Steiner Symmetrization and Applications

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`http://www.math.utah.edu/treiberg/Steiner/SteinerSlides.pdf`

3. Outline.

- Steiner Symmetrization.
- Symmetrization preserves area.
- Symmetrization reduces diameter.
- Isodiametric inequality.
- Symmetrization reduces length.
- Steiner's proof of the isoperimetric inequality.
- Existence of a solution of the isoperimetric problem.
- Other Geometric Problems solved by symmetrization.
- Proof that a circular tambourine has the lowest bassnote among tambourines of the same area.

4. Steiner Symmetrization setup

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise C^1 boundary. Let $L^{n-1} \subset \mathbb{R}^n$ be a hyperplane through the origin.

Rotate space so that L is the $x_n = 0$ hyperplane. For each $\mathbf{x} \in L$ let the perpendicular line through $x \in L$ be

$$G_x = \{\mathbf{x} + y\mathbf{e}_n : y \in \mathbb{R}\}.$$

Let $m_x = |\Omega \cap G_x|$ be the measure (total length) of the slice.
Replacing the slices by intervals centered on the L with the same length yields the symmetrized domain

$$S_L(\Omega) = \left\{\mathbf{x} + y\mathbf{e}_n : \mathbf{x} + z\mathbf{e}_n \in \Omega \text{ for some } z \text{ and } -\tfrac{1}{2}m_x \le y \le \tfrac{1}{2}m_x\right\}.$$

Let $\Pi : \mathbb{R}^n \to L$ be the orthogonal projection.

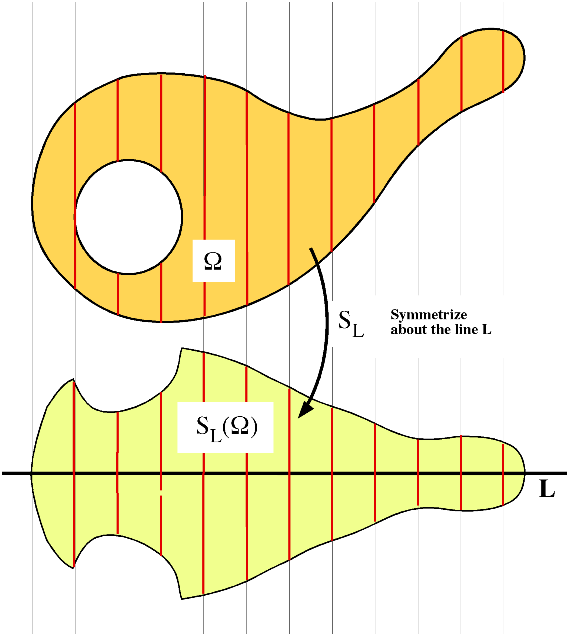

Ω
S_L
Symmetrize
about the line L
S_L(Ω)
L

Figure: Jakob Steiner (1796–1863)

"Calculating replaces thinking while geometry stimulates it".

Jakob Steiner, a self made Swiss farmer's son and contemporary of Gauss was the foremost "synthetic geometer." He hated the use of algebra and analysis and distrusted figures. He proposed several arguments to prove that the circle is the largest figure with given boundary length. Besides symmetrization, his four-hinge method has great intuitive appeal, but is limited to two dimensions. He published several proofs trying to avoid analysis and the calculus of variations.

Let $\omega = \Pi(\Omega)$ be the projection in L. The result follows from Fubini's Theorem.

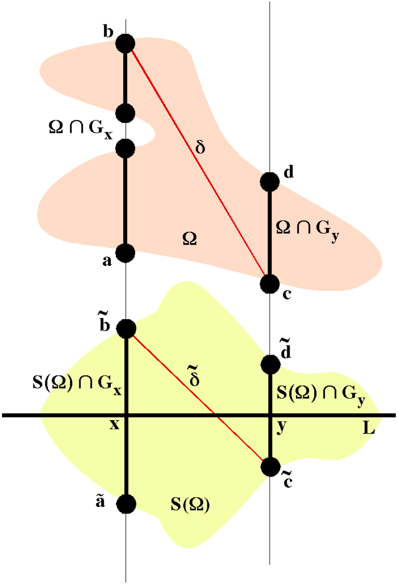
$$\begin{aligned} V(\Omega) &= \int_{(x_1,\dots,x_{n-1})\in\omega} \left(\int_{x_n \in G_x \cap \Omega} dx_n \right) dx_1\, dx_2 \cdots dx_{n-1} \\ &= \int_{(x_1,\dots,x_{n-1})\in\omega} m_{(x_1,\dots,x_{n-1,0})}\, dx_1\, dx_2 \cdots dx_{n-1} \\ &= \int_{(x_1,\dots,x_{n-1})\in\omega} \left(\int_{x_n \in G_x \cap S(\Omega)} dx_n \right) dx_1\, dx_2 \cdots dx_{n-1} \\ &= V(S_L(\Omega)). \end{aligned}$$

The diameter is the largest distance between points.

$$\operatorname{diam}(\Omega) = \sup_{\mathbf{x},\mathbf{y}\in\Omega} d(\mathbf{x},\mathbf{y}).$$

Choose $(\mathbf{x}, z)$ and $(\mathbf{y}, w)$ in $S(\Omega)$ where $\mathbf{x}, \mathbf{y} \in L$.
To see that their distance apart is at most $\operatorname{diam}(\Omega)$, we must show that

$$\begin{aligned}
\tilde{\delta}^2 &= d\big((\mathbf{x}, z), (\mathbf{y}, w)\big)^2 \\
&\le \left(\frac{m_x + m_y}{2}\right)^2 + \|\mathbf{x} - \mathbf{y}\|^2 \\
&\le \sup_{\substack{\mathbf{p} \in G_x \cap \Omega, \\ \mathbf{q} \in G_y \cap \Omega}} d(\mathbf{p}, \mathbf{q})^2 \\
&= \delta^2 \\
&\le \operatorname{diam}(\Omega)^2
\end{aligned}$$



Let $a = min(G_x \cap \Omega)$,
$b = \max(G_x \cap \Omega)$, $c = min(G_y \cap \Omega)$,
$d = \max(G_x \cap \Omega)$ so that

$$\delta^2 = (b-d)^2 + \|\mathbf{x}-\mathbf{y}\|^2 \leq \operatorname{diam}(\Omega)^2.$$

This is the longer diagonal so
$(b-c)^2 \geq (d-a)^2$.
By symmetrizing,
$2\tilde{b} = -2\tilde{a} = m_x \leq b - a$ and
$2\tilde{d} = -2\tilde{c} = m_y \leq d - c$
Thus

$$\begin{aligned} 4\tilde{\delta}^2 &= \left(2\tilde{b} + 2\tilde{d}\right)^2 + 4\|\mathbf{x}-\mathbf{y}\|^2 \\ &\leq (b-a+d-c)^2 + \|\mathbf{x}-\mathbf{y}\|^2 \\ &\leq (2[b-c])^2 + 4\|\mathbf{x}-\mathbf{y}\|^2 = 4\delta^2. \qquad \square \end{aligned}$$

Theorem (Bieberbach, 1915)

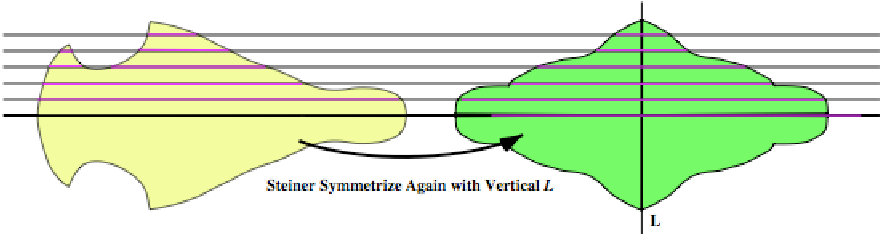
Let $K \subset \mathbb{R}^n$ be a compact domain. Then the volume satisfies

$$|K| \leq \frac{|B_1| \, diam(\Omega)^n}{2^n}. \tag{1}$$

Equality holds if and only if K is a closed ball of radius $\frac{1}{2}\, diam(\Omega)$.

Luen-Fai Tam's proof of (1) requires finitely many symmetrizations. Choose a family of n mutually perpendicular hyperplanes $\{L_i\}$ (*e.g.*, the coordinate hyperplanes) and perform Steiner symmetrizations.

$$K_n = S_{L_n} \circ S_{L_{n-1}} \circ \cdots \circ S_{L_1}(K)$$



These n symmetrizations result in a star-shaped K_n that is a centrally symmetric body. Since $d(\mathbf{x}, -\mathbf{x}) \leq \operatorname{diam}(\Omega)$ for every $\mathbf{x} \in K_n$ it follows that $K_n \subset \overline{B_r(0)}$ closed ball with radius $r = \frac{1}{2}\operatorname{diam}(\Omega)$, and (1) follows.

The proof that equality implies Ω is a ball is similar to the proof of equality in the isoperimetric inequality, which shall be discussed later. $\square$

Among all domains in the plane with a fixed boundary length, the circle has the greatest area. For simplicity we focus on domains bounded by simple curves.

Theorem (Isoperimetric Inequality)

- *Let $\gamma \subset \mathbf{E}^2$ be a simple closed curve in the plane whose length is L and that encloses an area A. Then the following inequality holds*

$$4\pi A \le L^2. \tag{2}$$

- *If equality holds in (2), then the curve γ is a circle.*

Simple means curve is assumed to have no self intersections.
A circle of radius r has $L = 2\pi r$ and encloses $A = \pi r^2 = \frac{L^2}{4\pi}$.
Thus II says if γ is a simple closed curve of length L and encloses an area A, then γ encloses an area no bigger than the area of the circle with the same length.

A set $K \subset \mathbf{E}^2$ is convex if for every pair of points $x, y \in K$, the straight line segment $\overline{xy}$ from x to y is also in K, *i.e.*, $\overline{xy} \subset K$.
The bounding curve of a convex set is automatically rectifiable. The *convex hull* of K, denoted $\hat{K}$, is the smallest convex set that contains K. This is equivalent to the intersection of all halfspaces that contain K,

$$\hat{K} = \bigcap_{\substack{C \text{ is convex} \\ C \supset K}} C = \bigcap_{\substack{H \text{ is a halfspace} \\ H \supset K}} H.$$

A halfspace is a set of the form $H = \{(x, y) \in \mathbf{E}^2 : ax + by \leq c\}$, where (a, b) is a unit vector and c is any real number.

Since $K \subset \hat{K}$ by its definition, we have $\hat{A} \geq A$.
Taking convex hull reduces the boundary length because the interior segments of the boundary curve, the components of $\gamma - \partial\hat{K}$ of γ are replaced by straight line segments in $\partial\hat{K}$.

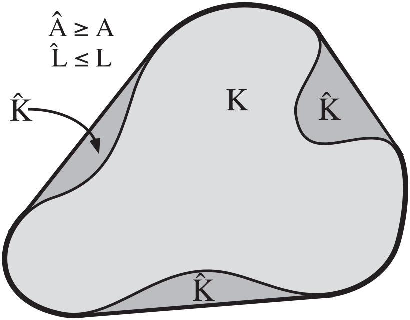


Figure: The region K and its convex hull $\hat{K}$.

Thus the isoperimetric inequality for convex sets implies

$$4\pi A \leq 4\pi \hat{A} \leq \hat{L}^2 \leq L^2.$$

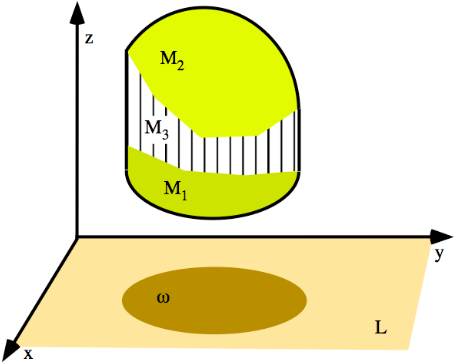
Furthermore, one may also argue that if $4\pi A = L^2$ implies $4\pi \hat{A} = \hat{L}^2$ so that if the second part of the isoperimetric inequality holds and $\hat{K}$ is a circle, then so is K. The basic idea is to consider the the extreme points $\partial^* \hat{K} \subset \partial \hat{K}$ of $\hat{K}$, that is points $x \in \partial \hat{K}$ such that if $x = \lambda y + (1 - \lambda) z$ for some $y, z \in \hat{K}$ and $0 < \lambda < 1$ then $y = z = x$. $\hat{K}$ is the convex hull of its extreme points. However, the extreme points of the convex hull lie in the curve $\partial^* \hat{K} \subset \gamma \cap \partial \hat{K}$. $\hat{K}$ being a circle implies that every boundary point is an extreme point, and since they come from γ, it means that γ is a circle.

Theorem

Let $K \subset \mathbb{R}^n$ be a compact set and S_L Steiner symmetrization with respect to a hyperplane L.

- *If $K_1 \subseteq K_2$ then $S_L(K_1) \subseteq S_L(K_2)$.*
- *If $0 \in L$ then $S_L(B_r(0)) = B_r(0)$.*
- *If $0 \in L$ and $\lambda > 0$ then $S_L(\lambda K) = \lambda S_L(K)$.*
- *If K is convex the $S_L(K)$ is convex.*
- *If K is a polyhedron then so is $S_L(K)$.*
- *If K has piecewise smooth boundary then so does $S_L(K)$.*

For simplicity, we assume that $K \subset \mathbb{R}^3$ is convex, compact, positive volume and with piecewise smooth boundary. A general set may be approximated by piecewise smooth sets (or even polyhedra). The proof for general sets that area decreases under symmetrization is similar to the convex case.



Let $\omega = (\Pi(K))^{\circ}$ be the interior of the projection. Split ∂K into three pieces, lower part M_1, upper part M_2 and side M_3.

18. Proof of Area Reduction.

By convexity

the lower surface M_1 is the graph of the convex function $z = u_1(x, y)$;
upper surface M_2 is the graph of the concave function $z = u_2(x, y)$.

Symmetrization preserves convexity with

upper surface $\tilde{M}_2$ given by the concave function $\tilde{u}_2 = \frac{1}{2}(u_2 - u_1)$.

M_3 consists of vertical segments so $\mathrm{A}(M_3) = \mathrm{A}(S_L(M_3))$.

The area of M_1 is given by

$$\mathrm{A}(M_1) = \int_\omega \sqrt{1 + \left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2}\, dx\, dy$$

Write $p_1 = \frac{\partial u_1}{\partial x}$, $p_2 = \frac{\partial u_1}{\partial y}$, $q_1 = \frac{\partial u_2}{\partial x}$, $q_2 = \frac{\partial u_2}{\partial y}$.

$$\begin{aligned}
\mathrm{A}(\partial K) - \mathrm{A}(\partial S_L(K)) &= \int_\omega \sqrt{1 + p_1^2 + p_2^2} + \sqrt{1 + q_1^2 + q_2^2} \\
&\quad - 2 \int_\omega \sqrt{1 + \left(\frac{q_1 - p_1}{2}\right)^2 + \left(\frac{q_2 - p_2}{2}\right)^2} \\
&\geq 0,
\end{aligned}$$

by Minkowski's inequality.

Equality holds if and only if

$$q_1 = -p_1 \text{ and } q_2 = -p_2 \text{ almost everywhere.}$$

Then by integrating that $u_2 = -u_1 = \tilde{u}_2$ on ω up to additive constant.

Lemma (Minkowski's Inequality.)

For any numbers a_i and b_i,

$$\sqrt{\sum_{i=1}^{p}(a_i + b_i)^2} \leq \sqrt{\sum_{i=1}^{p} a_i^2} + \sqrt{\sum_{i=1}^{p} b_i^2}.$$

Equality holds if and only if $(a_1, \ldots, a_p)$ is proportional to $(b_1, \ldots, b_p)$.

Thus we have proved:

Theorem

Let $K \subset \mathbb{R}^3$ be compact, convex with $|K| > 0$ and piecewise smooth boundary, zero center of mass. Let $0 \in L$ be a plane. Then

$$A(\partial S_L(K)) \leq A(\partial K). \tag{3}$$

Equality in (3) holds iff L is a plane of symmetry and $S_L(K) = K$.

Thus if Ω is such a compact convex body that has least boundary area among all bodies with volume V, then $S_L(\Omega)$ can't have smaller area and equality holds in (3). Thus Ω has L as a plane of symmetry. As L was arbitrary, Ω is symmetric with restect to all L, thus is the ball.

Theorem (Isoperimetric Inequality)

Let $K \subset \mathbb{R}^3$ be compact, convex, $|K| > 0$ and piecewise smooth boundary. Let $\bar{B}_r \subset \mathbb{R}^3$ be a closed ball such that $|B_r| = |K|$. Then

$$A(\partial \bar{B}_r) = 6^{\frac{2}{3}} \pi^{\frac{1}{3}} |K|^{\frac{2}{3}} \le A(\partial K) \tag{4}$$

Equality in (4) holds if and only if K is a closed ball of radius r.

Figure: L. Dirichlet (1805–1859)

As O. Perron put it, it's like trying to argue that 1 is the greatest natural number by showing that for every other number x there is a larger one, namely x^2.

This is not a proof because there is no demonstration of the existence of a minimizing figure. Dirichlet tried several times to persuade Steiner that his proofs were incomplete on that account. But Steiner insisted that this was self evident. However, in one of the 1842 papers Steiner conceded

> *"and the proof is readily made if one assumes that there is a largest figure."*

Existence was first proved by Weierstraß(1875) and Schwarz(1884) using calculus of variations. Caratheodory and Study(1909) gave rigorous treatment of Steiner's method without the calculus of variations.

Let $\mathcal{X}_0$ denote the space of nonempty compact sets in $\mathbb{R}^n$.

For $r > 0$ the r-neighborhood F_r of a set $F \in \mathcal{X}_0$ is given by

$$F_r = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, F) \leq r\}.$$

The Hausdorff distance between $E, F \in \mathcal{X}_0$ is given by

$$d_H(E, F) = \inf\{r > 0 : \; E \subseteq F_r \text{ and } F \subseteq E_r\}$$

$(\mathcal{X}_0, d_H)$ is a complete metric space.

Diameter is a continious function: If a sequence of compact sets

$$A_n \to A_0 \text{ in } d_H \text{ as } n \to \infty \quad \Longrightarrow \quad \lim_{n\to\infty} \operatorname{diam}(A_n) = \operatorname{diam}(A_0).$$

Volume (Lebesgue measure) is a upper-semicontinious function: If a sequence of compact sets

$$A_n \to A_0 \text{ in } d_H \text{ as } n \to \infty \quad \Longrightarrow \quad \limsup_{n\to\infty} |A_n| \leq |A_0|.$$

Steiner symmetrization is also only a semicontinuous transformation.

$$A_n \to A \text{ and } S(A_n) \to \tilde{A} \text{ as } n \to \infty \quad \Longrightarrow \quad \tilde{A} \subset S(A).$$

Theorem

Let $\mathcal{Q} \subset \mathcal{X}_0$ be an infinite family of nonempty compact sets which are uniformly bounded: all sets $Q \in \mathcal{Q}$ are contained in some fixed ball $Q \subset U$. Then there exists a subsequence $\{Q_n\} \subset \mathcal{Q}$ and a compact set $Q_0 \in \mathcal{X}_0$ such that $Q_0 \subset U$ and

$$Q_n \quad \xrightarrow{d_H} \quad Q_0 \quad \text{as } n \to \infty$$

Let $\{L_i\}_{i=1,\dots,k}$ be a finite set of hyperplanes containing 0 and S_i be Steiner symmetrization relative to L_i. A multiple symmetrization is the composite

$$S^* = S_k \circ S_{k-1} \circ \cdots \circ S_1.$$

For $\Omega \in \mathcal{X}_0$ let $\mathfrak{S}$ denote the set of all $S^*(\Omega)$ multiple symmetrizations of Ω.

Theorem

Let Ω be a nonempty compact set and $\mathfrak{S}$ the family of all multiple symmetrizations of Ω. Then there is a subsequence $\{\Omega_n\} \subset \mathfrak{S}$ and a closed ball $\bar{B}$ such that $|\bar{B}| = |\Omega|$ and

$$\Omega_n \to \bar{B} \text{ as } n \to \infty.$$

Let $\mathcal{K}_0 \subset \mathcal{X}_0$ denote the space of convex compact sets with $|K| > 0$.

Both volume and surface area are now continuous functions: If a sequence of convex sets $\{K_n\} \subset \mathcal{K}_0$ such that

$$K_n \to K_0 \text{ in } d_H \text{ as } n \to \infty \quad \Longrightarrow \begin{cases} \lim\limits_{n\to\infty} \mathrm{V}(K_n) = \mathrm{V}(K_0). \\ \\ \lim\limits_{n\to\infty} \mathrm{A}(\partial K_n) = \mathrm{A}(\partial K_0). \end{cases}$$

Steiner Symmetrization is also now a continuous transformation.

$$K_n \to K_0 \text{ as } n \to \infty \quad \Longrightarrow \quad \begin{cases} K_0 \in \mathcal{K}_0 \text{ and} \\ \\ S(K_n) \to S(K_0) \text{ as } n \to \infty. \end{cases}$$

Fix $V > 0$ and let

$$A_0 = \inf\{\mathrm{A}(\partial K) : \; K \in \mathcal{K}_0 \text{ such that } |K| = V\}.$$

Choose a minimizing sequence of $K_n \in \mathcal{K}_0$ such that $|K_n| = V$ and

$$\lim_{n\to\infty} \mathrm{A}(\partial K_n) = A_0.$$

By approximation, we may arrange that ∂K_n are piecewise smooth. Let $r > 0$ be the radius of a ball such that $|\bar{B}_r| = V$. By the sphericalization theorem, we may choose a multiple symmetrization S_i^* such that $T_i = S_i^*(K_i) \subset \bar{B}_{r+\frac{1}{i}}(0)$. Since area is decreased,

$$A_0 \le \mathrm{A}(\partial T_i) \le \mathrm{A}(\partial K_i)$$

so that T_i is also a minimizing sequence

$$\lim_{n\to\infty} \mathrm{A}(\partial T_n) = A_0.$$

Now, by the Blaschke-Hadwiger selection theorem, there is a subsequence $\{i'\} \subset \{i\}$ such that

$$T_{i'} \to T_0 \text{ as } i' \to \infty \quad \text{and} \quad T_0 \subset \bar{B}_{r+\varepsilon} \text{ for all } \varepsilon > 0.$$

Now, by continuity of convex sets,

$$|T_0| = \lim_{i' \to \infty} |T_{i'}| = V$$
$$\mathrm{A}(\partial T_0) = \lim_{i' \to \infty} \mathrm{A}(\partial T_{i'}) = A_0.$$

Thus $T_0 \in \mathcal{K}_0$ is the desired minimizer.
Incidentally, since $T_0 \subset \bar{B}_{r+\varepsilon}$ for all $\varepsilon > 0$ we see that $T_0 = \bar{B}_r(0)$.

For any nonempty compact $\Omega \subset \mathbb{R}^n$, the circumradius is the radius of the smallest ball that contains Ω

$$r_{\mathrm{circ}}(\Omega) = \inf\{r > 0 : \Omega \subset \bar{B}_r(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

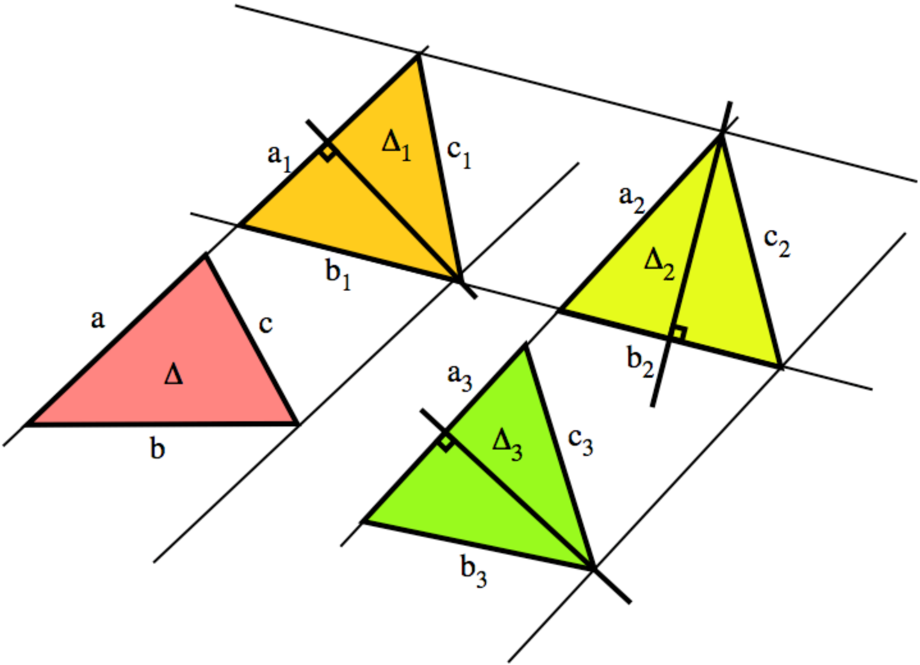
Theorem

Among all triangles with a given area A, the equilateral triangle, and only the equilateral triangle, has the smallest boundary length, diameter and circumradius.

Theorem

Among all quadrilaterals with a given area A, the square, and only the square, has the smallest boundary length, diameter and circumradius.

30. Diagram of proof of triangle theorem



Theorem (Blaschke (1923))

Let $K \subset \mathbb{R}^2$ be a compact, convex set and $T(K) \subset K$ its largest included triangle. Then

$$4\pi|T(K)| \geq 3\sqrt{3}|K|$$

with equality only for ellipses.

Theorem (Winternitz (1921))

Let $K \subset \mathbb{R}^2$ be a compact, convex set with $K| > 0$. Let L be a line through its center of mass. Let M_1 and M_2 be the components of $K - L$. Then

$$\frac{4}{5} \leq \frac{|M_1|}{|M_2|} \leq \frac{5}{4}.$$

Equality holds if and only if K is a triangle and L is parallel to one of its sides.

Theorem (Sylvester (1885))

Let $K \subset \mathbb{R}^2$ be a compact, convex set with $|K| > 0$. Let $\Delta(\mathbf{x}, \mathbf{y}, \mathbf{z})$ denote the triangle with vertices $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Then the ratio of the expected area of a random triangle to the area of the domain satisfies

$$\frac{35}{48\pi^2} \leq \frac{1}{|K|^4} \int_K \int_K \int_K |\Delta(\mathbf{x}, \mathbf{y}, \mathbf{z})| \, d\mathbf{x} \, d\mathbf{y} \, d\mathbf{z} \leq \frac{1}{12}.$$

The left equality holds if and only if K is an ellipse. The right equality holds if and only if K is a triangle.

Theorem (Groß(1918))

Let $K \subset \mathbb{R}^2$ be a compact, convex set. Let Δ denote the triangle with least area that contains K. Then

$$2|K| \geq |\Delta|.$$

Equality holds if and only if K is a parallelogram.

33. Fundamental frequency of a tambourine

Let $\Omega \subset \mathbb{R}^n$ be a compact domain with piecewise smooth boundary. Let $u(\mathbf{x}, t)$ be the vertical displacement of a membrane stretched across Ω. The vibrating membrane satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta\, u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \qquad \text{for } \mathbf{x} \in \Omega \text{ and } t \geq 0;$$

$$u = 0, \qquad \text{for } \mathbf{x} \in \partial\Omega \text{ and } t \geq 0.$$

If there is a solution obtained by separating variables

$$u(\mathbf{x}, t) = U(\mathbf{x})\, T(t)$$

for some constant λ

$$\frac{T''}{T} = -\lambda = \frac{\Delta\, U}{U}.$$

The time function satisfies $T'' + \lambda T = 0$ which gives sinusoidal dependence in time if $\lambda > 0$: there are constants a, b so that

$$T(t) = a \cos\left(\sqrt{\lambda} t\right) + b \cos\left(\sqrt{\lambda} t\right).$$

Thus the frequencies $\sqrt{\lambda}$ are determined by the eigenvalues of the domain. Those are numbers λ such that there are non-vanishing functions $U \in C^2(\Omega)$ such that

$$\begin{aligned} \Delta\, U + \lambda U &= 0, \qquad \text{for } \mathbf{x} \in \Omega; \\ U &= 0, \qquad \text{for } \mathbf{x} \in \partial\Omega. \end{aligned} \tag{5}$$

By multiplying by U and integrating by parts,

$$\lambda \int_\Omega U^2 = \int_\Omega |D\, U|^2$$

which implies $\lambda > 0$.

It turns out that the smallest eigenvalue λ_1 or the fundamental frequency is determined as minimizer of the energy

$$\lambda_1(\Omega) = \inf_{U \in PC_0^1(\Omega)} \mathcal{R}(U); \qquad \text{where} \quad \mathcal{R}(U) = \frac{\int_\Omega |D\,U|^2}{\int_\Omega U^2}$$

$\mathcal{R}(U)$ is called the Rayleigh Quotient. It turns out that the function U that minimizes the Rayleigh quotient satisfies (5) and can be taken positive $U > 0$ in Ω°.

Since the eigenvalue minimizes the Rayleigh Quotient, we can get an upper estimate from any function $V \in PC_0^1(\Omega)$ (functions that are piecewise C^1 and vanish on $\partial\Omega$.) we get the upper bound

$$\lambda_1(\Omega) \le \mathcal{R}(V).$$

Let $U \in PC_0^1(\Omega)$ be a positive function which vanishes on $\partial\Omega$. Then consider the compact set in the next higher dimension between 0 and U:

$$\mathcal{G}(U) = \{(\mathbf{x}, y) \in \mathbb{R}^n \oplus \mathbb{R} : \mathbf{x} \in \Omega,\ 0 \le y \le U(x)\}.$$

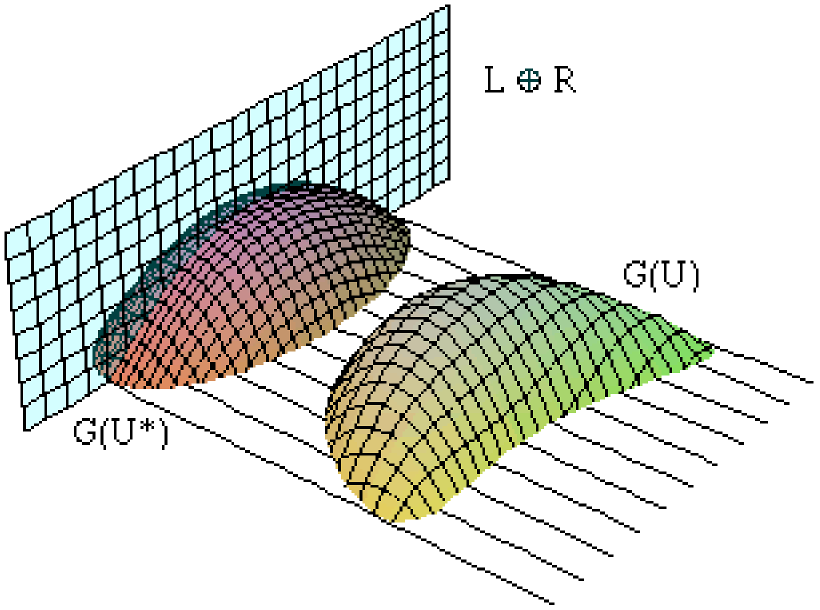
If $L \subset \mathbb{R}^n$ is an $(n-1)$-plane through 0 then $L \oplus \mathbb{R}$ is a vertical n-plane in $\mathbb{R}^n \oplus \mathbb{R}$. Symmetrizing gives a domain of the same type:

$$\mathcal{G}(U^*) = S_{L\oplus\mathbb{R}}\big(\mathcal{G}(U)\big)$$

The bottom surface of $\mathcal{G}(U^*)$ is just $S_L(\Omega)$. The upper surface of $\mathcal{G}(U^*)$ is the graph $y = U^*(\mathbf{x})$ which is symmetric with respect to the $L \oplus \mathbb{R}$. If L is given by $x_n = 0$ then U^* decreases with x_n on $x_n > 0$ and is an even function

$$U^*(x_1, \dots, x_{n-1}, -x_n) = U^*(x_1, \dots, x_{n-1}, x_n).$$

37. Symmetrizing $\mathcal{G}(U)$ diagram



38. Symmetrization preserves L^2

Observe that if U^* is the symmetrized function, then

$$\mathcal{G}\big((U^*)^2\big) = S_{L\oplus\mathbb{R}}\big(\mathcal{G}(U^2)\big)$$

Because symmetrization preserves volume in $\mathbb{R}^{n+1}$,
the L^2 integrals are preserved under symmetrization:

$$\int_\Omega U^2 = \big|\mathcal{G}(U^2)\big| = \big|\mathcal{G}\big((U^*)^2\big)\big| = \int_{S_L(\Omega)} (U^*)^2.$$

First observe that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_A\left(\sqrt{1+\varepsilon^2|Du|^2}-1\right)=\int_A|Du|^2. \tag{6}$$

Then if U^* is the symmetrized function, then $\mathcal{G}(\varepsilon U^*)=S_{L\oplus\mathbb{R}}\big(\mathcal{G}(\varepsilon U)\big)$. Since symmetrization reduces the surface area, for all $\varepsilon\geq 0$,

$$\int_{S_L(\Omega)}\sqrt{1+\varepsilon^2|D\,U^*|^2}\leq\int_\Omega\sqrt{1+\varepsilon^2|D\,U|^2}$$

with equality at $\varepsilon=0$. Thus there is an inequality for derivatives at $\epsilon=0$ which implies by (6), the energy is reduced by symmetrization:

$$\int_{S_L(\Omega)}|D\,U^*|^2\leq\int_\Omega|D\,U|^2.$$

Let $U \in PC_0^1(\Omega)$ be a positive eigenfunction so $\lambda_1(\Omega) = \mathcal{R}(U)$. Its symmetrization U^* may not be an eigenfunction of $S_L(\Omega)$ but can play the role of V in the upper bound. Using the fact that the L^2 is preserved and energy is diminished under symmetrization,

$$\lambda_1\big(S_L(\Omega)\big) \le \frac{\int_{S_L(\Omega)} |D\,U^*|^2}{\int_{S_L(\Omega)} (U^*)^2} \le \frac{\int_\Omega |D\,U|^2}{\int_\Omega U^2} = \lambda_1(\Omega).$$

It turns out that equality holds if and only if Ω is symmetric about L.

Theorem (Faber & Krahn (1923))

Among all compact domains $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary and $|\Omega| = A$, the ball and only the ball has the smallest fundamental tone.

Thanks!