Constructing Polyhedra

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- A. D. Alexandrov, *Convex Polyhrdra,* Springer, Berlin 2005; orig. Gosadarstv, Izdat. Tekhn.-Teor. Lit., Moscow–Leningrad, 1950 (Russian).
- T. Bonnesen & W. Fenchel, *Theorie der Konvexen Körper,* Chelsea, New York, 1971; orig. Springer, Berlin 1934.
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- Convex Polygons and Polyhedra
- Minkowski's Problem about Polyhedra with Given Areas
- Alexandrov's Mapping Lemma
- Proof of the Area Theorem using the Mapping Lemma
  - Proof of Uniqueness by Minkowski's Mixed Volume Inequality
  - Proof of Continuity and Closed Graph
- Minkowski's Mixed Volume Inequality
  - Mixed Volumes as Coefficients
  - Derivative of Volume
  - Minkowski's Inequality from Brunn-Minkowski Inequality
- Kneser & Süss proof of the Brunn-Minkowski Inequality
- Other Problems Soluble by Alexandrov's Mapping Lemma
  - Polyhedra with Vertices on Given Rays
  - Polyhedra with Given Net.

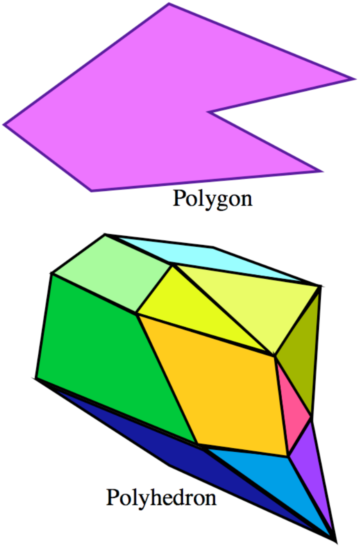

Figure: Polyhedra.

A polygon is a connected open plane set whose boundary consists finitely many different line segments or rays glued end to end. A polyhedron is a piecewise flat surface in three space consisting of finitely many planar polygons glued pairwise along their sides. A polyhedron is assumed to be closed: each side of every polygon is glued to the side of another polygon.

We shall abuse notation and call polyhedron together with its interior a "polyhedron."

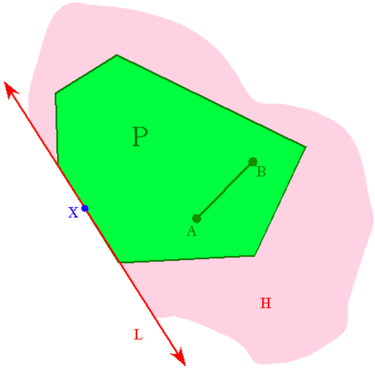

Figure: Convex Polyhedron.

A polyhedron is convex if there is a supporting halfspace at every boundary point X. That is, there is a hyperplane L through X such that the polyhedron is in one of the closed halfspaces bounded by L.
A polytope is a convex polyhedron.
A convex body is a compact convex set that contains interior points.

Equivalently, P is convex if for every pair of points in P, the straight line segment between the points lies entirely inside P.

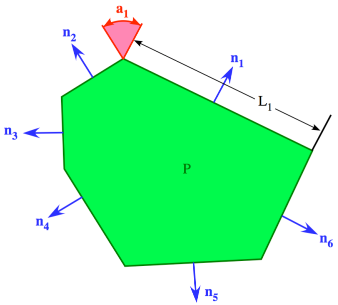

Figure: Dimensions of a Polygon.

Associated to each edge F_i of a polygon (or top dimensional face of a polyhedron) is a perpendicular outward unit normal vector $\mathbf{n}_i$.

Each face has a length (or area) L_i. The angle at the vertex, $\mathbf{a}_i$ is the angle between neighboring normals (the spherical angle or area in the unit sphere of the convex hull of neighboring normals.)

For a closed polygon, the total angle over all V vertices is the angle of the circle

$$\sum_{i=1}^{V} \mathbf{a}_i = 2\pi.$$

(For polyhedra, the sum of vertex angles is the area of the unit sphere which is 4π in $\mathbb{R}^3$.)

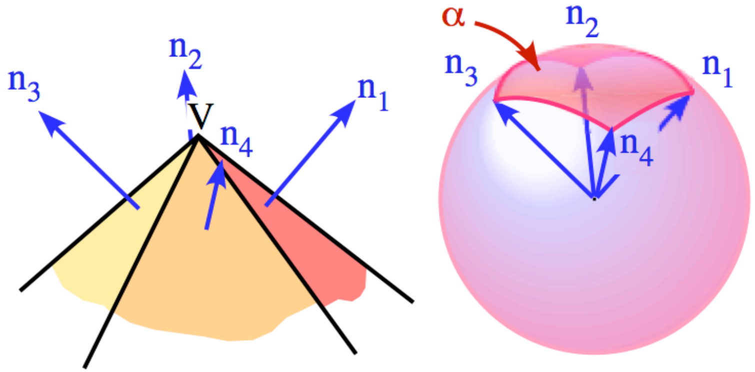

Figure: Spherical Angle α of a Vertex V.

Translate normal vectors n_i to the origin and view them as vectors of the unit sphere. The spherical angle α of a vertex V is the area of the spherical convex hull of the normal vectors of faces adjacent to V.

Figure: Hermann Minkowski 1864–1909.
"Mich interessiert alles, was konvex ist!" H. Minkowski.

Minkowski was the most respected graduate student in Göttingen when Hilbert began his studies.

Minkowski studied number theory using geometrical methods, culminationg in his *Geometrie der Zahlen* (1896). His highly innovative ideas contributed to the development of synthetic geometry and convexity theory.

He provided the first proof for the "Minkowski Problem" for polyhedra using a variational argument.

Minkowski asked a reconstruction question for polyhedra $P \subset \mathbb{R}^3$.

> *Given distinct outward normal vectors* $\mathcal{N} = \{\mathbf{n}_i\}_{i=1,\dots,V}$ *and positive numbers* $\{\beta_i\}_{i=1,\dots,V}$*, does there exist a convex polyhedrom* P *whose unit normals are exactly the vectors* $\mathcal{N}$ *and whose side areas at corresponding sides is*
>
> $$A_i = \beta_i, \qquad \textit{for all } i = 1, \dots, V?$$

It turns out that the normals and lengths have to satisfy geometric necessary conditions.

Hemisphere Condition for Normals. The set of normals cannot lie in any single closed semicircle (hemisphere).

For example if this were not the case, there is a unit vector $\mathbf{w}$ such that

$$\mathbf{n}_i \bullet \mathbf{w} \geq 0, \qquad \text{for all } i = 1, \ldots, V.$$

Then the polygon would not be closed in the $-\mathbf{w}$ direction and there would have to be sides of infinite length.

Eqivalently, the positive cone of the normals

$$\text{Cone}_+(\mathcal{N}) = \left\{ \sum_{i=1}^{V} \theta_i \mathbf{n}_i : \theta_i \geq 0 \right\}$$

would have to be the entire space $\mathbb{R}^3$.

Both Sides Equal Condition for Area. For each unit vector $\mathbf{w}$, the area of the sides facing in the $\mathbf{w}$ direction have the same shadow as the sides facing the $-\mathbf{w}$ direction. In other words

$$\sum_{i=1}^{V} A_i\ (\mathbf{n}_i \bullet \mathbf{w}) = 0, \qquad \text{for all unit vectors } \mathbf{w} \in \mathbb{S}^2$$

where $\mathbb{S}^2$ is the 2 sphere, the set of unit vectors of $\mathbb{R}^3$ based at the origin. The area of the projection of the ith face to $\mathbf{n}^\perp$ is $A_i\ (\mathbf{n}_i \bullet \mathbf{w})$. It is positive for faces on the $\mathbf{w}$ side of P, and negative for those of the other side. Note that this condition would fail if the Hemisphere Condition did not hold.

Because we may write every $\mathbf{w}$ in a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbb{R}^3$, the the Both Sides Equal Condition reduces to 3 equations

$$\sum_{i=1}^{V} A_i\ (\mathbf{n}_i \bullet \mathbf{e}_j) = \left(\sum_{i=1}^{V} A_i \mathbf{n}_i \right) \bullet \mathbf{e}_j = 0, \qquad \text{for } j = 1, 2, 3. \tag{1}$$

Theorem (Construction Theorem for Minkowski's Problem for Polyhedra)

Suppose we are given $V \geq 3$ distinct vectors $\mathcal{N} = \{\mathbf{n}_i\}_{i=1,\ldots,V}$ in $\mathbb{S}^2$ and positive numbers $\{\beta_i\}_{i=1,\ldots,V}$ that satisfy the vector Both-Sides-Equal Condition

$$\sum_{i=1}^{V} \beta_i \; \mathbf{n}_i = \mathbf{0}.$$

Then there exist a convex polyhedrom $\mathcal{P}$ with V sides whose outer unit normals are exactly the vectors $\mathcal{N}$ and whose side areas at corresponding sides are the given

$$A_i = \beta_i, \qquad \text{for all } i = 1, \ldots, V.$$

There is only one solution up to translation.

Assuming there are solutions with given Minkowski Data, the solutions are uniquely determined (up to translation). In $d = 2$ this is elementary.

Theorem (Uniqueness Theorem for Minkowski's Problem for Polygons)

Suppose we have two convex polygons P and P' with $V \geq 2$ vertices whose outer normals $\mathbf{n}_i = \mathbf{n}'_i$ and corresponding lengths $L_i = L'_i$ coincide. Then P' is a translation of P.

Proof. Assume that the sides are numbered consecutively counterclockwise. The vertices are determined by the side lengths and vertex angles. Denoting by $\mathcal{R}$ the $+90^\circ$ rotation, the vector on the side is $F_j = L_j\,\mathcal{R}\mathbf{n}_j$ and $X_{j+1} = X_j + F_j$ for $j = 1, \dots V$, where $X_{V+1} = X_1$ since the polygon closes. Since L_j and $\mathbf{n}_j$ are the same for both polygons, the translation is just the difference of starting vertices

$$X'_j = X_j + (X'_1 - X_1)$$

for all $j = 1, \dots, V$.

Figure: Aleksandr Danilovich Aleksandrov 1912–1999.

A. D. Alexandrov was a highly original geometer began in the era when Riemannian Geometry was being extended to high dimensions. He founded the Leningrad School of geometry. He spearheaded the development of synthetic methods in geometry and introduced geometric methods in the solution of nonlinear elliptic PDE's.

Among mathematicians trained by Alexandrov's School are some of the leaders of modern geometry: Pogorelov, Gromov and Perelman.

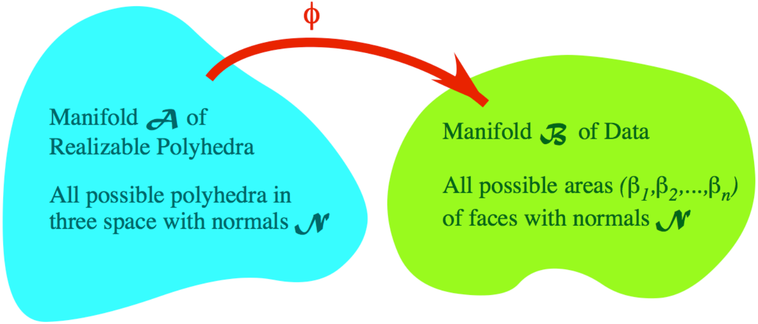

Figure: Setup for Minkowski's Problem: Show ϕ is Onto.

ϕ is the map that assigns to an actual polyhedron in three space the vector of the areas of each of its faces.

A topoligcal space $\mathcal{T}$ is *d-dimensional manifold* if every point possesses a neighborhood which is homeomorphic to an open ball of $\mathbb{R}^d$. Examples: $\mathbb{S}^d$ or an open set in $\mathbb{R}^d$.

Alexandrov's Minkowski Problem solution depends on Mapping Lemma.

Theorem (Alexandrov's Mapping Lemma)

Suppose $\varphi : \mathcal{A} \to \mathcal{B}$ is a mapping between n-dimensional manifolds that satisfies the following conditions

1. *Every component of $\mathcal{B}$ contains image points of $\mathcal{A}$.*
2. *φ is one-to-one.*
3. *φ is continuous*
4. *φ has closed graph: if $\{B_j\} \subset \mathcal{B}$ is a sequence consisting of image points $B_j = \varphi(A_j)$ for some $A_j \in \mathcal{A}$ which converges $B_j \to B$ in $\mathcal{B}$ as $j \to \infty$, then there exists $A \in \mathcal{A}$ with $\varphi(A) = B$ and a subsequence A_{i_m} of $\{A_i\}$ which converges to A as $m \to \infty$.*

Then φ is onto, i.e., $\varphi(\mathcal{A}) = \mathcal{B}$.

Open and Closed Argument.

Proof. By (4), the inverse function is continuous. As φ is also continuous, it is also a homeomorphism. By the Invariance of Domain Theorem, which is proved in an algebraic topology course, such as Math 6520, the image $\varphi(\mathcal{A})$ is open in $\mathcal{B}$. By (4), $\varphi(\mathcal{A})$ is also closed. By (1), every component of $\mathcal{B}$ has image points of $\mathcal{A}$. Thus if $\mathcal{B}'$ is any component of $\mathcal{B}$ then since $\varphi(A) \cap \mathcal{B}$ is a nonempty, open and closed subset of $\mathcal{B}'$, it must agree with it $\varphi(A) \cap \mathcal{B} = \mathcal{B}'$. Since this holds for all components, we conclude $\varphi(\mathcal{A}) = \mathcal{B}$. $\square$

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:-o A differential equator's mantra. :-|

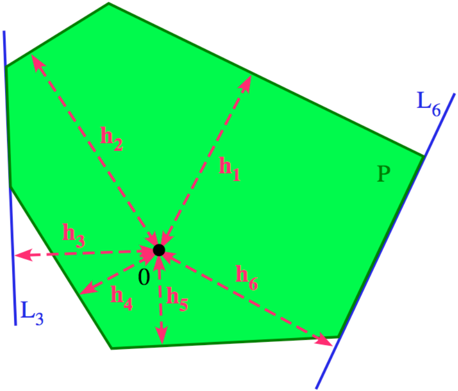


Figure: Support Distances.

One way to determine the ith face of a convex polyhedron is by its support distance which is the signed distance h_i of the plane of the face from the origin. If the origin is interior to the body then all $h_i > 0$. If $X \in L_i$ is a point in the ith face, then

$$h_i = X \bullet \mathbf{n}_i.$$

The polyhedron may be expressed as the intersection of halfspaces

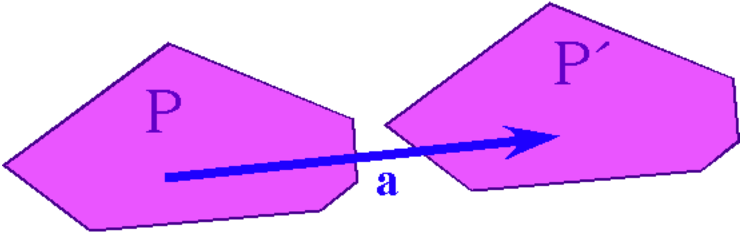
$$P = \bigcap_{j=1}^{V} \left\{ X \in \mathbb{R}^d : X \bullet \mathbf{n}_j \le h_j \right\} \tag{2}$$

Suppose P is a polyhedron with V faces in the directions $\mathbf{n}_i$ and support numbers h_i. Suppose that we translate P by the vector $\mathbf{a}$. The translated polyhedron is

$$P' = P + \mathbf{a} = \{X + \mathbf{a} : X \in P\}.$$

P' has the same normal directions. But if X is a point on the jth face of p then $X' = X + \mathbf{a}$ is a point on the jth face of P'. Hence

$$h_j' = X' \bullet \mathbf{n}_j = (X + \mathbf{a}) \bullet \mathbf{n}_j = h_i + \mathbf{a} \bullet \mathbf{n}_j. \tag{3}$$



Suppose we are given a finite collection of $V \geq 4$ distinct unit vectors $\mathbf{n}_i$ that satisfy the Hemisphere Condition.

We define $\mathcal{A}$, the manifold of realizations (polyhedra that exist in $\mathbb{R}^3$).

We define $\mathcal{B}$, the manifold of configurations, the manifold of Minkowski Data (boundary areas satisfying the Both-Sides-Equal Condition).

For $\mathbf{h} \in \mathcal{A}$, we define the mapping $\varphi(\mathbf{h}) \in \mathcal{B}$ as the vector of support distances in the $\mathbf{n}_i$ directions for the polyhedron $P(\mathbf{h})$.

The Minkowski Theorem holds if every configuration $\mathbf{b} \in \mathcal{B}$ is realizable: There is an polyhedron $\mathbf{a} \in \mathcal{A}$ such that $\varphi(\mathbf{a}) = \mathbf{b}$. In other words, if

$$\varphi(\mathcal{A}) = \mathcal{B}.$$

Each polyhedron is determined by V support numbers by (2). For arbitrary $\mathbf{h}$, the polyhedron $P(\mathbf{h})$ gotten by intersecting halfspaces may be empty or may not have a 2 dimensional face for every normal $\mathbf{n}_j$. We call such $P(\mathbf{h})$ not full.

We regard as all translates as equivalent, so we divide out by all translations. Let $M : \mathbb{R}^3 \to \mathbb{R}^V$ be the full-rank linear transformation

$$M(\mathbf{a}) = (n_1 \bullet \mathbf{a}, \ldots, \mathbf{n}_V \bullet \mathbf{a}).$$

For the translation $\mathbf{a} \in \mathbb{R}^3$, by (3), the two support vectors differ by $\mathbf{h}' = \mathbf{h} + M(\mathbf{a})$.

$$\mathcal{A} = \{\mathbf{h} \in \mathbb{R}^V : P(\mathbf{h}) \text{ is full.}\}/M(\mathbb{R}^3)$$

$P(\mathbf{h})$ being full is an open condition, since small perturbations of $\mathbf{h}$ do not destroy faces of $P(\mathbf{h})$ for full $P(\mathbf{h})$. Thus $\mathcal{A}$ is a quotient of an open set of V dimensions by a 3-dimensional linear subspace. Thus, it is a (linear) manifold of dimension $V - 3$.

The vector of facial areas is a vector of V positive numbers $\beta_j \in \mathbb{R}_+^V$. But not all positive vectors satisfy necessary conditions, so we restrict areas to those that satisfy the Both-Sides-Equal Condition

$$\mathcal{B} = \left\{ (\beta_1, \ldots, \beta_V) \in \mathbb{R}_+^V : \sum_{j=1}^{V} \beta_j \mathbf{n}_j = \mathbf{0} \right\}$$

There are 3 linear equations that hold on an open subset of V dimensions. Thus $\mathcal{B}$ is a (linear) manifold of dimension $V - 3$.

Verify Assumption (1).

The manifold $\mathcal{B}$ is the intersection of the positive orthant in $\mathbb{R}^V$, an open convex cone, with the $V-3$ dimensional subspace, the kernel of $N : \mathbb{R}^V \to \mathbb{R}^3$ given for $\mathbf{b} = (\beta_1, \dots, \beta_V) \in \mathbb{R}^V$ by

$$N(\mathbf{b}) = \sum_{j=1}^{V} \beta_j \mathbf{n}_j.$$

Hence $\mathcal{B}$ is a convex open set in the subspace, which is connected (one component!)

To show (1), we show $\varphi(\mathcal{A}) \neq \emptyset$. But for $\mathbf{u} = (1, \dots, 1) \in \mathcal{A}$, $P(\mathbf{u})$ is full. So $\varphi(\mathbf{u}) \in \mathcal{B}$. The vectors $\mathbf{n}_i$ are interior points of the ith faces.

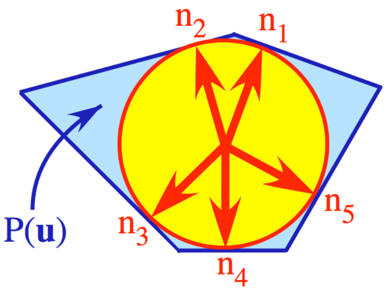
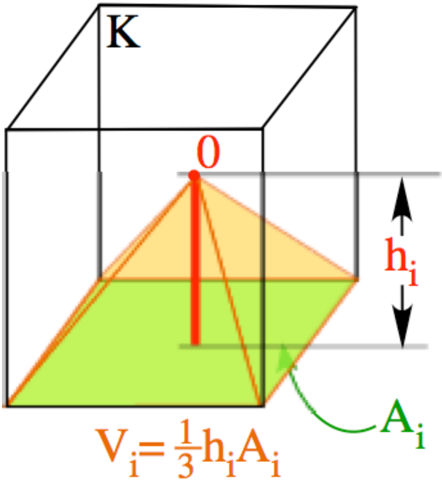


Figure: Polyhedron $P(\mathbf{u})$, where $\mathbf{u} = (1, 1, 1, 1, 1)$.



Let K be a compact convex polyhedron whose faces have normal vectors n_i, support numbers h_i and face areas A_i. Translate so that $0 \in K$. Then $h_i \geq 0$. The volume may be expressed as the sum of volumes of pyramids for each face, the face is the base and 0 is the apex. Thus the height is h_i and

$$V(K) = \frac{1}{3} \sum_{i=1}^{n} h_i A_i.$$

Verify Assumption (2): all polyhedra with equal corresponsing face areas are translates of one another. Thus φ is one-to-one.

Theorem (Uniqueness Theorem for Minkowski's Problem for Polyhedra)

Suppose we have two bounded convex polyhedra P and P' in $\mathbb{R}^3$ whose outer normals $\mathbf{n}_i = \mathbf{n}'_i$ and corresponding areas $A_i = A'_i > 0$ coincide. Then P' is a translation of P.

Proof. Our proof depends on an inequality of Minkowsi.

Theorem (Minkowski's Inequality)

Suppose we have two bounded convex polyhedra P and P' in $\mathbb{R}^3$ with the same normals $\{\mathbf{n}_i\}_{i=1,\dots,V}$. Let A_i, h_i and A'_i, h'_i denote areas and support numbers in the $\mathbf{n}_i$ directions. (Some areas may be zero.) Then the mixed volume $V(P, P', P')$ satisfies

$$V(P, P', P') = \frac{1}{3} \sum_{i=1}^{V} h_i A'_i \geq V(P)^{1/3} V(P')^{2/3}. \tag{4}$$

If equality holds, then P and $P' = cP + \mathbf{a}$ are homothetic translates.

Minkowski's theorem holds for general closed convex sets K and K', not just polyhedrons.

After two applications we'll define mixed volume, show it equals sum in (4) and derive Minkowski's from Brunn-Minkowski's Inequality.

Suppose that P and P' are two solutions of the Minkowski Problem. Then $A_i = A_i' = \beta_i$ for each i. Then the mixed volume

$$V(P, P', P') = \frac{1}{3}\sum_{i=1}^{V} h_i A_i' = \frac{1}{3}\sum_{i=1}^{V} h_i A_i = V(P).$$

From Minkowski's Inequality,

$$V(P) = V(P, P', P') \geq V(P)^{1/3} V(P')^{2/3}$$

follows

$$V(P) \geq V(P').$$

By reversing the roles of P and P', we find by the same argument that $V(P') \geq V(P)$. It follows that $V(P) = V(P')$ and equality holds in (4). By the uniqueness statement in Minkowski's Theorem, P and P' are homothetic translates. But since they have the same volume, $c = 1$ so P and P' are translates and correspond to the same point in $\mathcal{A}$, the realization manifold of polyhedra. $\square$

Minkowski's inequality gives us the Isoperimetric Inequality, that says that the largest the volume convex set can have for a given surface area is the volume attained attained by the ball.

If $\mathcal{U} \subset \mathbb{R}^3$ is the unit ball and K is a compact convex set, then, as we shall see, the surface area is given by the mixed volume

$$3V(\mathcal{U}, K, K) = A(\partial K).$$

By Minkowski's Inequality

$$\tfrac{1}{3}\,\mathrm{A}(\partial K) = V(\mathcal{U}, K, K) \geq V(\mathcal{U})^{1/3} V(K)^{2/3} = \left(\tfrac{4}{3}\pi\right)^{1/3} V(K)^{2/3}.$$

In other words,

$$A(\partial K)^{3/2} \geq 6\sqrt{\pi}V(K). \qquad (5)$$

Equality holds iff K is a ball. For the ball $\mathcal{B}$ of radius R,

$$A(\partial\mathcal{B})^{3/2} = (4\pi R^2)^{3/2} = 6\sqrt{\pi}(\tfrac{4}{3}\pi R^3) = 6\sqrt{\pi}V(\mathcal{B}).$$

Verify Assumption (3).

For $\mathbf{h} \in \mathcal{A}$, the polygon $P(\mathbf{h})$ is full. If we fix h_i with $i \neq j$ and vary h_j, the edge moves parallel in and out continuously with h_j. As the edges of the jth face are cut by planes of some other faces, its length varies continuously. It follows that $P(\mathbf{h})$ is a continuous vector function.

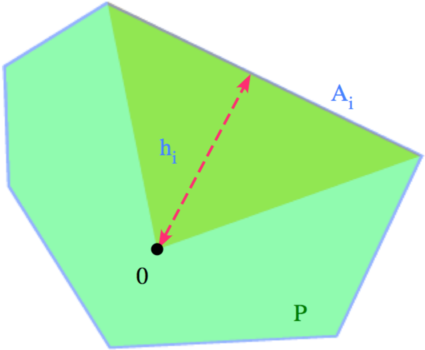


Figure: h_i =Ht. of Pyramid with Base= A_i.

Assume 0 is interior point of P.

Let P be a polyhedron with side areas A_i and support numbers h_i. Assume all side areas are bounded: $0 < \alpha \le A_i \le \beta$. Then

$$\mathrm{A}(\partial P) = \sum_{j=1}^{n} A_j \le n\beta.$$

The volume of the pyramid from the origin to the ith face is less than the total volume

$$\frac{1}{3} h_i A_i \le \frac{1}{3} \sum_{i=1}^{n} h_i A_i = V(P).$$

By the isoperimetric inequality (5), the volume is less than the volume of a ball with the same surface area

$$V(P) \le \frac{1}{6\sqrt{\pi}} \mathrm{A}(\partial P)^{3/2}.$$

It follows that

$$\frac{1}{3} h_i \alpha \le \frac{1}{3} h_i A_i \le \frac{1}{6\sqrt{\pi}} (n\beta)^{3/2}.$$

from which it follows that

$$h_i \le \frac{n^{3/2}\beta^{3/2}}{2\sqrt{\pi}\alpha}.$$

Verify Assumption (4).

It is a compactness argument. Assume that $A_i \in \mathcal{A}$ are polygons with face areas $B_i = \varphi(A_i)$ that converge $B_i \to B$ as $i \to \infty$ (B is positive!)

By translating A_i we may assume that 0 is interior to A_i. A convergent sequence is bounded, $0 < \alpha \le B_i \le \beta$ for all i. Hence by the geometric estimate for the jth support numbers of A_i,

$$0 \le h_{i,j} \le K$$

for all (i, j).

But a bounded sequence in a finite dimensional space is sequentially compact. There is a subsequence i_k such that

$$h_{i_k,j} \to h_{\infty,j} \qquad \text{as } k \to \infty$$

so $A_{i_k} \to A_\infty$ in $\mathcal{A}$. By continuity

$$B = \lim_{k\to\infty} B_{i_k} = \lim_{k\to\infty} \varphi\left(A_{i_k}\right) = \varphi\left(\lim_{k\to\infty} A_{i_k}\right) = \varphi(A_\infty).$$

To prove Minkowski's Inequality, we use the Brunn-Minkowski Inequality. The mixed volume of polyhedra is related to the Minkowski Addition.

Minkowski Addition of any two sets $A, B \subset \mathbb{R}^d$ is defined to be

$$A + B := \{x + y : x \in A \text{ and } y \in B\}.$$

For $s \geq 0$, the Minkowski Dilation by factor s is defined to be

$$sA = \{sx : x \in A\}.$$

For example, the Minkowski sum of dilated rectangles is a rectangle

$$\begin{aligned} s([0,a] \times [0,b]) + t([0,c] \times [0,d]) \\ = [0, sa + tc] \times [0, sb + td]. \end{aligned} \tag{6}$$

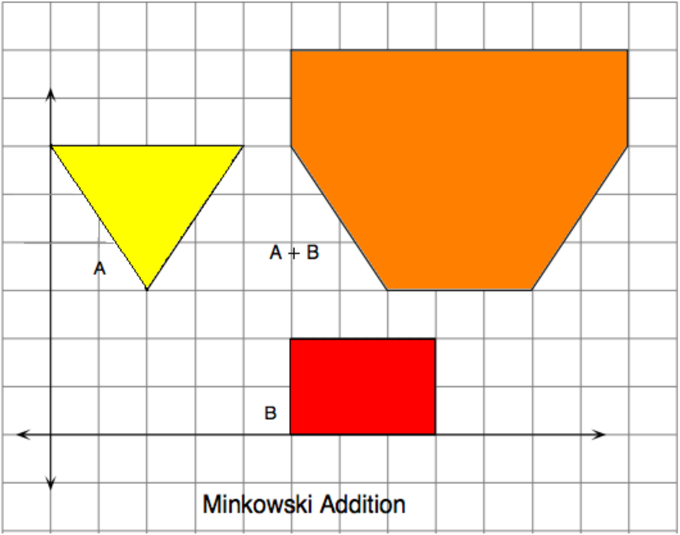


Figure: Minkowski Addition of a Triangle and a Rectangle in the Plane.

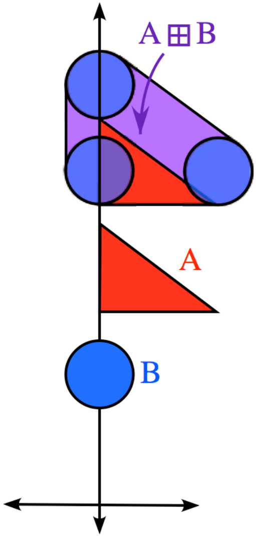


Figure: Minkowski Addition as Smear.

Minkowski Addition may be written

$$A + B = \bigcup_{x \in B} A + \{x\}.$$

It is the union of all translates of A by points of B. The A set is smeared around by the B set.

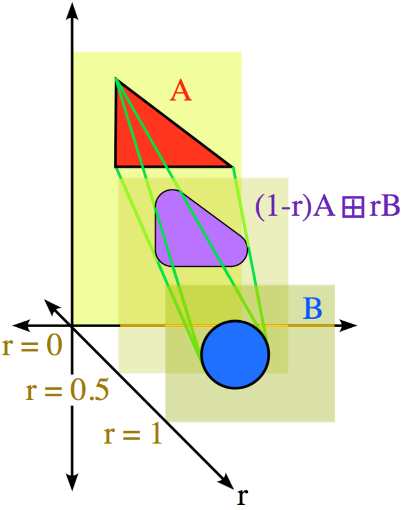


Figure: $(1-r)A+rB$ as section of cvx hull of $K = A \times \{0\} \cup B \times \{1\}$.

For $0 \le r \le 1$, $(1-r)A + rB$ linearly morphs A at $r = 0$ into B at $r = 1$.

In (x, y, z, r) space draw A and B in different hyperplanes

$$K = \{(x, y, z, 0) : (x, y, z) \in A\} \cup \{(x, y, z, 1) : (x, y, z) \in B\}$$

and consider the three dimensional convex hull $\hat{K}$ (outlined in green in the Figure.) Then the $r =$ const. section is the Minkowski convex combination

$$\{(x, y, z, r) : (x, y, z) \in \mathbb{R}^3\} \bigcap \hat{K} = \{(x, y, z, r) : (x, y, z) \in (1-r)A + rB\}.$$

Theorem (Polynomial Nature of Sum. [Due to Minkowski])

Let A and B be convex bodies in the $\mathbb{R}^3$. Then the volume of the linear combination $sA + tB$ is a cubic form for nonnegative s and t,

$$V(sA + tB) = a_{000}s^3 + 3a_{001}s^2t + 3a_{011}st^2 + a_{111}t^3.$$

Coefficients of s^3 and t^3 are evidently volumes $a_{000} = V(A)$ and $a_{111} = V(B)$. The cross terms define $V(A, B, B) := a_{011}$ and $V(A, A, B) = a_{001}$ called the mixed volumes.

For A, B, C, D convex sets and nonnegative s, t the following hold:

- $V(A, B, B) \geq 0$;
- $V(A, B, B) = V(B, B, A)$;
- If ρ is a rigid motion then $V(\rho A, \rho B, \rho B) = V(A, B, B)$;
- $V(A, A, A) = V(A)$;
- $V(sA + tB, C, C) = sV(A, C, C) + tV(B, C, C)$;
- If $A \subset B$ then $V(A, C, C) \leq V(B, C, C)$ and $V(A, A, C) \leq V(B, B, C)$.

We prove it for polyhedra. Let $\mathbf{n}_i$ denote all normals of both A or B. If P_i is the $\mathbf{n}_i$ boundary plane of X and h_i its support number,

$$V(X) = \frac{1}{3}\sum_{i=1}^{n} h_i\, A(P_i \cap X).$$

If $P_{i,j}$ is the $\mathbf{v}_j$ boundary line of $P_i \cap X$ in P_i and $h_{i,j}$ its support number,

$$A(P_i \cap X) = \frac{1}{2}\sum_{j=1}^{n_i} h_{i,j}\, L(P_{i,j} \cap X).$$

If $\mathbf{w}_k$ are the two directions in the line $P_{i,j}$ and $h_{i,j,k}$ support numbers of the endpoints of $P_{i,j} \cap X$,

$$L(P_{i,j} \cap X) = h_{i,j,1} + h_{i,j,2}.$$

Let h'_i, h''_i and h_i denote support numbers of A, B and $X = sA + tB$, resp., in the n_i direction. They satisfy

$$h_i = \sup_{sa+tb\in X} (sa + tb) \cdot \mathbf{n}_i = s \sup_{a\in A} a \cdot \mathbf{n}_i + t \sup_{b\in B} x \cdot \mathbf{n}_i = sh'_i + th''_i.$$

Similarly in the parallel planes P_i, P'_i, P''_i and lines $L_{i,j}$, $L'_{i,j}$, $L''_{i,j}$

$$h_{i,j} = sh'_{i,j} + th''_{i,j}; \qquad h_{i,j,k} = sh'_{i,j,k} + th''_{i,j,k}.$$

Combining, we see that the volume of $sA + tB$ is a cubic polynomial

$$V(X) = \frac{1}{6} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{\ell=1}^{2} (sh'_i + th''_i)(sh'_{i,j} + th''_{i,j})(sh'_{i,j,k} + th''_{i,j,k}).$$

Moreover, since the volume, area and length are unchanged by translation, we may move the origins independently in each or the planes and lines so that they are in the interior of the A, B, $P_i \cap A$, $P_i \cap B$, $P_{i,j} \cap A$ and $P_{i,j} \cap B$. Hence all coefficients h so also a_{ijk} are positive. □

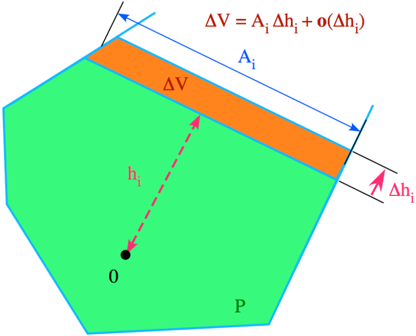


Figure: Volume increase due to increasing Δh_i

Let P be a polyhedron with face directions $\mathbf{n}_i$, support numbers h_i and face areas A_i. The increase in volume by increasing the support number from h_i to $h_i + \Delta h_i$ is thickness times area plus higher order terms.

$$\Delta V = A_i \, \Delta h_i + \mathbf{o}(\Delta h_i).$$

Hence

$$\frac{\partial V}{\partial h_i} = A_i.$$

Lemma

For convex polyhedra A and B we have $V(A, A, B) = \frac{1}{3}\sum_{i=1}^{n} h_i'' A_i'$.

Proof. $V(A, A, B) = a_{001}$ where

$$V(sA + tB) = a_{000}s^3 + 3a_{001}s^2t + 3a_{011}st^2 + a_{111}t^3.$$

We differentiate volume in two ways.

$$\frac{\partial}{\partial t}\Big|_{t=0} V(A + tB) = 3a_{001}.$$

Also, using the chain rule,

$$\begin{aligned}\frac{\partial}{\partial t}\Big|_{t=0} V(A + tB) &= \frac{\partial}{\partial t}\Big|_{t=0} V(P(h' + th'')) \\ &= \sum_{i=1}^{n} h_i'' \frac{\partial}{\partial h_i} V(P(h' + th''))\Big|_{t=0} \\ &= \sum_{i=1}^{n} h_i'' A_i'. \quad \square\end{aligned}$$

The theorem of Hermann Karl Brunn (1862–1939) says that since the Minkowski addition tends to "round out" the figures being added, the volume of the added figure exceeds the volume of the summands.

Theorem (Brunn-Minkowski)

Let $A, B \in \mathbb{R}^d$ be proper convex polyhedra. Then,

$$V(A+B)^{\frac{1}{d}} \geq V(A)^{\frac{1}{d}} + V(B)^{\frac{1}{d}}. \tag{7}$$

Equality holds if and only if A and B are homothetic. Two figures A and B are homothetic, i. e., they are similar and are similarly situated, which means there is a translation and dilation so that

$$A = rB + \{x\}.$$

The inequality (7) is due to Brunn. Minkowski proved that equality implies homothety.

The theorem continues to hold if A and B are arbitrary bounded measurable sets.

44. Minkowski's Inequality follows from the Brunn-Minkowski Theorem.

Pf. To show for all proper polyhedra A, B,

$$V(A,A,B) \geq V(A)^{2/3} V(B)^{1/3} \tag{4}$$

and "=" implies A, B homothetic. Using $V((1-t)A + tB) =$

$V(A)(1-t)^3 + 3V(A,A,B)(1-t)^2 t + 3V(A,B,B)(1-t)t^2 + V(B)t^3$,

by Brunn-Minkowski, $f : [0,1] \to \mathbb{R}$ is nonnegative and concave down,

$$f(t) = V((1-t)A + tB)^{\frac{1}{3}} - (1-t)V(A)^{\frac{1}{3}} - tV((B)^{\frac{1}{3}}$$

Differentiating at $t = 0$, the result follows from

$$0 \leq \left.\frac{\partial f}{\partial t}\right|_{t=0} = V(A)^{-\frac{2}{3}}[-V(A) + V(A,A,B)] + V(A)^{\frac{1}{3}} - V(B)^{\frac{1}{3}}.$$

Equality in (4) implies $f(t) \equiv 0$ so "=" holds in the Brunn-Minkowski Theorem, whence A and B are homothetic. $\square$

Theorem (Brunn-Minkowski)

Let $A, B \in \mathbb{R}^d$ *be proper convex sets and* $0 \leq \lambda \leq 1$. *Then,*

$$V((1-\lambda)A + \lambda B)^{\frac{1}{d}} \geq (1-\lambda)V(A)^{\frac{1}{d}} + \lambda V(B)^{\frac{1}{d}}.$$

Equality holds if and only if A *and* B *are homothetic.*

Without loss of generality we may assume that volumes are both one and then the general case follows. Indeed, if $\lambda \in (0,1)$, we let

$$\bar{\lambda} = \frac{\lambda V(B)^{\frac{1}{d}}}{(1-\lambda)V(A)^{\frac{1}{d}} + \lambda V(B)^{\frac{1}{d}}}$$

then applying (7) to unit volume $V(A)^{-\frac{1}{d}}A$ and $V(B)^{-\frac{1}{d}}B$ gives

$$V\left((1-\bar{\lambda})\frac{A}{V(A)^{\frac{1}{d}}} + \bar{\lambda}\frac{B}{V(B)^{\frac{1}{d}}}B\right)^{\frac{1}{d}} \ge \frac{(1-\bar{\lambda})}{V(A)^{\frac{1}{d}}}V(A)^{\frac{1}{d}} + \frac{\bar{\lambda}}{V(B)^{\frac{1}{d}}}V(B)^{\frac{1}{d}} = 1.$$

But the left side is

$$V\left(\frac{(1-\lambda)V(A)^{\frac{1}{d}}\frac{A}{V(A)^{\frac{1}{d}}} + \lambda V(B)^{\frac{1}{d}}\frac{B}{V(B)^{\frac{1}{d}}}}{(1-\lambda)V(A)^{\frac{1}{d}} + \lambda V(B)^{\frac{1}{d}}}\right)^{\frac{1}{d}} = \frac{V\left((1-\lambda)A + \lambda B\right)^{\frac{1}{d}}}{(1-\lambda)V(A)^{\frac{1}{d}} + \lambda V(B)^{\frac{1}{d}}}$$

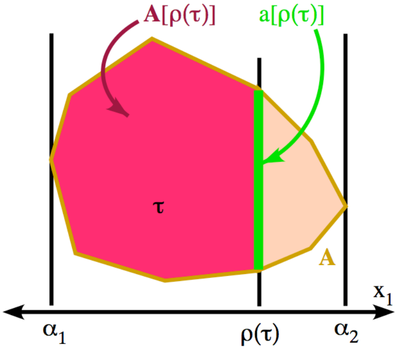


Figure: Left Portion of A.

The idea is to prove the inequality inductively on dimension. Let the convex bodies $A, B \subset \mathbb{R}^d$ with $V(A) = V(B) = 1$.

Choose a direction, say the x_1 axis. The projection of A is $[\alpha_1, \alpha_2]$ on the axis. Cut the body along the $x_1 = \xi$ plane. Define the left portion and face by

$$A[\xi] = \{x \in A : x_1 \le \xi\},$$
$$a[\xi] = \{x \in A : x_1 = \xi\}.$$

Similarly for B. Let $\tau \in [0, 1]$ denote the volume of the portion and $\rho(\tau)$ and $\sigma(\tau)$ denote the corresponding x_1 coordintes

$$\tau = V(A[\rho(\tau)]) = V(B[\sigma(\tau)]).$$

Rewriting volumes $\tau = V(A[\rho(\tau)]) = V(B[\sigma(\tau)])$,

$$\tau = \int_{\alpha_1}^{\rho(\tau)} v(a[\xi])\, d\xi = \int_{\beta_1}^{\sigma(\tau)} v(b[\eta])\, d\eta,$$

where v is the $d-1$ dimensional volume. Differentiating, we find

$$1 = v(a[\rho(\tau)])\frac{d\rho}{d\tau} = v(b[\sigma(\tau)])\frac{d\sigma}{d\tau}. \tag{8}$$

Base Case. For $d = 1$, if $A = [\alpha_1, \alpha_2]$ and $B = [\beta_1, \beta_2]$ then $(1-\lambda)A + \lambda B = [(1-\lambda)\alpha_1 + \lambda\beta_1, (1-\lambda)\alpha_2 + \lambda\beta_2]$ so that Thus the volume of the Minkowski sum

$$\begin{aligned} V((1-\lambda)A + \lambda B) &= (1-\lambda)\alpha_2 + \lambda\beta_2 - [(1-\lambda)\alpha_1 + \lambda\beta_1] \\ &= (1-\lambda)[\alpha_2 - \alpha_1] + \lambda[\beta_2 - \beta_1] \\ &= (1-\lambda)V(A) + \lambda V(B). \end{aligned}$$

Induction Case. For $d > 1$ we assume that Brunn-Minkowski holds for $d-1$. Let $\gamma(\tau) = (1-\lambda)\rho(\tau) + \lambda\sigma(\tau)$. The Minkowski sum

$$S_\lambda = (1-\lambda)A + \lambda B$$

is defined for $x_1 \in [(1-\lambda)\alpha_1 + \lambda\beta_1, (1-\lambda)\alpha_2 + \lambda\beta_2]$. Its $\gamma(\tau)$ slice contains the Minkowski sum of the sections

$$s_\lambda[\gamma(\tau)] \supset (1-\lambda)a[\rho(\tau)] + \lambda b[\sigma(\tau)]. \tag{9}$$

Its volume is given by the integral

$$V(S_\lambda) = \int_{(1-\lambda)\alpha_1+\lambda\beta_1}^{(1-\lambda)\alpha_2+\lambda\beta_2} v(s_\lambda[\zeta])\, d\zeta.$$

Change variables to $\zeta = \gamma(\tau)$ using (8). By (9) and the induction hypothesis,

$$
\begin{aligned}
V(S_\lambda) &= \int_0^1 v\Big(s_\lambda[\gamma(\tau)]\Big)\frac{d\gamma}{d\tau}\,d\tau \\
&\geq \int_0^1 v\Big((1-\lambda)a[\rho(\tau)] + \lambda b[\sigma(\tau)]\Big)\left((1-\lambda)\frac{d\rho}{d\tau} + \lambda\frac{d\sigma}{d\tau}\right)\,d\tau \\
&\geq \int_0^1 \left((1-\lambda)v(a[\rho(\tau)])^{\frac{1}{d-1}} + \lambda v(b[\sigma(\tau)])^{\frac{1}{d-1}}\right)^{d-1} \\
&\qquad \cdot \left(\frac{1-\lambda}{v(a[\rho(\tau)])} + \frac{\lambda}{v(b[\sigma(\tau)])}\right)\,d\tau \\
&\geq 1 = (1-\lambda)V(A)^{\frac{1}{d}} + \lambda V(B)^{\frac{1}{d}},
\end{aligned}
$$

using Jensen's Inequality and unity of volumes.

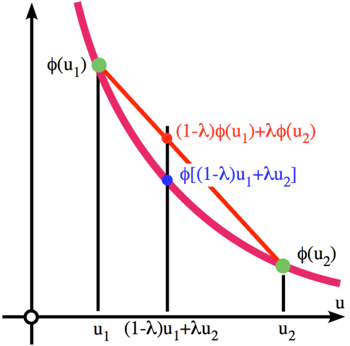


Figure: Convex $\phi(u) = u^{-\frac{1}{d-1}}$.

The inequality between heights of the curve and chord is called Jensen's Inequality.

Jensen's Inequality for $\phi(u) = u^{-\frac{1}{d-1}}$ says

$$(1-\lambda)\phi(u_1) + \lambda\phi(u_2) \ge \phi((1-\lambda)u_1 + \lambda u_2).$$

Let $u_1 = \frac{1}{v(a[\rho(\tau)])}$ and $u_2 = \frac{1}{v(b[\sigma(\tau)])}$. Hence

$$(1-\lambda)\left(\frac{1}{v(a)}\right)^{-\frac{1}{d-1}} + \lambda\left(\frac{1}{v(b)}\right)^{-\frac{1}{d-1}}$$
$$\ge \left(\frac{1-\lambda}{v(a)} + \frac{\lambda}{v(b)}\right)^{-\frac{1}{d-1}}$$

and the desired inequality follows.

Equality implies $u_1 = u_2$.

To see that equality in (7) implies A and B are homothetic, it suffices to show in case $V(A) = V(B)$ that A and B are translates.

To this end we translate A and B so that they have a common center of mass since (7) is independent of translation. Expressing the x_1 coordinate of the center of mass, by (8),

$$\begin{aligned} &\int_{\alpha_1}^{\alpha_2} \xi\, v(a[\xi])\, d\xi = \int_0^1 \rho(\tau) v(a[\rho(\tau)]) \frac{d\rho}{d\tau}\, d\tau = \int_0^1 \rho(\tau)\, d\tau \\ &= \int_{\beta_1}^{\beta_2} \xi\, v(b[\xi])\, d\xi = \int_0^1 \sigma(\tau) v(b[\sigma(\tau)]) \frac{d\sigma}{d\tau}\, d\tau = \int_0^1 \sigma(\tau)\, d\tau \end{aligned} \tag{10}$$

Equality in (7) implies equality in Jesen's Inequality. Thus for every $\tau \in (0,1)$ we have $v(a[\rho(\tau)]) = v(b[\sigma(\tau)])$. It follows from (8) that

$$\rho(\tau) - \sigma(\tau) = \mathit{const.},$$

and from (10) that the *const*. $= 0$.

Finally, since $\rho(\tau) = \sigma(\tau)$ for $0 < \tau < 1$, it follows that

$$\alpha_2 = \lim_{\tau \to 1} \rho(\tau) = \lim_{\tau \to 1} \sigma(\tau) = \beta_2.$$

Thus in the direction of the x_1-axis, the support planes of A and B coincide, and thus the support numbers in this direction are equal.

Since we could have chosen any direction for the x_1-axis, the support numbers of A and B are identical in all directions, hence A and B are identical, as claimed. □

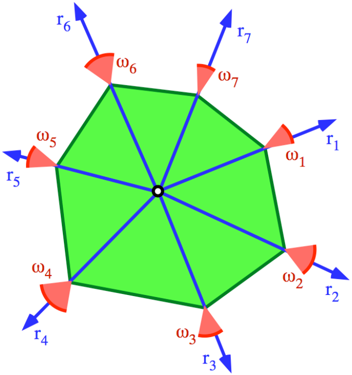


Figure: Find Polyhedron with Vertices on Rays with Given Angles

Polyhedra with Vertices on Given Rays.

Suppose $r_1, \ldots, r_m$ are rays emanating from the origin such that they don't all lie in any halfspace. Suppose we are given positive numbers $\omega_1, \ldots, \omega_m$. The problem is to find a convex polyhedron with one vertex on each ray such that the spherical angle at the vertex on r_i is given by ω_i.

The spherical angle at the vertex is the area of the region in the unit sphere which is the convex hull of the normals of the neighboring faces. In two dimensions, it is the angle between neighboring normals.

Let $\Omega_{i_1,\ldots,i_k}$ denote the angle at the origin of the infinite cone which is the convex hull of the rays $\mathcal{H}(r_{i_1},\ldots,r_{i_k})$.

Theorem (Alexandrov)

Let $r_1,\ldots,r_m \in \mathbb{R}^3$ be rays emanating from the origin such that they don't all lie in a halfspace. That the numbers $\omega_1,\ldots,\omega_m$ be the spherical angles at the vertices of a polytope whose vertices lie on the rays $r_1,\ldots,r_m$ it is necessary and sufficient that

1. *All $\omega_i > 0$.*
2. $\sum_{i=1}^m \omega_i = 4\pi$.
3. *For every subset of rays $r_{i_1},\ldots,r_{i_k}$ contained in some halfspace, there holds $\sum_p \omega_{i_p} > \Omega_{i_1,\ldots,i_k}$ where the sum is over all rays r_p not in the convex hull $\mathcal{H}(r_{i_1},\ldots,r_{i_k})$.*

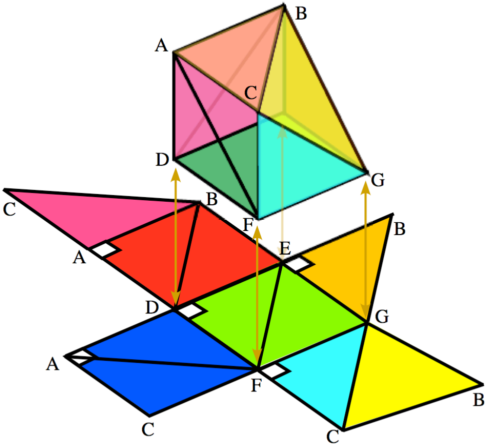


Figure: Find Polyhedron with Given Net

Weyl's Problem for Polyhedra

Cut apart the surface of a polytope in $\mathbb{R}^3$, unroll it onto the plane and cut the resulting development into polygons. The result is a net, the gluing plan how to reassemble the polygons back into the polytope.

Is it possible to start with a collection of polygons in the plane and instructions giving which polygon sides to glue, and then find a convex polyhedron whose surface has this gluing plan?

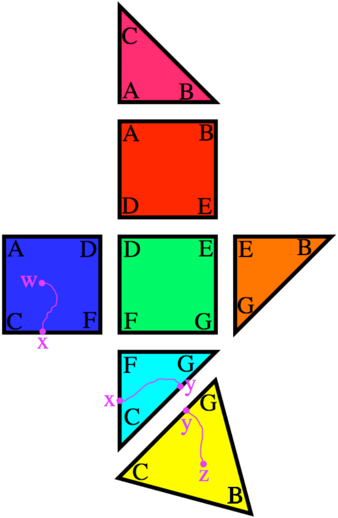


Figure: Net or Gluing Plan

The gluing plan for a polyhedron satisfies necessary conditions. We are given finitely many polygons in the plane, labeled at the corners.

1. Each side of any polygon is identified with exactly one other side. e.g., *EB* occurs exactly twice: as a side of the red square and as a side of the orange triangle. The identified sides have equal lengths and the orientation is preserved. Call this local planarity. This implies no boundary.

2. For any points, say *w* and *z* in the polygons, there is a connecting path that may cross polygon to polygon at identified points on the sides *wxyz*. Call this connectedness.

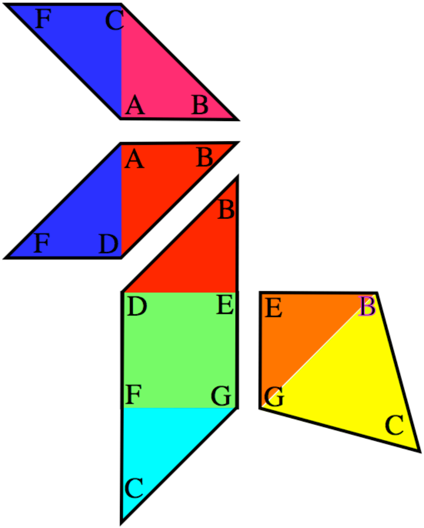


Figure: Another Net for the same Polyhedron

There are many other nets that give the same polyhedron. Here we give another for our truncated cube example.

Note that interior points of a side may be "corners" of a polygon, such as E and F in the red-green-cyan parallelogram. It may also happen that sides of the same polygon get identified.

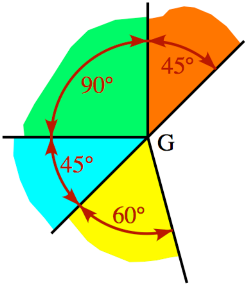


Figure: Intrinsic Angle at G.

The intrinsic angle depends only on the net. It is not the spherical angle.

Gluing the sides together, *i.e.*, taking the union of polygons and identifying corresponding sides and corners is the identification space. It inherits the local Euclidean structure (lengths, angles, areas) of the polygons. Call the identified sides edges and identified corners vertices.

3. At each vertex V, the total angle at the adjacent corners is at most 2π. e.g., at the vertex G in the diagram, the sum of angles is

$$\frac{\pi}{4}+\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi}{3}=\frac{4\pi}{3}\le 2\pi.$$

This condition is called nonnegative curvature.

4. The surface of a convex polyhedron is homeomorphic to the sphere. The polygons of the boundary satisfy the Euler condition

$$\chi = v - e + f = 2,$$

where v is the number of identified vertices, e is the number of edges and f is the number of polygons. In our truncated cube example, $v = 7$, $e = 12$ and $f = 7$ so that $v - e + f = 7 - 12 + 7 = 2$. χ is called the Euler Characteristic.

Theorem

The identification space of a connected, planar gluing plan is homeomorphic to the sphere if and only if $v - e + f = 2$.

To simplify the statement, we consider the double cover of a planar convex polygon, *i.e.*, two congruent convex polygons lying on top of each other sewn along their sides as a closed convex polyhedron.

Theorem (Weyl)

Suppose finitely many polygons are given in the plane and a gluing plan that is locally planar, connected, with nonconvex vertices and such that the Euler condition holds. Then there closed convex polyhedron that realizes the gluing plan. This polyhedron is unique up to rigid motion and reflection.

Here is Minkowsi's proof of (7) using induction. The inequality is proved for finite unions of rectangular boxes first and then a limiting process gives the general statement. Suppose that $A = \cup_{i=1}^{n} R_i$ and $B = \cup_{j=1}^{m} S_j$ where R_i and S_j are pairwise disjoint open rectangles, that is $R_i \cap R_j = \emptyset$ and $S_i \cap S_j = \emptyset$ if $i \neq j$. The proof is based on induction on $\ell = m + n$.

For $\ell = 2$ there are two boxes $R = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ and $R = (c_1, d_1) \times (c_2, d_2) \times (c_3, d_3)$ so $R + S$
$= (a_1 + c_1, b_1 + d_1) \times (a_2 + c_2, b_2 + d_2) \times (a_3 + c_3, b_3 + d_3)$. Then $V(R) = \prod_{i=1}^{3} \ell_i$, $V(S) = \prod_{i=1}^{3} w_i$ and $V(R + S) = \prod_{i=1}^{3}(\ell_i + w_i)$ where $\ell_i = b_i - a_i$ and $w_i = d_i - c_i$. Using Arithmetic-Geometric Inequality,

$$\frac{V(R)^{\frac{1}{3}} + V(S)^{\frac{1}{3}}}{V(R+S)^{\frac{1}{3}}} = \frac{\prod \ell_i^{\frac{1}{3}} + \prod w_i^{\frac{1}{3}}}{\prod(\ell_i + w_i)^{\frac{1}{3}}} = \prod_{i=1}^{3} \left(\frac{\ell_i}{\ell_i + w_i}\right)^{\frac{1}{3}} + \prod_{i=1}^{3} \left(\frac{w_i}{\ell_i + w_i}\right)^{\frac{1}{3}}$$

$$\leq \frac{1}{3} \left(\sum_{i=1}^{3} \frac{\ell_i}{\ell_i + w_i} + \sum_{i=1}^{3} \frac{w_i}{\ell_i + w_i} \right) = 1.$$

Now assume the induction hypothesis: suppose that (7) holds for $A = \cup_{i=1}^{n} R_i$ and $B = \cup_{j=1}^{m} S_j$ with $m + n \le \ell - 1$. For A and B so that $m + n = \ell$, we may arrange that $n \ge 2$. Then some vertical or horizontal plane, say $x = x_1$, can be placed between two rectangles. Let $R_i' = R_i \cap \{(x, y) : x < x_1\}$ and $R_i'' = R_i \cap \{(x, y) : x > x_1\}$ and put $A' = \cup_i R_i'$ and $A'' = \cup_i R_i''$. By choice of the plane, the number of nonempty rectangles in $\#A' < n$ and $\#A'' < n$, but both A' and A'' are nonempty. Select a second plane $x = x_2$ and set $S_i' = S_i \cap \{(x, y) : x < x_2\}$ and $S_i'' = S_i \cap \{(x, y) : x > x_2\}$ and put $B' = \cup_i S_i'$ and $B'' = \cup_i S_i''$. Note that $\#B' \le m$ and $\#B'' \le m$. x_2 can be chosen so that the area fraction is preserved

$$\theta = \frac{V(A')}{V(A') + V(A'')} = \frac{V(B')}{V(B') + V(B'')}.$$

By definiton of Minkowski sum, $A + B \supset A' + B' \cup A'' + B''$. Furthermore, observe that $A' + B'$ is to the left and $A'' + B''$ is to the right of the plane $x = x_1 + x_2$, so they are disjoint sets. Now we may use the additivity of area and the induction hypothesis on $A' + B'$ and $A'' + B''$.

$$
\begin{aligned}
V(A + B) &\geq V(A' + B') + V(A'' + B'') \\
&\geq \left(V(A')^{\frac{1}{3}} + V(B')^{\frac{1}{3}} \right)^3 + \left(V(A'')^{\frac{1}{3}} + V(B'')^{\frac{1}{3}} \right)^3 \\
&= \theta \left(V(A)^{\frac{1}{3}} + V(B)^{\frac{1}{3}} \right)^3 + (1 - \theta) \left(V(A)^{\frac{1}{3}} + V(B)^{\frac{1}{3}} \right)^3 \\
&= \left(V(A)^{\frac{1}{3}} + V(B)^{\frac{1}{3}} \right)^3 .
\end{aligned}
$$

Thus the induction step is complete. □

Finally every compact region can be realized as the intersection of a decreasing sequence of open sets $A_n \supset A_{n+1}$ so that $A = \cap_n A_n$. A_n can be taken as the interiors of a union of finitely many closed squares. For each $\varepsilon = 2^{-n} > 0$ consider the closed squares in the grid of side ε which meet the set. Then the interior of the union of these squares is A_n. Removing the edges of the squares along gridlines A_n' results in a set with the same area. The result follows since Lebesgue measure of the limit is limit of the Lebesgue measure for decreasing sequences. Since the Minkowski sum of a decreasing set of opens is itself a decreasing set of opens, it follows that

$$\begin{aligned} V(A+B)^{\frac{1}{3}} &= \lim_{n\to\infty} V(A_n+B_n)^{\frac{1}{3}} \geq \lim_{n\to\infty} V(A_n'+B_n')^{\frac{1}{3}} \\ &\geq \lim_{n\to\infty} \left(V(A_n')^{\frac{1}{3}} + V(B_n')^{\frac{1}{3}} \right) = V(A)^{\frac{1}{3}} + V(B)^{\frac{1}{3}} \end{aligned}$$

and we are done. $\square$

Theorem (Arithmetic-Geometric Inequality)

Let $x_i \geq 0$ for $i = 1, \ldots, n$. Then

$$\text{Geometric Mean} = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} x_i = \text{Arithmetic Mean.}$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

If $x_1 = x_2 = \cdots = x_n = c$ then both sides equal c so equality holds. Let $S = \sum_{i=1}^{n} x_i$. We maximize

$$f(y) = \prod_{i=1}^{n} y_i$$

subject to $y_i \geq 0$ and $\sum_{i=1}^{n} y_i = S$. We have $y_i \leq S$ so the function is to be maximized over closed and bounded subset of $\mathbb{R}^n$. As f is continuous, it has a maximum. If $S = 0$ the maximum is zero at the origin.

If $S > 0$ then $f > 0$ and the maximum occurs in the interior of the orthant. The Lagrange Multiplier method says the maximum occurs at critical pints of the function

$$\mathcal{L} = f(y) - \mu \left(\sum_{i=1}^{n} y_i - S \right)$$

At the maximum point z,

$$0 = \frac{\partial \mathcal{L}}{\partial y_i} = \frac{f(z)}{z_i} - \mu$$

so that $\mu z_i = f(z)$ hence all z_i are equal and $\mu > 0$. Adding, $z_i = S/n$ so $\mu S/n = f(z) = S^n/n^n$. Thus $\mu = (S/n)^{n-1}$. It follows that

$$\left(\prod_{i=1}^{n} y_i \right)^{\frac{1}{n}} = f(y)^{\frac{1}{n}} \le f(z)^{\frac{1}{n}} = \frac{1}{n} S = \frac{1}{n} \sum_{i=1}^{n} y_i.$$

Equality holds iff $y_1 = y_2 = \cdots = y_n$. $\square$

Thanks!