{S_0}{S_1 + c_2}\right\} \le e^u \le \left(e^{|u_0|_0} + \frac{S_1}{c_2}\right) e^{c_2\tau}. \tag{86}$$

If $c_2 < S_1$ then u is bounded above

$$e^u \;\le\; S^{\frac{S_1}{k_1}} \left(e^{|u_0|_0} + \frac{S_1}{S_1 - c_2}\right). \tag{87}$$

If β satisfies the additional condition

$$\left|\frac{\beta_\tau}{\beta}\right| \le c_4 e^{-k_1\tau}. \tag{88}$$

for some $c_4 > 0$ and $c_2 < S_1$, and f decays

$$|f| \le c_5 e^{-k_2\tau} \tag{89}$$

for some constants c_5 and k_2, then the solution tends to zero at best like $e^{-S_0\tau}$, namely

$$\begin{aligned}(e^{-|u_0|_0} - 1)e^{-S_1\tau + c_7\min\{\tau, 1/k_1\}} - c_5c_6e^{c_7/k_1}\mathcal{V}(\tau) \le \\ \le \;\; e^u - 1 \;\le\; (e^{|u_0|_0} - 1)e^{-S_1\tau + (c_4+S_0)\min\{\tau,1/k_1\}} + c_5c_6e^{(c_4+S_0)/k_1}\mathcal{V}(\tau)\end{aligned}$$

where the bound $1/c_6 \le e^u \le c_6$ is given by (86) and (87), $c_7 = S_0 + c_4c_6^{1+k_1/S_1}$ and

$$\mathcal{V} = \begin{cases} \dfrac{e^{-k_2\tau} - e^{-S_1\tau}}{S_1 - k_2}, & \text{if } S_1 \ne k_2, \\[2ex] \tau e^{-S_1\tau}, & \text{if } S_1 = k_2. \end{cases}$$

Proof. Bounds are gotten by using the maximum principle to find ODEs that guarantee that their solutions $M(\tau)$ are sub or supersolutions for u. For an upper bound in (86) we suppose that the initial data $|u_0|_0 \le M(0)$ and that if (τ_0, θ_0) is the first point where $u = M$ then $M_\tau \ge u_\tau$ at that point. There $u_\theta = 0$ and $u_{\theta\theta} \le 0$ or in other words the curvature $\kappa \ge e^{-M}$ (which is positive.) Since β is increasing, we obtain at (θ_0, τ_0)

$$\begin{aligned} u_\tau &= e^{-M}\beta\left(\frac{\mathcal{R}}{\kappa}\right) - \beta(\mathcal{R}) + f \\ &\le e^{-M}\beta(\mathcal{R}e^M) - \beta(\mathcal{R}) + f \\ &\le S_1e^{-M} + c_2 = M_\tau. \end{aligned} \tag{90}$$

Integrating this linear ODE in e^M gives the first upper bound. If $c_2 < S_1$ then (90) yields

$$u_\tau - S_1e^{-M} \;\le\; g \quad = \quad c_2 - \frac{S_1}{1 + (S-1)e^{-k_1\tau}}.$$

Using

$$\frac{S_1}{k_1}\ln s + (S_1 - c_2)\tau \le -\int_0^\tau g = \frac{S_1}{k_1}\ln\left(1 - s + se^{k_1\tau}\right) - c_3\tau \le (S_1 - c_3)\tau$$

in the solution

$$e^M \le e^{|u_0|_0}\exp\left(\int_0^\tau g\right) + S_1\int_0^\tau \exp\left(\int_{\tau'}^\tau g\right)\, d\tau'$$

yields the estimate (87).

Observe that assuming (88),

$$\beta'(\mathcal{R})\mathcal{R} = \frac{\beta_\tau(\mathcal{R})}{\beta(\mathcal{R})} \le c_4 e^{-k_1\tau} \le c_4\left(\frac{\mathcal{R}}{R_0}\right)^{-k_1/S_1}$$

we have

$$\frac{d}{dx}\beta(\mathcal{R}x) = \beta'(\mathcal{R}x)\mathcal{R} \le \frac{c_4}{x}\left(\frac{\mathcal{R}x}{R_0}\right)^{-k_1/S_1} \le c_4 S x^{-1-k_1/S_1}e^{-k_1\tau} \tag{91}$$

which decreases in τ so that we can refine the inequality (90) to

$$\begin{aligned}
e^M M_\tau &\le \beta(\mathcal{R}) + c_4 e^{-(1+k_1/S_1)M}e^{-k_1\tau}(e^M - 1) - \beta(\mathcal{R})e^M + fe^M \\
(e^M - 1)_\tau &\le (-\beta(\mathcal{R}) + c_4 e^{-k_1\tau})(e^M - 1) + c_5 e^{-k_2\tau}e^M \\
y_\tau &\le (-S_1 + (c_4 + S_0)e^{-k_1\tau})y + c_5 c_6 e^{-k_2\tau},
\end{aligned}$$

where c_6 is an upper bound for e^M given by (87). For convenience, let $E[c](x) = \int_0^x e^{cx}\, dx$ for any real number c. Integrating yields

$$y \le e^{-S_1\tau}\left(y_0 e^{(c_4+S_0)E[k_1](\tau)} + c_5 c_6 e^{(S_0+c_4)/k_1}E[S_1 - k_2](\tau)\right).$$

The inequality (90) follows.

Consider now the subsolution $u \ge m$. Again, let θ_0 be the point where $u(\cdot, \tau_0)$ is minimum and equals m. There $u_\theta = 0$ and $u_{\theta\theta} \ge 0$, or what is the same, $\kappa \le e^{-u}$. Since α is decreasing,

$$\begin{aligned}
u_\tau &= e^{-m}\alpha\left(\frac{\kappa}{\mathcal{R}}\right) - \beta(\mathcal{R}) + f \\
&\ge e^{-m}\beta\left(\mathcal{R}e^m\right) - \beta(\mathcal{R}) + f \\
&\ge S_0 e^{-m} - (c_2 + S_1) = m_\tau.
\end{aligned}$$

Integrating the linear equation in e^m gives the left inequality of (86).

Finally, assuming (88) and using (91) then the inequality becomes

$$\begin{aligned}
(e^m - 1)_\tau &\ge \beta(\mathcal{R}e^m) - \beta(\mathcal{R})e^m - fe^m \\
&\ge -\beta(\mathcal{R})(e^m - 1) + c_4(e^m - 1)e^{-k_1\tau}e^{-(1+k_1/S_1)m} - c_5 e^m e^{-k_2\tau} \\
&\ge -\left(S_1 - (S_0 + c_4 c_6^{1+k_1/S_1})e^{-k_1\tau}\right)(e^m - 1) - c_5 c_6 e^{-k_2\tau}
\end{aligned}$$

where $e^{-m} \le c_6$ is from (87). Integrating the inequality yields (90). □

Next we consider apriori gradient estimates. The computation seems to be facilitated by making the change of variables

$$v = \mathrm{Atn}(u_\theta) \tag{92}$$

Then we rewrite the curvature of $X = e^{u(\theta)})(\cos\theta, \sin\theta)$ as

$$\begin{aligned} \kappa &= e^{-u}(1 - v_\theta)\cos v, \\ \kappa_\theta &= -e^{-u}[v_{\theta\theta}\cos v + (1 - v_\theta^2)\sin v], \end{aligned}$$

Thus curvature is positive provided that $v_\theta < 1$.

From this we can compute the evolution equation for v by differentiating the equation for u

$$\begin{aligned} u_t &= \beta\left(\frac{\mathcal{R}}{\kappa}\right)e^{-u}\sec v - \beta(\mathcal{R}) + f \\ v_t &= u_{x\tau}\cos^2 v \\ &= -\beta'\left(\frac{\mathcal{R}}{\kappa}\right)\frac{\kappa_\theta \mathcal{R} e^{-u}\cos v}{\kappa^2} - \beta\left(\frac{\mathcal{R}}{\kappa}\right)e^{-u}(1 - v_\theta)\sin v + f_\theta \\ &= \beta'\left(\frac{\mathcal{R}}{\kappa}\right)\frac{\mathcal{R}\left[v_{\theta\theta} + (1 - v_\theta^2)\tan v\right]}{(1 - v_\theta)^2} - \beta\left(\frac{\mathcal{R}}{\kappa}\right)e^{-u}(1 - v_\theta)\sin v + f_\theta \end{aligned}$$

Also we can write the evolution equation for $w = v_\theta$.

$$\begin{aligned} w_\tau \;=\; & \beta'\left(\frac{\mathcal{R}}{\kappa}\right)\frac{\mathcal{R}w_{\theta\theta}}{(1-w)^2} + \beta''\left(\frac{\mathcal{R}}{\kappa}\right)\frac{\mathcal{R}^2 e^u\left[w_\theta + (1 - w^2)\tan v\right]^2\sec v}{(1-w)^4} \\ & +\beta'\left(\frac{\mathcal{R}}{\kappa}\right)\frac{2\mathcal{R}\left[w_\theta + (1 - w^2)\tan v\right]}{(1-w)^3} \\ & +\beta'\left(\frac{\mathcal{R}}{\kappa}\right)\frac{\mathcal{R}\left[w_\theta + (1 - w^2)\tan^2 v - (w + w^2)\sec^2 v\right]}{1-w} \\ & +\beta\left(\frac{\mathcal{R}}{\kappa}\right)e^{-u}\left[(1 - w)\sin v\tan v + w_\theta\sin v - (1 - w)w\cos v\right] + f_{\theta\theta} \end{aligned}$$

A similar equation holds for $z = v_{\theta\theta}$

$$z_\tau = \beta'\left(\frac{\mathcal{R}}{\kappa}\right)\frac{\mathcal{R}z_{\theta\theta}}{(1-w)^2} - \beta\left(\frac{\mathcal{R}}{\kappa}\right)e^{-u}z\cos v + \cdots + f_{\theta\theta\theta}$$

where the dots correspond to terms which are at most first derivatives of w and that have coefficients involving derivatives of β, v or w. The pattern continues for more space and time derivatives. For example, the evolution of the time derivative $y = u_\tau$ satisfies

$$\begin{aligned} z_\tau \;=\; & \beta'\left(\frac{\mathcal{R}}{\kappa}\right)\frac{\mathcal{R}e^{-u}\left[y_{\theta\theta} + e^{-u}y(1 + u_\theta^2 - u_{\theta\theta})\right]}{\kappa^2(1 + u_\theta^2)} \\ & -\beta\left(\frac{\mathcal{R}}{\kappa}\right)ye^{-u}\sqrt{1 + u_\theta^2} + b_1 u_x y_\theta + b_2 e^{-k_1\tau} + f_\tau \end{aligned}$$

where $b_i(u, u_\theta, u_{\theta\theta}, \tau)$ are bounded functions.

We have an apriori gradient estimate.

Lemma 23. *Suppose $\beta \in C^1$ satisfies (84) and (88) and $f \in C^{1,0}(Q)$ so that*

$$|f|_0 < S_1, \qquad |f|_1 \leq c_7 e^{-k_2\tau}.$$

If $u \in C^{3,1}(Q)$ is a solution to (83) with initial condition $u(\theta, 0) = u_0(\theta)$ then if $|u_0|_1$ and c_2 are sufficiently small, then there is a bound on u_θ. There are constants τ_1, C_1,C_2 c_8, c_9 depending on c_4, k_1, R_0, S_0, S_1

and $|u_0|_0\ |u_{0,\theta}| \le C_1$ *and* $c_7 \le C_2$ *then* u_θ *is bounded and tends to zero exponentially. The estimate holds in two parts: For* $0 \le \tau \le \tau_1$,

$$\frac{|u_\theta|}{\sqrt{1+u_\theta^2}} \le \left(\frac{|u_{0,\theta}|_0}{\sqrt{1+|u_{0,\theta}|_0^2}} + \frac{c_7}{c_9+k_2} \right) e^{c_9\tau}.$$

For $\tau_1 \le \tau$,

$$\frac{|u_\theta|}{\sqrt{1+u_\theta^2}} \le \left(\frac{|u_{0,\theta}|_0}{\sqrt{1+|u_{0,\theta}|_0^2}} + \frac{c_7}{c_9+k_2} \right) e^{c_9\tau_1} e^{-c_8 S_0(\tau-\tau_1)/2} + c_7 \mathcal{W}(\tau)$$

where

$$\mathcal{W}(\tau) = \begin{cases} \dfrac{2e^{-k_2\tau}}{c_8 S_0 - 2k_2}, & \text{if } c_8 S_0 > 2k_2, \\ \tau e^{-2k_2\tau}, & \text{if } c_8 S_0 = 2k_2, \\ \dfrac{2e^{-k_2\tau_1} e^{-c_8 S_0(\tau-\tau_1)/2}}{2k_2 - c_8 S_0}, & \text{if } c_8 S_0 < 2k_2. \end{cases}$$

Proof. We define a supersolution M to (92) as the solution to an ODE with $M(0) = |u_0|_1$ and verify that $M \ge v$ using the maximum principle. Since $u(-\theta,\tau)$ solves the problem with f replaced by $f(-\theta,\tau)$, a lower bound is also given by $-M$. Let (θ_0,τ_0) be the first point where $v = M$ where $v_\theta = 0$ and $v_{\theta\theta} \le 0$. At such a point curvature is positive so we may write β. By Lemma (22.) there are constant so that $0 < c_8 \le e^{-u} \le c_8^{-1}$ and $\beta'\mathcal{R} \le c_4 e^{-k_1\tau}$. Hence (91) implies

$$\beta'(\mathcal{R}e^u \sec v)\mathcal{R} \quad \le \quad c_4 \left(\frac{R_0}{\mathcal{R}}\right)^{\frac{k_1}{S_1}} \left(\frac{\cos v}{c_8}\right)^{1+\frac{k_1}{S_1}} \le c_4 S \left(\frac{\cos v}{c_8}\right)^{1+\frac{k_1}{S_1}} e^{-k_1\tau}.$$

Thus at the maximum point there is c_9 so that

$$\begin{aligned} v_\tau &\le \beta'(\mathcal{R}e^u \sec v)\mathcal{R}\tan v - \beta e^{-u}\sin v + f_\theta \\ &\le c_9 e^{-k_1\tau}\sin v - c_8 S_0 \sin v + c_7 e^{-k_2\tau}. \end{aligned}$$

A simple estimate is to consider first τ in the interval $[0,\tau_1]$ where

$$2c_9 e^{-k_1\tau_1} = c_8 S_0.$$

If $\tau_1 < 0$ we set $\tau_1 = 0$ and skip to the second interval. On this interval, multiplying by cosine,

$$(\cos v)v_\tau \le c_9 \sin M + c_7 e^{-k_2\tau} = (\cos M)M_\tau$$

Integrating we find

$$\sin M \le \left(\sin(|v_0|_0) + \frac{c_7}{c_9+k_2}\right) e^{c_9\tau}.$$

For $|u_{0,\theta}|_0$ and c_7 sufficiently small $\sin M < 1$ and so u_θ remains finite. For concreteness, this holds provided $3|v_0|_0 e^{c_9\tau_1} \le 1$ and $3c_7 e^{c_9\tau_1} \le c_9 + k_2$.

For $\tau \ge \tau_1$ we have the inequality

$$(\cos v)v_\tau \le -\frac{1}{2}c_8 S_0 \sin M + c_7 e^{-k_2\tau} = (\cos M)M_\tau$$

whose integral satisfies in the notation of Lemma (22.),

$$\begin{aligned}\sin M &\le \sin(M(\tau_1))e^{-c_8 S_0(\tau-\tau_1)/2} + \\ &+c_7\left(E\left[\frac{c_8 S_0}{2}-k_2\right](\tau) - E\left[\frac{c_8 S_0}{2}-k_2\right](\tau_1)\right)e^{-c_8 S_0\tau/2}.\end{aligned}$$

This implies the estimate. Choosing c_7 smaller if necessary implies $M < 1$ for all time and $u_\theta \to 0$ exponentially. In particular we require that c_7 also satisfy $3c_7\tau_1 < e^{k_2\tau_1}$ if $c_8 S_0 = 2k_2$ and $6c_7 < |c_8 S_0 - 2k_2|e^{k_2\tau_1}$ otherwise. □

Along the same lines we show that convexity can be proved for solutions of the equation. We obtain higher derivative estimates.

Lemma 24. *For j, k natural numbers suppose $\beta \in C^{j+k}$ satisfies (84), (88) and*

$$\left|\left(\frac{\beta_\tau}{\beta}\right)_\tau\right|_{j,k} \le c_4 e^{-k_1\tau}. \tag{93}$$

Assume $f \in C^{j,1}(Q)$ so that

$$2|f|_0 < S_1, \qquad |f|_{j,k} \le c_7 e^{-k_2\tau}.$$

For any $0 < \delta < 1$ there are positive constants τ_1, C_1, C_2, C_3 and c_8, c_9, c_{11} depending on c_4, δ, k_1, R_0, S_0, S_1 and $|u_0|_1$ so that if $|v_0|_{j,1} \le C_1$, $c_7 \le C_2$ and $k_2 \le C_3$ then $v_\theta < \delta$ and tends to zero exponentially. c_{11} tends to zero as $C_1 + C_2 \to 0$. The estimate is given in two intervals. For $0 \le \tau \le \tau_1$,

$$|v|_{j,k} \le (|v_0|_{j,k} + c_{11})\, e^{c_9\tau}.$$

For $\tau_1 \le \tau$,

$$|v|_{j,1} \le (|v_0|_{j,k} + c_{11})\, e^{c_9\tau_1} e^{-c_8(\tau-\tau_1)/2} + c_{11}e^{-k_2\tau}\min\{\tau-\tau_1, 1\}.$$

Proof. We proceed as before one derivative at a time starting with the first. The ODE for a supersolution M will be valid as long as $M \le \delta$ (so that the curvature is positive.) We show that if the initial data and right side are small enough, $M \le \delta$ for all τ. By lemmas (22.) and (23.), for the correct choice of C_1 and C_2 we can arrange that

$$|e^u - 1| + |\tan v| \le c_5 e^{-k_2\tau} \le \delta.$$

Computing as before,

$$\beta''(\mathcal{R})\mathcal{R}^2 = \left(\frac{\beta_\tau}{\beta}\right)_\tau - \beta'(\mathcal{R})\mathcal{R}$$

thus,

$$\beta''\left(\frac{\mathcal{R}}{\kappa}\right)\mathcal{R}^2 \le c_6 e^{-k_1\tau}.$$

At a possible point (θ_0, τ_0) where a solution first touches $w(\theta_0,\tau_0) = M(\tau) \le \delta$ we have $w_\theta = 0$ and $w_{\theta\theta} \le 0$. The terms which aren't linear in w contain factors $\sin v$, $\tan v$ and $f_{\theta\theta}$ which decay $c_{10}e^{-k_2\tau}$ where c_{10} can be made small by requiring C_1 and C_2 to be small.The term $\beta e^{-u}(1-w)\cos^p v \ge c_8 > 0$ is uniformly positive. The remaining linear term is $\beta'\mathcal{R}\sec^2 v(1-w)^{-1}(1+w)w \le c_9 w e^{-k_1\tau}$. Thus at the maximum point,

$$w_\tau \le -c_8 w + c_9 w e^{-k_1\tau} + c_{10}e^{-k_2\tau}$$

which is exacly the differential inequality discussed in Lemma (23.).

The lower bound is obtained in the same way. δ is unnecessary as the denominators help in this case. The equation for the next higher derivative and time derivative has the same form and is handled in the same way, each time using the previous decaying terms to handle the junk terms. Since $j+k$ is finite, the procedure stops. □

It follows that for small data, an initially convex curve remains convex. Notice that for second derivatives and higher, the

Lemma 25. *Suppose $\beta \in C^1(I)$ so that $\beta' > 0$ and*

$$s \leq \frac{S_0}{\beta(\mathcal{R}(\tau))} \leq s + (1-s)e^{k_1\tau}$$

for all $\tau \in I$. Then

$$R_0 s^{S_1/k_1} e^{S_1\tau} \leq \mathcal{R} \leq R_0 e^{S_1\tau}$$

Proof. $\mathcal{R}$ satisfies

$$\frac{S_1 e^{k_1\tau}}{e^{k_1\tau} + S - 1} \leq \frac{d\mathcal{R}}{\mathcal{R}d\tau} = \beta(\mathcal{R}) \leq S_1$$

Thus integrating yields

$$\frac{S_1}{k_1} \ln\left(\frac{e^{k_1\tau} + S - 1}{S}\right) \leq \ln\frac{\mathcal{R}}{R_0} \leq S_1\tau$$

and the lemma follows. □

The condition corresponds to the Hölder continuity of α at zero. Note that if $\alpha(\kappa) = S_1 - a\kappa + \mathbf{o}(\kappa)$ is C^1 in a neighborhood of zero then β satisfies the better inequalities

$$S_1 - ae^{-S_1\tau} \cong \beta \geq \frac{S_1}{1 + (S-1)e^{-k_1\tau}} \geq S_1 - S_1(S-1)e^{-k_1\tau}$$

thus

$$k_1 \cong S_1.$$

12. Long time solvability the nonlinear equation.

In this section, we show that for sufficiently small c_1, the strength of the fluid effects compared to the combustion the nonlinear problem (83) admits an infinte time solution $u \in C^2(\mathbf{S}^1 \times [0,\infty)$ for arbitrary, but small initial data $u_0 \in C^2$. For some small constant k_2 depending on $\beta \in C^{2+\delta}[0,\infty)$ and c_1, we shall show that the solution satisfies $|u|_{2,1} \leq c_2 e^{-k_2\tau}$ for all $(\theta,\tau) \in Q = \mathbf{S}^1 \times [0,\infty)$. Let $Q_T = \mathbf{S}^1 \times [0,T]$. This implies that a solution is strongly asymptotically stable. In other words, in regimes where combustion dominates, any small perturbation of the rising circle is asymptotic in the sense that in the original time and radius variables,

$$\left|\frac{R(\theta,t) - \mathcal{R}(t)}{\mathcal{R}(t)}\right|_{2,1} \leq c_2 t^{-k_2/S_1}$$

as $t \to \infty$.

We prove that in any finite time interval, the Cauchy problem admits a solution. This is done by fixed point theorem in an appropriate weighted Hölder space. Since we have derived estimates independent of the time interval solutions dont degenerate with time.

We begin by defining our time weighted Banach spaces. Let $\wp \in C[0,\infty)$ be a weight function. We have in mind

$$\wp(\tau) = e^{-k_2\tau}$$

where $k_2 > 0$. Then define the weighted Banach space

$$C_{\wp}^{k+\delta,(k+\delta)/2}(Q) = \left\{ u \in C_{loc}^{k+\delta,(k+\delta)/2}(Q) : |u|_{\wp,k+\delta,(k+\delta)/2} < \infty \right\}$$

where the norm

$$|u|_{\wp,k+\delta,(k+\delta)/2,Q} := \sup_{\tau \geq 0} \wp(\tau)|u|_{C^{k+\delta,(k+\delta)/2}(\mathbf{S}^1 \times \{\tau\})}$$

The apriori estimates say that rate of decay of the solution is the least of the rate of the decay of the inhomogeneous term, and a rate determined by the equation. The constants k_2 is chosen to be smaller than the rates arising from finitely many estimates for $\partial_\theta^j \partial_\tau^k u \in C_{\wp}^{1+\delta,(1+\delta)/2}$. Then setting $f = \mathcal{N}[u]$ which is in $C_{\wp}^{1+\delta/2,1/2+\delta/4}$ then the etimates say that solutions of (94) satisfy $|w|_{\wp,j,k} \leq c_{13}|v|_{\wp,j,k}$ for $j = 0, 1, 2$ for some constant c_{13} depending on δ, β, δ, and R_0.

We state and prove the existence theorem.

Theorem 26. *Let $\beta \in C^{5+\delta}([0,\infty)$ satisfy conditions (84),*

$$\left| \frac{\beta_\tau}{\beta} \right|_5 \leq c_4 e^{-k_1 \tau}$$

where $0 < S_1 < S_2$ and $0 < c_4, k_1, R_0$. Then there is a constants c_3, c_4, c_5, c_6 depending on c_2, S_1, S_2, k_1, R_0 so that if $\delta \leq c_6$, $|u_0|_{2+\delta} \leq c_3$, $c_1 \leq c_4$ and $k_2 \leq c_5$ then there is a solution $u \in C_{\wp}^{2+\delta,1+\delta/2}(Q)$ to (83) such that $u(\theta, 0) = u_0(\theta)$.

Proof. Choose $T < \infty$ We set up a Shauder Fixed Point Theorem argument in Q_T. Let $v \in \mathcal{B}$ where $\mathcal{B} = \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} \leq r\}$ is a ball of radius $r \leq 1$ in $\mathcal{X} = C_{\wp}^{2+\delta,(1+\delta)/2}(Q_T)$. Define a mapping $\mathcal{T} : \mathcal{B} \to \mathcal{X}$ where $w = \mathcal{T}v$ is defined to be the unique solution of the inhomogeneous mean curvature flow equation

$$w_\tau = \alpha \left(\frac{\kappa[w]}{\mathcal{R}(\tau)} \right) e^{-w} \sqrt{1 + w_\theta^2} - \beta(\mathcal{R}(\tau) + c_1 \mathcal{N}[v](\theta, \tau). \tag{94}$$

Since $v \in C^{2+\delta,(1+\delta)/2}(Q_T)$ it follows from Lemma (21.) that $\mathcal{N}[v], (\mathcal{N}[v])_\theta \in C^{\epsilon,\epsilon/2}$ where $\epsilon = \delta/2$. Furthermore, $|\mathcal{N}[u]|_{\mathcal{X}} \leq c_5 r$ where c_5 is a universal constant. By lemmata (22.), and (23.), we know that any solution w remains bounded in C^1. Hence the equation is uniformly elliptic along solutions. Thus, by cutting off growth of $|w_\theta|$ above its bound in the equation, we obtain a uniformly elliptic equation whose ellipticity constants depend on T. By the existence theorem for the one dimensional fully nonlinear uniformly elliptic equations, (*e.g.* [10]), the equation has a solution $w \in C^{2+\gamma,1+\gamma/2}$ where γ depends on T. The equation for w_θ is now also uniformly elliptic. Again, by the unique existence theorem there is a solution $w_\theta \in C^{2+\gamma',1+\gamma'/2}(Q_T)$ which equals the derivative of w by uniqueness. γ' depends on T here also. Note that the time Hölder constant $< \mathcal{N} >_\tau^{(1+\gamma)/2} \leq c|u|_{2+\delta,1+\delta/2}$. Also since w_θ has a Hölder continuous time derivative, the left side of the equation for w is differentiable in time, thus the solution $w \in C^{3+\gamma,(3+\gamma)/2}$. In any case, the mapping $\mathcal{T}$ is compact.

Thus we need to prove that any fixed points of the mapping

$$\mathcal{T}u = c_1 \mathcal{N}[u]$$

are strictly within a fixed ball, and then the result follows from the Schauder fixed point theorem. The idea is to use bootstrapping on the solution to prove that u is sufficiently differentiable.

Since $u \in C_{\wp}^{2,1}$, equations for derivatives are uniformly parabolic on $[0,T]$. Start with $u \in C^{k+\gamma,(k+\gamma)/2}(\mathbf{S}^1 \times [T_1, T])$. We have $\mathcal{N}[u] \in C^{k-1+\gamma',(k+\gamma')/2}(\mathbf{S}^1 \times (T_1, T])$. This implies $u \in C^{k+1+\gamma'',(k+1+\gamma'')/2}(\mathbf{S}^1 \times [T_1+\epsilon, T])$. Repeating five times proves that $\partial_\theta^j \partial_\tau^k u$ for $j + k \leq 3$, $k \leq 2$ are $C^{2,1}(\mathbf{S}^1 \times [5\epsilon, T])$ and therefore the gradient estimates apply. The inital values for w,w_θ, $w_{\theta\theta}$ and therefore u_τ are given by v. The regularity at $\tau = 0$ is handled letting the flow smooth the solution for $\tau \in [0,1]$ and assuming $\gamma(1) \geq \delta$ and $\epsilon = 0.1$. Thus the fundamental inequality implies for constants c_9 depending on δ. β, R_0, k_1, $c-4$ that for $0 \leq \tau \leq 1$

$$|u|_{2+\delta,1+\delta/2} \leq |u_0|_{2+\delta} e^{c_9 \tau} + c_1 c_4 c_9$$

and

$$\sum_{j+k\leq 3,\ k\leq 2} |\partial_\theta^j \partial_\tau^k u|_{0,\mathbf{S}^1\times\{5\epsilon\}} \leq c_9 |u_0|_{2+\delta} + c_1 c_9.$$

For $\tau \geq 1$, using the the estimates on higher derivatives, Lemma (24.),

$$\begin{aligned} |u|_{2+\delta,1+\delta/2} &\leq \sum_{j+k\leq 3,\ k\leq 2} |\partial_\theta^j \partial_\tau^k u|_0 \\ &\leq c_9(|u_0|_{2+\delta} e^{c_9\tau} + c_1 c_4 c_9) e^{-S_1\tau + \min\{\tau-1, 1/k_1\}} + c_1 c_9 e^{-k_2}. \end{aligned}$$

In particular, for an appropriate choice of c_1 and $|u_0|_{2+\delta}$ small enough then

$$|u|_{\wp,2+\delta,1+\delta/2} < r$$

for any fixed point. □

13. Appendix A. Derivation of the linearized equation about the rising circle.

We wish to linearize the equations around the rising circle solution $\Gamma(t)$. Thus suppose $X(t,\theta;\varepsilon) \in C^1(\mathbf{R} \times \mathbf{S}^1 \times (-\delta,\delta))$ is a parameterization such that $\hat{X}(t,\theta;0) = X(t,\theta)$. Thus we replace $R(t)$ by $\hat{R} = R(t) + \varepsilon u(t,\theta)$ and take

$$\hat{X}(t,\theta;\varepsilon) = ((R(t) + \varepsilon u(t,\theta))\cos\theta, (R(t) + \varepsilon u(t,\theta))\sin\theta + F(t))$$

where F and R are as before. This is general since $\mathbf{n}$ is transverse to the circle at $\varepsilon = 0$. This fixes a parameterization for other ε. Let $\mathbf{t} = (-\sin\theta, \cos\theta)$ be the tangential vector field. Then the normal vector up to first order in ε,

$$\hat{\mathbf{n}} = \frac{(R+\varepsilon u)\mathbf{n} - \varepsilon u_\theta \mathbf{t}}{\sqrt{(R+\varepsilon u)^2 + \varepsilon^2 {u_\theta}^2}} = \mathbf{n} - \frac{\varepsilon u_\theta}{R}\mathbf{t} + \dots$$

so

$$\langle \hat{\mathbf{n}}, \partial_2 \rangle = \sin\theta - \frac{\varepsilon u_\theta}{R}\cos\theta + \dots$$

First we compute the linearization of $\mathcal{N}$. The normal velocity of $\hat{\Gamma}$ is

$$\hat{V} = \left\langle \hat{n}, \frac{\partial \hat{X}}{\partial t} \right\rangle = V + \varepsilon\left(u_t + \frac{g[\rho]\ell^2}{24\mu R} u_\theta \cos\theta \right) + \dots \tag{95}$$

Assume that the pressure is given by the harmonic functions of the form

$$\begin{aligned} \hat{P}_1 &= -g\rho_1 (r\sin\theta + F(t)) - \frac{g[\rho]R(t)^2}{2r}\sin\theta \\ &\quad + \varepsilon g[\rho] \sum_{k=1}^{\infty} r^{-k} \{a_k \cos k\theta + b_k \sin k\theta\} + \dots, \\ \hat{P}_2 &= -g\rho_1 (r\sin\theta + F(t)) - \frac{g[\rho]r}{2}\sin\theta + \\ &\quad + \varepsilon g[\rho] \left(c_0 + \sum_{k=1}^{\infty} r^k \{c_k \cos k\theta + d_k \sin k\theta\} \right) + \dots. \end{aligned}$$

Then the jump condition $[\hat{P}] = 0$ holds at $\varepsilon = 0$. Since

$$\begin{aligned}\hat{P}_2 - \hat{P}_1 &= -\frac{g[\rho]}{2}\left(r - \frac{R^2}{r}\right)\sin\theta + \varepsilon g[\rho]c_0 \\ &\quad + \varepsilon g[\rho]\sum_{k=1}^{\infty}\left\{c_k r^k - a_k r^{-k}\right\}\cos k\theta + \left\{d_k r^k - b_k r^{-k}\right\}\sin k\theta\end{aligned} \tag{96}$$

We also assume that $[\hat{P}]|_{r=\hat{R}} = 0$ for all ε, thus $d/d\varepsilon|_{\varepsilon=0}[\hat{P}] = 0$ implies

$$u\sin\theta = c_0 + \sum_{k=1}^{\infty}\left\{c_k R^k - a_k R^{-k}\right\}\cos k\theta + \left\{d_k R^k - b_k R^{-k}\right\}\sin k\theta \tag{97}$$

Now assume that we have the expansion

$$u(\theta) = u_0 + \sum_{k=1}^{\infty} u_k\cos k\theta + v_k\sin k\theta$$

then using identities like $2\cos k\theta\sin\theta = \sin(k+1)\theta - \sin(k-1)\theta$ to expand $u\sin\theta$ yields

$$\begin{aligned}2c_0 &= v_1, \\ 2Rc_1 - 2R^{-1}a_1 &= v_2, \\ 2R^j c_j - 2R^{-j}a_j &= v_{j+1} - v_{j-1} \qquad \text{for } j \ge 2, \\ 2R^j d_j - 2R^{-j}b_j &= u_{j-1} - u_{j+1} \qquad \text{for } j \ge 1.\end{aligned} \tag{98}$$

In polar coordinates, the gradient is $\nabla P = P_r\mathbf{n} + r^{-1}P_\theta\mathbf{t}$ so that

$$\left.\frac{\partial P}{\partial\hat{\mathbf{n}}}\right|_{r=\hat{R}} = \left\langle P_r\mathbf{n} + \frac{P_\theta}{r}\mathbf{t}, \mathbf{n} - \frac{\varepsilon u_\theta}{R}\mathbf{t} + \dots\right\rangle = P_r(\hat{R},\theta) - \frac{\varepsilon P_\theta(R,\theta)u_\theta}{R^2} + \dots$$

The first term is using (96), (97),

$$\begin{aligned}\left[\frac{\partial\hat{P}}{\partial r}\right] &= -g[\rho]\sin\theta + \frac{\varepsilon g[\rho]u}{R}\sin\theta \\ &\quad + \frac{\varepsilon g[\rho]}{R}\sum_{k=1}^{\infty}\left\{c_k R^k + a_k R^{-k}\right\}k\cos k\theta + \left\{d_k R^k + b_k R^{-k}\right\}k\sin k\theta \\ &= -g[\rho]\sin\theta + \frac{\varepsilon g[\rho]}{R}c_0 \\ &\quad + \frac{\varepsilon g[\rho]}{R}\sum_{k=1}^{\infty}\left\{(k+1)c_k R^k + (k-1)a_k R^{-k}\right\}\cos k\theta \\ &\quad + \frac{\varepsilon g[\rho]}{R}\sum_{k=1}^{\infty}\left\{(k+1)d_k R^k + (k-1)b_k R^{-k}\right\}\sin k\theta\end{aligned} \tag{99}$$

Since θ derivatives of (96) vanish at zeroth order at $r = \hat{R}$, by (99) and (96) the second summand of $[\partial\hat{P}/\partial\hat{\mathbf{n}}]$ vanishes up to the first oder term. Thus equating the jump conditions yields

$$-g[\rho]\langle\hat{\mathbf{n}}, \partial_2\rangle = -g[\rho]\left(\sin\theta - \frac{\varepsilon u_\theta}{R}\cos\theta + \dots\right) = \left[\frac{\partial\hat{P}}{\partial\hat{\mathbf{n}}}\right] = \left[\frac{\partial\hat{P}}{\partial r}\right] + \dots$$

Equating to (99) gives after expanding $u_\theta \cos\theta$,

$$\begin{aligned} 2c_0 &= v_1 \\ 2(j+1)R^j c_j + 2(j-1)R^{-j} a_j &= (j-1)v_{j-1} + (j+1)v_{j+1} \qquad \text{for } j \geq 1, \\ 2(j+1)R^j d_j + 2(j-1)R^{-j} b_j &= -(j-1)u_{j-1} - (j+1)u_{j+1} \end{aligned}$$

Solving these with (98) yields $a_1 = 0$, and for $j \geq 1$,

$$\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} = \frac{1}{2} \begin{pmatrix} R^j v_{j-1} & -R^j u_{j-1} \\ R^{-j} v_{j+1} & -R^{-j} u_{j+1} \end{pmatrix}$$

where we interpret $v_0 = 0$. Thus the harmonic functions take the form

$$\begin{aligned} \hat{P}_1 &= -g\rho_1 \left(r \sin\theta + F(t)\right) - \frac{g[\rho] R(t)^2}{2r} \sin\theta \\ &\quad + \frac{\varepsilon g[\rho]}{2} \left(-\frac{u_0 R}{r} \sin\theta + \sum_{k=2}^{\infty} \frac{R^k}{r^k} \left\{ v_{k-1} \cos k\theta - u_{k-1} \sin k\theta \right\} \right) + \ldots, \\ \hat{P}_2 &= -g\rho_1 \left(r \sin\theta + F(t)\right) - \frac{g[\rho] r}{2} \sin\theta \\ &\quad + \frac{\varepsilon g[\rho]}{2} \left(v_1 + \sum_{k=1}^{\infty} \frac{r^k}{R^k} \left\{ v_{k+1} \cos k\theta - u_{k+1} \sin k\theta \right\} \right) + \ldots. \end{aligned}$$

Finally, we get an expression for the hydrodynamic normal velocity

$$\begin{aligned} \hat{V}_h &= \langle \hat{\mathbf{n}}, \hat{\mathbf{w}} \rangle = -\frac{\ell^2}{12\mu} \left(\frac{\partial \hat{P}_2}{\partial \hat{\mathbf{n}}} + g\rho_2 \langle \hat{\mathbf{n}}, \partial_2 \rangle \right) \\ &= -\frac{g[\rho]\ell^2}{24\mu} \left(\sin\theta - \frac{\varepsilon u_\theta}{R} \cos\theta + \frac{\varepsilon}{R} \sum_{k=1}^{\infty} \left\{ v_{k+1} k \cos k\theta - u_{k+1} k \sin k\theta \right\} \right) \\ &= -\frac{g[\rho]\ell^2}{24\mu} \left(\sin\theta - \frac{\varepsilon}{2R} v_1 \right) \\ &\quad - \frac{\varepsilon g[\rho]\ell^2}{48\mu R} \sum_{k=2}^{\infty} (k-1) \left\{ (v_{k+1} - v_{k-1}) \cos k\theta - (u_{k+1} - u_{k-1}) \sin k\theta \right\} \end{aligned}$$

Let U be the harmonic function in the unit disk such that $U = u$ on $\mathbf{S}^1$. Let $\tilde{U}$ be a harmonic conjugate so that $\Phi := U + \tilde{U} i$ is analytic in the disk. The conjugate function of $\tilde{u}$ is the restriction of $\tilde{U}$ to the unit circle. It is defined for $u \in L^2(\mathbf{S}^1)$ and satisfies $\|\tilde{u}\| = \|u - u_0\|$. It is given by the Hilbert transform or series

$$\begin{aligned} \tilde{u} &= \frac{1}{2\pi}\, PV \int_{\mathbf{S}^1} u(\theta - \sigma) \cot\left(\frac{\sigma}{2}\right) d\sigma, \\ &= \sum_{k=1}^{\infty} u_k \sin k\theta - v_k \cos k\theta \end{aligned}$$

where the integral is in the principal value sense. $\tilde{u}$ is defined for $u \in L^1$ [7] and by Privalov's theorem, $\tilde{u} \in C^{k+\alpha}$ for $u \in C^{k+\alpha}$ [19]. Applied to our equation, the normal velocity may be expressed

$$\hat{V}_h = -\frac{g[\rho]\ell^2}{24\mu} \sin\theta + \frac{\varepsilon g[\rho]\ell^2}{24\mu R} \left(u_\theta \cos\theta - \frac{\partial}{\partial\theta} \left(\tilde{u} \sin\theta + (u - u_0) \cos\theta \right) \right) + \ldots$$

Consider the curvature term. In these coordinates,

$$\begin{aligned}\hat{\kappa} &= \frac{-\hat{R}\hat{R}_{\theta\theta}+2\hat{R}_\theta{}^2+\hat{R}^2}{\left(\hat{R}^2+\hat{R}_\theta{}^2\right)^{3/2}}\\ &= \frac{-\varepsilon(R+\varepsilon u)u_{\theta\theta}+2\varepsilon^2 u_\theta{}^2+(R+\varepsilon u)^2}{((R+\varepsilon u)^2+\varepsilon^2 u_\theta{}^2)^{3/2}}\\ &= \frac{1}{R}-\frac{u_{\theta\theta}+u}{R^2}\varepsilon+\dots\end{aligned}$$

Thus

$$\hat{V}_c=\mathcal{L}(\Gamma)=\alpha(\kappa)=\alpha\left(\frac{1}{R}\right)-\alpha'\left(\frac{1}{R}\right)\frac{(u_{\theta\theta}+u)}{R^2}\varepsilon+\dots.$$

Using (95)

$$\frac{\partial u}{\partial t}=\mathcal{V}u=-\frac{\alpha'(1/R)}{R^2}\left(\frac{\partial^2 u}{\partial\theta^2}+u\right)-\frac{g[\rho]\ell^2}{24\mu R}\frac{\partial}{\partial\theta}\left(\tilde{u}\sin\theta+(u-u_0)\cos\theta\right). \tag{100}$$

14. Appendix: Eigenvalues of the linearized operator and integral equations.

Now we shall estimate eigencvalues of (29). Suppose y is any nonzero eigenfunction corresponding to a real eigenvalue. If y is constant then $\lambda=-1$. If nonconstant, multiplying the equation and integrating

$$\begin{aligned}0 &= \int (y''+c_1\mathcal{M}y+(1+\lambda)y)\,y\\ &= -\int (y')^2+c_1\int y\mathcal{M}y+(1+\lambda)\int y^2\\ &\le -\|y'\|^2+c_1\|y\|\|y'\|+(1+\lambda)\|y\|^2\\ &\le -\|y'\|^2+\|y'\|^2+\frac{c_1{}^2}{4}\|y\|^2+(1+\lambda)\|y\|^2\end{aligned}$$

hence the eigenvalues are bounded below by $-1-c_1^2/4\le\lambda$.

We argue that the operator $L=\beta'(\partial_{\theta\theta}+1)+cR^{-1}\mathcal{M}$ is sectorial as a map $C^{2+\alpha}\to C^\alpha$ for each t. Making the change of variables as in (18), we show that $\mathcal{V}=\partial_{\theta\theta}+g\mathcal{M}$ is sectorial. Thus assume that $v\in C^{2+\alpha}\subset H^1$ is a complex eigenfunction

$$v_{\theta\theta}+g\mathcal{M}v+\lambda v=0$$

with eigenvalue λ.

$\mathcal{V}$ is a real operator $\mathcal{V}\bar{v}=\overline{\mathcal{V}v}$ but it isn't necessarily self adjoint. Thus, multiplying the eigenequation by the complex conjugate and the conjugated equation by the function and integrating we get,

$$\begin{aligned}\int \bar{v}v_{\theta\theta}+g\bar{v}\mathcal{M}v\,d\theta &= -\lambda\int\bar{v}v\,d\theta,\\ \int v\bar{v}_{\theta\theta}+gv\mathcal{M}\bar{v}\,d\theta &= -\bar{\lambda}\int\bar{v}v\,d\theta.\end{aligned}$$

Subtracting, using $v_0=0$ and $\|v\mathcal{M}v\|\le\|v_\theta\|\,\|v-v_0\|$, we get

$$\begin{aligned}\|v_\theta\|^2-g\|v_\theta\|\,\|v\| &\le \int \bar{v}_\theta v_\theta-g\Re e(\bar{v}\mathcal{M}v)\,d\theta=\Re e\lambda\|v\|^2\\ g\|v_\theta\|\,\|v\| &\ge g\left|\int \Im m\,(\bar{v}\mathcal{M}v)\;d\theta\right|=|\Im m\lambda|\|v\|^2.\end{aligned}$$

Thus using Wirtinger's inequality $\|v-v_0\| \le \|v_\theta\|$ we have for any $\varepsilon > 0$,

$$\begin{aligned}
|\Im m\lambda| \|v\|^2 &\le g\|v_\theta\| \, \|v\| \\
&\le \varepsilon \left(\|v_\theta\|^2 - g\|v_\theta\| \, \|v\| \right) + \frac{(1+\varepsilon)^2}{4\varepsilon} g^2 \|v\|^2 \\
&\le \left(\varepsilon \Re e \lambda + \frac{(1+\varepsilon)^2 g^2}{4\varepsilon} \right) \|v\|^2.
\end{aligned}$$

Minimizing over $\varepsilon > 0$ yields $4\Re e\lambda \ge -g^2$ and

$$2|\Im m\lambda| \le g^2 + g\sqrt{g^2 + 4\Re e\lambda}.$$

The discreteness of the spectrum depends on Fredholm properties of the corresponding integral equations. The methods of Sherman and Vekua, which coincide on the circle, are to represent the unknown function by the Cauchy type integral.

$$\Phi(z) = \mathcal{T}(\mu, c) = \frac{1}{2\pi i} \int_\gamma \frac{\mu(\sigma)}{\zeta^2} (\zeta - z) \ln \left(1 - \frac{z}{\zeta} \right) d\zeta + \frac{1}{2\pi} \int_\gamma \mu(\sigma) \, d\sigma + ci$$

where $\mu(\sigma)$ is an unknown Hölder continuous real function of arclength, $\zeta = e^{i\sigma}$ and c is a real constant. By Vekua's Theorem ([14], p.192), the second derivative of this function has Hölder continuous boundary values on $\mathbf{S}^1$ and μ and c are uniquely determined by Φ. This operator has a kernel function which is continuous up to the circle. Then for $|z| < 1$,

$$\begin{aligned}
\Phi'(z) &= -\frac{1}{2\pi i} \int_\gamma \frac{\mu(\sigma)}{\zeta^2} \left[1 + \ln \left(1 - \frac{z}{\zeta} \right) \right] d\zeta, \\
\Phi''(z) &= \frac{1}{2\pi i} \int_\gamma \frac{\mu(\sigma) \, d\zeta}{\zeta^2 (\zeta - z)}.
\end{aligned}$$

The first integral has an integrable kernel function with logarithmic singularity. Thus the limit $z \to z_0 \in \gamma$ is obtained by substituting z_0 for z. The second is singular. However, in the Hilbert problem, this reduces also to a Fredholm type kernel function. Inserting the boundary values into (31), using the Sokhotski-Plemelj formula for the boundary values of the leading order term yields

$$\Re e[z^2 \Phi''(z)] = \Re e \left[\frac{\mu(\theta)}{2} + \frac{1}{2\pi i} \int_\gamma \frac{z^2 \mu(\sigma) \, d\zeta}{\zeta^2 (\zeta - z)} \right].$$

This is transformed to a regular integral using

$$\int_\gamma \frac{z^2 \mu(\sigma) \, d\zeta}{\zeta^2 (\zeta - z)} = \int_\gamma \frac{z^2 - \zeta^2}{\zeta^2 (\zeta - z)} \mu(\sigma) \, d\zeta + \int_\gamma \frac{\mu(\sigma)}{\zeta - z} \, d\zeta.$$

But the real part of the Cauchy formula may be expressed as a double layer potential (Poisson kernel function)

$$\Re e \left[\frac{1}{2\pi i} \int_\gamma \frac{\mu(\sigma)}{\zeta - z} \, d\zeta \right] = \frac{1}{2\pi} \int_\gamma \mu(\sigma) \frac{\cos \angle(\mathbf{n}, \mathbf{r})}{r} \, d\sigma$$

where $\mathbf{r} = z - \zeta$. This operator is also Fredholm. Thus as $z \to \gamma$,

$$\begin{aligned}
\Re e \left[\frac{1}{2\pi i} \int_\gamma \frac{\mu(\sigma)}{\zeta - z} \, d\zeta \right] &\to \frac{1}{2\pi} \int_\gamma \cos \left(\frac{\theta - \sigma}{2} \right) \mu(\sigma) \, d\sigma, \\
\Re e \left[\frac{1}{2\pi i} \int_\gamma \frac{z^2 - \zeta^2}{\zeta^2 (\zeta - z)} \mu(\sigma) \, d\zeta \right] &\to -\frac{1}{2\pi} \int_\gamma \cos^2 \left(\frac{\theta - \sigma}{2} \right) \mu(\sigma) \, d\sigma.
\end{aligned}$$

Also

$$
\begin{aligned}
1+\ln\left(1-\frac{z}{\zeta}\right) &= \ln\left[2e\sin\frac{|\theta-\sigma|}{2}\right]+i\,\mathrm{sgn}(\theta-\sigma)\left(\frac{\pi}{2}-\frac{|\theta-\sigma|}{2}\right),\\
\left(1-\frac{z}{\zeta}\right)\ln\left(1-\frac{z}{\zeta}\right) &= 2\sin\frac{|\theta-\sigma|}{2}\left\{\sin\frac{|\theta-\sigma|}{2}\ln\left(2\sin\frac{|\theta-\sigma|}{2}\right)\right.\\
&\qquad \left.-\cos\frac{|\theta-\sigma|}{2}\left(\frac{\pi}{2}-\frac{|\theta-\sigma|}{2}\right)\right\}\\
&+\ 2i\sin\frac{\theta-\sigma}{2}\left\{\ln\left(2\sin\frac{|\theta-\sigma|}{2}\right)\cos\frac{|\theta-\sigma|}{2}\right.\\
&\qquad \left.+\sin\frac{|\theta-\sigma|}{2}\left(\frac{\pi}{2}-\frac{|\theta-\sigma|}{2}\right)\right\}.
\end{aligned}
$$

This means that we get a Fredholm integral equation for μ. (31).

$$\mathcal{N}\mu := \mu(\theta)+\int_\gamma K(\theta,\sigma;c,\lambda)\mu(\sigma)\,d\sigma = 0 \tag{101}$$

where μ and c are unknowns. The adjoint integral equation is defined to be

$$\mathcal{N}^*\nu := \nu(\theta)+\int_\gamma K(\sigma,\theta;c_1,\lambda)\nu(\sigma)\,d\sigma. \tag{102}$$

We can be more specific about the dependence on constants in the kernel function K. In particular, K may be decomposed

$$K(\theta,\sigma;c_1,\lambda)=K_1(\theta,\sigma)+c_1K_2(\theta,\sigma)+(1+\lambda)K_3(\theta,\sigma).$$

For $w = z/\zeta$ it is convenient to observe that analytic functions f satisfy $f(\bar w)=\overline{\bar f(w)}$. For the functions we have, the coefficients of the power series are real, thus $\bar f = f$ and we may deduce symmetry properties. The kernel functions K_1 and K_3 are symmetric and $K_2(\theta,\sigma) = -K_2(\sigma,\theta)$ is antisymmetric. The Fredholm theorems apply to these operators. $\eta\in\mathbf{C}$ denotes an eigenvalue of $\mathcal{N}$ whenever there is a nontrivial null space $\mathcal{E}_\eta$ of the operator $\mathcal{N}+\eta$. The Fredholm theory tell us that there are at most countably many eigenvalues η and that each eigenspace $\mathcal{E}_\eta$ is finite dimensional. Moreover, the set of eigenvalues is discrete and can only accumulate at -1 as formulated. The dimensions of the eigenspaces of the equation and adjoint equation coincide, so that if η is an eigenvalue of $\mathcal{N}$ it is an eigenvalue of $\mathcal{N}^*$. The alternative holds. If η is not an eigenvalue of $\mathcal{N}'$ then $\mathcal{N}\mu+\eta\mu = f(\theta)$ can be solved uniquely for any f. If η is an eigenvalue, then it can be solved only if $f\perp\mathcal{E}_\eta$ and the dimension of the solution space equals $\dim\mathcal{E}_\eta$.

Theorem 27. *The set of eigenvalues λ of the equation (29) forms a countable discrete set of real numbers which tend to $+\infty$. For each eigenvalue λ there is a finite dimensional space of eigensolutions.*

Proof. The relation between solutions is the following. If Hölder y solves (29), then $\Phi = Y + i\tilde Y$ solves (31) where by choice, $\tilde y$ has zero average so $\tilde Y(0) = 0$ so $\Phi = \mathcal{T}(\mu, 0)$. Now $\Phi = \mathcal{T}(\mu, c)$ is a one to one correspondence such μ is Hölder. Thus solutions of (29) are in one to one correspond to solutions of (101) via $y = \Re e\,\mathcal{T}(\mu, 0)$. If zero is not an eigenvalue of $\mathcal{N}'$, then the equation $\mathcal{N}\mu = 0$ has only zero solution.

Let

$$
\begin{aligned}
\mathcal{K}_1\mu(\theta) &:= \mu(\theta)+\int_\gamma\left(K_1(\theta,\sigma)+c_1K_2(\theta,\sigma)-c_1^2K_3(\theta,\sigma)\right)\mu(\sigma)\,d\sigma,\\
\mathcal{K}_2\mu(\theta) &:= \int_\gamma K_3(\theta,\sigma)\mu(\sigma)\,d\sigma.
\end{aligned}
$$

Consider the equation $\mathcal{K}\mu = f$. The solubility depends on the $\mathcal{K}'$ whose kernel function is by symmetry $K_1 - c_1 K_2 - c_1^2 K_3$. In particular this is the integral equation corresponding to the differential equations

$$y'' - c_1 \mathcal{M} y - c_1^2 y = 0.$$

Multiplying by y, integrating,

$$\begin{aligned} 0 &= -\|y'\|^2 - c_1 \int_\gamma y \mathcal{M} y - c_1^2 \|y\|^2 \\ &\leq -\|y'\|^2 + c_1 \|y\| \|y'\| - c_1^2 \|y\|^2 \end{aligned}$$

which implies y is constant zero. Thus $\mathcal{K}$ is invertible with a resolvent which is also a Fredholm operator. Our eigenvalue problem is equivalent to the Fredholm equation

$$\mu + (c_1^2 + 1 + \lambda)\mathcal{K}_1^{-1}\mathcal{K}_2 \mu = 0.$$

Thus the set of eigenvalues of $\mathcal{K}_1^{-1}\mathcal{K}_2$ is a discrete sequence tending to infinity and eigenspaces are finite dimensional. Moreover, there is a Fredholm alternative [5].

We confirmed the reality assumption by the Hilbert boundary problem solution. The lower estimate then tells us the eigenvalues run off to infinity.

References

[1] M. Abid & P. D. Romney, *Propogation rates of buouyant chemical fronts in aqueous solution,* Fall technical meeting, Combustion Institute, Western States Section, Los Angeles, CA, Oct. 28–29, 1996.

[2] P. Constantin, P. D. Lax, & A. Majda *A simple one-dimensional model for the three dimensional vorticity equation,* Comm. Pure Appl. Math, **36**, (1985), pp. 715–724.

[3] G. Folland, *Introduction to partial differential equations,* Princeton University Press, Princeton, NJ, 1976, p. 178.

[4] A. Friedman, *Partial differential equatins of parabolic type,* Prentice Hall, Englewood Cliffs, NJ., 1964; republ. by Krieger, Malabar, FL., 1983.

[5] F. D. Gakhov, *Boundary value problems,* Fizmatgiz, Moscow, 1963 (Russian); English trans. orig. publ. Pergammon, New York, 1966; republ. by Dover, Mineola, NY., 1990.

[6] G. H. Hardy, J. E. Littlewood & G. Pólya, *Inequalities,* Cambridge University Press, Cambridge, 1991, 2nd ed., p. 261.

[7] Y. Katznelson, *An introduction to harmonic analysis,* Wiley, New York, 1968; corrected republ. by Dover, New York, 1976, p. 79.

[8] K. Knopp, *Theory and application of infinite series,* 2nd. ed., Blackie & Son Ltd., London, 1951, republ. Dover, Mineola, N.Y., 1990, p. 469.

[9] N. V. Krylov *Lectures on elliptic and parabolic equations in Hölder Spaces,* American Mathematical Society, Providence, 1996.

[10] G. M. Lieberman, *Second order parabolic equations,* World Scientific, Singapore,1996.

[11] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhäuser, Basel, 1995.

[12] O. A. Layženskaja,, V.A. Sollonikov & N. N. Uralceva, *Linear and quasilinear equations of parabolic type,* Nauka, Moscow, 1967 (Russian); English trans. American mathematical Society, Providence, 1968.

[13] R. Mc Owen, *Partial differential equations: methods and applications,* Prentice Hall, Upper Saddle River, N. J., 1996.

[14] N. I. Muskhelishvili, *Singular integral equations: boundary problems of function theory and their application to mathematical physics,* 2nd ed., Gostekhizdat, Moscow, 1946 (Russian); English trans. orig. publ. Noordhoff & Gronigen, Holland, 1953; republ. by Dover, Mineola, NY., 1992.

[15] S. Saks & A. Zygmund, *Analytic Functions,* 3rd ed., Elsevier Publishing - Polish Scientific Publishers, New York, 1971; p. 324.

[16] A. N. Varchenko & P. I. Etingof *Why the boundary of a round drop becomes a curve of order four,* University Lecture Series 3, Americam Mathematical Society, Providence, 1992.

[17] I. N. Vekua, *Generalized analytic functions,* Fizmatgiz, Moscow, 1959 (Russian); Engl. trans. Pergammon Press Ltd., London, 1962.

[18] J. Zhu, *A numerical study of chemical front propogation in a Hele-Shaw flow under Buouyancy effects.* Physics of Fluids, 1998.

[19] A. Zygmund, *Trigonometric series* 2nd. ed., Cambridge University Press, Cambridge, 1979, p. 115–121.

Department of Mathematics
University of Utah
Salt Lake City, UTAH 84112-0090
`treiberg@math.utah.edu`