University of Utah Undergraduate Mathematics Colloquium

Mixed Area and the Isoperimetric Inequality

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- URL of Beamer Slides:

"Mixed Area and the Isoperimetric Inequality"

http://www.math.utah.edu/~treiberg/MixedAreaSlides.pdf

3. References..

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4. Outline.

Most everything we say about Mixed Areas is also true also for higher dimensional Mixed Volumes, and is part of the field of Convex Geometry.

- 1 Convex Sets & Minkowski Addition.
- 2 Minkowski's Theorem & Mixed Area.
- **3** Support Functions.
- 4 Steiner's Formula.
- 5 Brunn-Minkowski Inequality.
 - Proof by Wirtinger's Inequality.
- 6 Isoperimetric Inequality.
- 7 A Better Isoperimetric Inequality for Polygons.
- 8 Stabiliy Versions of Brunn's and Isoperimetric Inequalities.
 - Sharpened Wirtinger's Inequality.

5. Convex Sets.

Let $S \subset \mathbb{E}^2$ be a compact (closed and bounded) set.

A set is convex if for every pair of points in the set $x, y \in S$, the line segment from x to y is also contained in the set.

A compact convex set is called a convex body. The set of all convex bodies in \mathbb{E}^2 is denoted $\mathcal{K}^2.$



Figure: Convex Body S in \mathbb{E}^2 .

6. Minkowski Addition of Sets.



Figure: Hermann Minkowski 1864–1909. *"Mich interessiert alles, was konvex ist!"* H. Minkowski. Minkowski Addition of any two sets $A, B \subset \mathbb{E}^2$ is defined to be

 $A \boxplus B := \{x + y : x \in A \text{ and } y \in B\}.$

For $s \ge 0$, the Minkowski Dilation by factor s is defined to be

$$sA = \{sx : x \in A\}$$

For example, the Minkowski sum of dilated rectangles is a rectangle

 $s([0, a] \times [0, b]) \boxplus t([0, c] \times [0, d])$ = [0, sa + tc] × [0, sb + td]. (1)

7. Minkowski Addition of Sets.



Figure: Minkowski Addition of a Triangle and a Rectangle.

8. Kinematic interpretation of Minkowski Addition.



Minkowski Addition may be written

$$A \boxplus B = \bigcup_{x \in B} A \boxplus \{x\}.$$

It is the union of all translates of *A* by points of *B*. The *A* set is smeared around by the *B* set.

Figure: Minkowski Addition as Smear.

9. Morphing interpretation of Minkowski Addition.



Figure: $(1 - r)A \boxplus rB$ as sections of convex hull of $K = A \times \{0\} \cup B \times \{1\}$.

For $0 \le r \le 1$, $(1-r)A \boxplus rB$ linearly morphs A at r = 0 into B at r = 1.

In (x, y, r) space draw A and B in different planes

$$\mathcal{K} = \{(x, y, 0) : (x, y) \in A\} \cup \\ \{(x, y, 1) : (x, y) \in B\}$$

and consider the three dimensional convex hull \hat{K} (outlined in green in the Figure.) Then the r = const. section is the Minkowski convex combination

$$\{(x, y, r) : (x, y) \in \mathbb{R}^2\} \bigcap \hat{K} \\= \{(x, y, r) : (x, y) \in (1-r)A \boxplus rB\}.$$

Theorem (Minkowski)

Let A and B be convex bodies in the plane. Then the area of the linear combination $sA \boxplus tB$ is a quadratic form for nonnegative s and t,

$$\mathcal{A}(sA \boxplus tB) = a_{11}s^2 + 2a_{12}st + a_{22}t^2.$$

Coefficients of s^2 and t^2 are evidently areas $a_{11} = \mathcal{A}(A)$ and $a_{22} = \mathcal{A}(B)$. The cross term defines $\mathcal{A}(A, B) := a_{12}$, called the mixed area.

For $A, B, C \in \mathcal{K}^2$ and nonnegative s, t the following hold:

•
$$\mathcal{A}(A,B) \geq 0;$$

•
$$\mathcal{A}(A,B) = \mathcal{A}(B,A);$$

• If ρ is a rigid motion then $\mathcal{A}(\rho A, \rho B) = \mathcal{A}(A, B)$;

•
$$\mathcal{A}(A, A) = \mathcal{A}(A);$$

- $\mathcal{A}(sA \boxplus tB, C) = s \mathcal{A}(A, C) + t \mathcal{A}(B, C);$
- If $A \subset B$ then $\mathcal{A}(A, C) \leq \mathcal{A}(B, C)$.



Figure: Addition of Set and Segment.

Let $K \in \mathcal{K}^2$ and $L = \{r(\cos \theta, \sin \theta) : 0 \le r \le 1\}$, a line segment in the θ direction. The sum is the smear of sK along tL. The area of $sK \boxplus tL$ consists of the sK part and the rest, whose area is $tw(\theta + \frac{\pi}{2})$, where $w(\theta + \frac{\pi}{2})$ is the width of the body K projected to the perpendicular line.

$$\begin{aligned} \mathcal{A}(sK \boxplus tL) &= \\ &= \mathcal{A}(K)s^2 + 2 \mathcal{A}(K,L)st + \mathcal{A}(L)t^2 \\ &= \mathcal{A}(K)s^2 + w(\theta + \frac{\pi}{2})st + 0. \end{aligned}$$

Hence $\mathcal{A}(K,L) = \frac{1}{2}w(\theta + \frac{\pi}{2}).$

12. Coordinates of a line.

An unoriented line in the plane is determined by two numbers, p the distance to the origin and θ , the direction to the closest point.

The variable range is $0 \le p$ and $0 \le \theta < 2\pi$.

Equivalently, we may take the range $-\infty < \tilde{p} < \infty$ and $0 \le \eta < \pi$.



Figure: (p, θ) coordinates for the line L.

The equation of the line $L(p, \theta)$ in Cartesian coordinates is $\cos(\theta)x + \sin(\theta)y = p$

(2)

13. Support function and width.



Figure: Width and support function of convex Ω in θ direction.

For $\theta \in [0, 2\pi)$, the support function, $h(\theta)$, is the largest p such that $L(p, \theta) \cap \Omega \neq \emptyset$. The width is $w(\theta) = h(\theta) + h(\theta + \pi)$.

14. Length of a convex curve in terms of its support function.



Figure: Tangent Velocity Vector Z'.

For a C^2 strictly convex Ω , $h(\theta)$ determines the position $Z(\theta)$ uniquely. $N(\theta) = (\cos \theta, \sin \theta)$ is the outward normal vector. Thus $N' = (-\cos \theta, \sin \theta)$ and N'' = -N. $Z(\theta)$ is the position so $Z'(\theta)$ is the velocity with $N \cdot Z' = 0$. The support function $h(\theta)$ satisfies

 $h(\theta) = Z(\theta) \cdot N(\theta).$

Differentiating wrt θ ,

$$h' = Z \cdot N' + Z' \cdot N = Z \cdot N'.$$

Hence Z = hN + h'N'. The speed is $\frac{ds}{d\theta} = |Z'| =$

= |h'N + hN' + h''N' - h'N| = h + h''.

Also the curvature is

$$rac{d heta}{ds}=\kappa=rac{1}{h+h''}>0.$$



Figure: Augustin Louis Cauchy 1789–1857

Theorem (Cauchy's Formula [1841])

Let Ω be a bounded convex domain. Then

$$\mathsf{L}(\partial\Omega) = \int_0^{2\pi} h(\theta) \, d\theta = \int_0^{\pi} w(\theta) \, d\theta.$$
 (3)

Thus the average width of
$$\Omega$$
 is $\frac{\mathcal{L}(\partial \Omega)}{\pi}$.

$$\begin{split} \mathsf{L}(\partial\Omega) &= \int_{\partial\Omega} ds = \int_0^{2\pi} h + h'' \, d\theta = \\ &= \int_0^{2\pi} h(\theta) \, d\theta = \int_0^{\pi} h(\theta) + h(\theta + \pi) \, d\theta \\ &= \int_0^{\pi} w(\theta) \, d\theta. \quad \Box \end{split}$$

16. Area in terms of support function.

Theorem

Suppose Ω is a compact, convex domain with a C^2 boundary. Then

$$\mathcal{A}(\Omega) = \frac{1}{2} \int_0^{2\pi} h \, ds = \frac{1}{2} \int_0^{2\pi} h^2 - \dot{h}^2 \, d\theta.$$
(4)



Figure: Area in Polar Coordinates.

Since
$$\frac{ds}{d\theta} = h + \ddot{h}$$
 so
 $\mathcal{A}(\Omega) = \int_{\Omega} dA$
 $= \frac{1}{2} \int_{0}^{2\pi} h \, ds$
 $= \frac{1}{2} \int_{0}^{2\pi} h(h + \ddot{h}) \, d\theta$
 $= \frac{1}{2} \int_{0}^{2\pi} h^2 - \dot{h}^2 \, d\theta.$

17. Support Function of a Minkowski Sum.

Suppose $A, B \in \mathcal{K}^2$ and $s, t \ge 0$. Let $h_A(\theta)$ and $h_B(\theta)$ be their support functions. Let $U(\theta) = (\cos \theta, \sin \theta)$. Since the support function may be written

$$h_A(heta) = \max_{z \in A} \ z \cdot U(heta)$$

and

$$sA \boxplus tB = \{sx + ty : x \in A \text{ and } y \in B \}$$

we have

$$h_{sA\boxplus tB}(\theta) = sh_A(\theta) + th_B(\theta).$$

The support function of a Minkowski linear combination is the linear combination of support functions.

E.g., since the support function of a point $\{(a, b)\}$ is $a \cos \theta + b \sin \theta$, the support function of the translation of a body K by (a, b) is

$$h_{K\boxplus\{(a,b)\}}(\theta) = h_A(\theta) + a\cos\theta + b\sin\theta.$$

18. Proof of Minkowski's Theorem and a Formula for Mixed Area.

Minkowski proved his theorem first for polygons where it is easier and then approximated convex bodies by polygons. For piecewise C^1 convex bodies, $A, B \in \mathcal{K}^2$ and $s, t \ge 0$, Minkowski's theorem is an immediate consequence of the area formula.

$$\begin{aligned} \mathcal{A}(sA \boxplus tB) &= \frac{1}{2} \int_{0}^{2\pi} h_{sA \boxplus tB}^{2} - \dot{h}_{sA \boxplus tB}^{2} \, d\theta \\ &= \frac{1}{2} \int_{0}^{2\pi} (sh_{A} + th_{B})^{2} - \left(s\dot{h}_{A} + t\dot{h}_{B}\right)^{2} \, d\theta \\ &= \left(\frac{1}{2} \int_{0}^{2\pi} h_{A}^{2} - \dot{h}_{A}^{2} \, d\theta\right) s^{2} + \left(\int_{0}^{2\pi} h_{A}h_{B} - \dot{h}_{A}\dot{h}_{B} \, d\theta\right) st \\ &+ \left(\frac{1}{2} \int_{0}^{2\pi} h_{B}^{2} - \dot{h}_{B}^{2} \, ds\right) t^{2} \\ &= \mathcal{A}(A)s^{2} + 2 \, \mathcal{A}(A, B)st + \mathcal{A}(B)t^{2}. \end{aligned}$$

Thus the area is quadratic. Moreover,

$$\mathcal{A}(A,B) = rac{1}{2} \int_0^{2\pi} h_A h_B - \dot{h}_A \dot{h}_B \, d\theta.$$

For piecewise C^2 strictly convex bodies A, B, convexity implies $h_B + h''_B > 0$. Since we may translate the body without changing the mixed area, we may assume that the origin $0 \in A$ so that $h_A \ge 0$. Positivity of the mixed area follows. Integrating by parts, using that h_A and h_B are 2π -periodic,

$$egin{aligned} \mathcal{A}(A,B) &= rac{1}{2} \int_{0}^{2\pi} h_A h_B - \dot{h}_A \dot{h}_B \, d heta \ &= rac{1}{2} \int_{0}^{2\pi} h_A \left(h_B + \ddot{h}_B
ight) \, d heta \ &\geq 0. \end{aligned}$$

The other properties we listed are proved similarly.

20. Steiner's Formulas.

The *r*-Parallel Set Ω_r of a convex body Ω is the set of points within distance $r \ge 0$ of Ω .

$$\Omega_r = \{x \in \mathbb{E}^2 : \operatorname{dist}(x, \Omega) \leq r\} = \Omega \boxplus rB$$

where B is the closed unit ball.

Theorem (Steiner's Formulas)

The area and boundary length of the r-parallel set of a convex body Ω are

$$\mathcal{A}(\Omega_r) = \mathcal{A}(\Omega) + \mathcal{L}(\partial\Omega)r + \pi r^2, \quad (5)$$

$$\mathcal{L}(\partial\Omega_r) = \mathcal{L}(\partial\Omega) + 2\pi r, \quad (6)$$

$$\mathcal{L}(\partial A) = 2 \mathcal{A}(A, B). \quad (7)$$



Figure: Jakob Steiner 1796–1863.

 $\mathcal{A}(A \boxplus rB) = \mathcal{A}(A) + 2 \mathcal{A}(A, B)r + \mathcal{A}(B)r^{2}$

so Steiner's Formula (5) implies (7).

21. Steiner's Formula is Obvious for Polygons.



Figure: Parallel set P_r of polygon has area $\mathcal{A}(P) + \mathcal{L}(\partial P)r + \pi r^2$.

22. Proof of Steiner's Formulas.



We indicate the proof if $\partial\Omega$ is C^1 . Let $h(\theta)$ be the support function of Ω and r the support function of rB centered at the origin. Hence the support function of $\Omega \boxplus rB$ is h + r. The area and length formulas give

$$egin{aligned} \mathcal{A}(\Omega_r) &= rac{1}{2} \int_0^{2\pi} (h+r)^2 - \dot{h}^2 \, d heta \ &= rac{1}{2} \int_0^{2\pi} h^2 - \dot{h}^2 \, d heta + rac{1}{2} \int_0^{2\pi} 2hr + r^2 \, d heta \ &= \mathcal{A}(\Omega) + \mathcal{L}(\partial\Omega)r + \pi r^2; \end{aligned}$$

rB

$$\mathcal{L}(\partial\Omega_r) = \int_0^{2\pi} h + r \, d\theta$$

= $\int_0^{2\pi} h \, d\theta + 2\pi r$
= $\mathcal{L}(\partial\Omega) + 2\pi r$.

Figure: Parallel set Ω_r .

23. Wirtinger's Inequality.

Wirtinger's Inequality bounds the L^2 norm of a function by the L^2 norm of its derivative. It is also known as the Poincaré Inequality in higher dimensions. We state stronger hypotheses than necessary.



Figure: Wilhelm Wirtinger 1865–1945.

Theorem (Wirtinger's inequality)

Let $f(\theta)$ be a piecewise $C^1(\mathbb{R})$ function with period 2π (for all θ , $f(\theta + 2\pi) = f(\theta)$). Let \overline{f} denote the mean value of f

$$ar{f} = rac{1}{2\pi} \int_0^{2\pi} f(heta) \, d heta.$$

Then

$$\int_{0}^{2\pi} \left(f(\theta) - \overline{f}
ight)^2 \ d\theta \leq \int_{0}^{2\pi} \left(f'(\theta)
ight)^2 \ d\theta.$$

Equality holds iff for some constants a, b,

$$f(\theta) = \overline{f} + a\cos\theta + b\sin\theta.$$

24. Proof of Wirtinger's Inequality.

Idea: express f and f' in Fourier series. Since f' is bounded and f is continuous, the Fourier series converges at all θ

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \{a_k \cos k\theta + b_k \sin k\theta\}$$

where the Fourier coefficients are determined by formally multiplying by $\sin m\theta$ or $\cos m\theta$ and integrating to get

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos m\theta \, d\theta, \qquad b_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin m\theta \, d\theta,$$

hence $2\overline{f} = a_0$. Since the sines and cosines are complete, the Parseval equation holds

$$\int_{0}^{2\pi} \left(f - \bar{f} \right)^{2} = \pi \sum_{k=1}^{\infty} \left(a_{k}^{2} + b_{k}^{2} \right).$$
(8)

Formally, this is the integral of the square of the series, where after multiplying out and integrating, terms like $\int \cos m\theta \sin k\theta = 0$ or $\int \cos m\theta \cos k\theta = 0$ if $m \neq k$ drop out and terms like $\int \sin^2 k\theta = \pi$ contribute π to the sum.

25. Proof of Wirtinger's Inequality..

The Fourier Series for the derivative is given by

$$f'(heta) \sim \sum_{k=1}^{\infty} \{-ka_k \sin k heta + kb_k \cos k heta\}$$

Since f' is square integrable, Bessel's inequality gives

$$\pi \sum_{k=1}^{\infty} k^2 \left(a_k^2 + b_k^2 \right) \le \int_0^{2\pi} (f')^2.$$
(9)

Wirtinger's inequality is deduced form (8) and (9) since

$$\int_0^{2\pi} (f')^2 - \int_0^{2\pi} \left(f - \bar{f} \right)^2 \ge \pi \sum_{k=2}^\infty (k^2 - 1) \left(a_k^2 + b_k^2 \right) \ge 0.$$

Equality implies that for $k \ge 2$, $(k^2 - 1)(a_k^2 + b_k^2) = 0$ so $a_k = b_k = 0$, thus f takes the form $f(\theta) = \overline{f} + a\cos\theta + b\sin\theta$.

The theorem of Hermann Karl Brunn (1862–1939) says that since the Minkowski addition tends to "round out" the figures being added, the area of the added figure exceeds the area of the summands.

Theorem (Brunn's Inequality)

Let $A, B \in \mathbb{E}^2$ be arbitrary bounded measurable sets in the plane. Then,

$$\sqrt{\mathcal{A}(A \boxplus B)} \ge \sqrt{\mathcal{A}(A)} + \sqrt{\mathcal{A}(B)}.$$
 (10)

Equivalently, the mixed area satisfies

$$\mathcal{A}(A,B)^2 \ge \mathcal{A}(A) \mathcal{A}(B). \tag{11}$$

Minkowski proved that equality holds if and only if A and B are homothetic. Two figures A and B are *homothetic* if and only if they are similar and are similarly situated, which means there is a point x and $r \ge 0$ so that $A = rB \boxplus \{x\}$.

27. Proof of Brunn's Inequality from Wirtinger's inequality.

First, the expressions (10) and (11) are equivalent because

$$\mathcal{A}(A \boxplus B) = \mathcal{A}(A) + 2 \mathcal{A}(A, B) + \mathcal{A}(B),$$
$$\left(\sqrt{\mathcal{A}(A)} + \sqrt{\mathcal{A}(B)}\right)^2 = \mathcal{A}(A) + 2\sqrt{\mathcal{A}(A)\mathcal{A}(B)} + \mathcal{A}(B).$$

In case of C^1 convex bodies, let h_A and h_B be the support functions. Their averages are proportional to boundary lengths L_A , L_B by Cauchy's Formula

$$\frac{1}{2\pi} \int_0^{2\pi} h_A \, d\theta = \frac{L_A}{2\pi}, \qquad \qquad \frac{1}{2\pi} \int_0^{2\pi} h_B \, d\theta = \frac{L_B}{2\pi}.$$

Let $f(\theta) = L_B h_A(\theta) - L_A h_B(\theta)$. Its average

$$\frac{1}{2\pi}\int_0^{2\pi}f\,d\theta=0.$$

28. Proof of Brunn's Inequality from Wirtinger's inequality.

Applying Wirtinger's Inequality to f,

$$L_B^2 \int h_A^2 - 2L_A L_B \int h_A h_B + L_A^2 \int h_B^2 =$$

$$\int (L_B h_A - L_A h_b)^2 =$$

$$\int f^2 \leq \int \dot{f}^2$$

$$= \int (L_B \dot{h}_A - L_A \dot{h}_b)^2$$

$$= L_B^2 \int \dot{h}_A^2 - 2L_A L_B \int \dot{h}_A \dot{h}_B + L_A^2 \int \dot{h}_B^2$$

Thus

$$L_B^2\left(\int h_A^2 - \dot{h}_A^2\right) + L_A^2\left(\int h_B^2 - \dot{h}_B^2\right) \le 2L_A L_B\left(\int h_A h_B - \dot{h}_A \dot{h}_B\right)$$
$$L_B^2 \mathcal{A}(A) + L_A^2 \mathcal{A}(B) \le 2L_A L_B \mathcal{A}(A, B).$$
(12)

29. Proof of Brunn's Inequality from Wirtinger's inequality...

Subtracting $2L_A L_B \sqrt{\mathcal{A}(A) \mathcal{A}(B)}$ yields $L_B^2 \mathcal{A}(A) - 2L_A L_B \sqrt{\mathcal{A}(A) \mathcal{A}(B)} + L_A^2 \mathcal{A}(B)$ $\leq 2L_A L_B \mathcal{A}(A, B) - 2L_A L_B \sqrt{\mathcal{A}(A) \mathcal{A}(B)}$

so inequality (11) follows

$$0 \leq \left(L_B\sqrt{\mathcal{A}(A)} - L_A\sqrt{\mathcal{A}(B)}\right)^2 \leq 2L_A L_B\left(\mathcal{A}(A, B) - \sqrt{\mathcal{A}(A)\mathcal{A}(B)}\right).$$

Equality in Brunn's Inequality implies equality in Wirtinger's Inequality. So using $\bar{f} = 0$,

$$L_b h_A - L_A h_B = f = a_1 \cos \theta - b_1 \sin \theta$$

or in other words, A and B are homothetic up to translation.

30. Isoperimetric Inequality.

Among all domains in the plane with a fixed boundary length, the circle has the greatest area. For simplicity we focus on domains bounded by simple curves.

Theorem (Isoperimetric Inequality.)

Let C be a simple closed curve in the plane whose length is L and that encloses an area A. Then the following inequality holds

$$4\pi A \le L^2. \tag{13}$$

2 If equality holds in (13), then the curve C is a circle.

Simple means curve is assumed to have no self intersections. A circle of radius r has $L = 2\pi r$ and encloses $A = \pi r^2 = \frac{L^2}{4\pi}$. Thus the Isoperimetric Inequality says if C is a simple closed curve, then C encloses an area no bigger than the area of the circle with the same boundary length. The convex hull of K, denoted \hat{K} , is the smallest convex set that contains K. This is equivalent to the intersection of all halfspaces that contain K,

$$\hat{K} = \bigcap_{\substack{\Omega \text{ is convex} \\ \Omega \supset K}} \Omega = \bigcap_{\substack{H \text{ is a halfspace} \\ H \supset K}} H.$$

A halfspace is a set of the form $H = \{(x, y) \in \mathbb{E}^2 : ax + by \le c\}$, where (a, b) is a unit vector and c is any real number.

32. Reduce proof of Isoperimetric Inequality to convex domain case.

Since $K \subset \hat{K}$ by its definition, we have $\mathcal{A}(\hat{K}) \geq \mathcal{A}(K)$.

Taking convex hull reduces the boundary length because the interior segments of the boundary curve, the components of $\partial K - \partial \hat{K}$ of ∂K are replaced by straight line segments in $\partial \hat{K}$. Thus also $\mathcal{L}(\partial \hat{K}) \leq \mathcal{L}(\partial K)$.



Figure: The region K and its convex hull \hat{K} .

Thus the isoperimetric inequality for convex sets implies

$$4\pi A \leq 4\pi \hat{A} \leq \hat{L}^2 \leq L^2.$$

Furthermore, one may also argue that equality $4\pi A = L^2$ implies equality $4\pi \hat{A} = \hat{L}^2$ in the isoperimetric inequality for convex sets so that \hat{K} is a circle. But then so is K.

The basic idea is to consider the the extreme points $\partial^* \hat{K} \subset \partial \hat{K}$ of \hat{K} , that is points $x \in \partial \hat{K}$ such that if $x = \lambda y + (1 - \lambda)z$ for some $y, z \in \hat{K}$ and $0 < \lambda < 1$ then y = z = x. \hat{K} is the convex hull of its extreme points. However, the extreme points of the convex hull lie in the curve $\partial^* \hat{K} \subset \partial K \cap \partial \hat{K}$. \hat{K} being a circle implies that every boundary point is an extreme point, and since they come from ∂K , it means that ∂K is a circle.

34. Minkowski's proof of Isoperimetric Inequality from Brunn's Ineq.

The Isoperimetric Inequality follows from Steiner's Formula and Brunn's Inequality. Let Ω be a convex body with piecewise C^1 boundary and B the closed unit ball. Then by (7) and (11),

$$egin{aligned} \mathcal{L}(\partial\Omega)^2 &= 4\,\mathcal{A}(\Omega,B)^2\ &\geq 4\,\mathcal{A}(\Omega)\,\mathcal{A}(B)\ &= 4\pi\,\mathcal{A}(\Omega). \end{aligned}$$

as desired.

Equality implies equality in Brunn's Inequality. Thus up to translation, Ω is homothetic to B, or h_{Ω} is constant so Ω is a ball.

Another proof using (12) is just as fast:

$$egin{aligned} 4\pi^2\,\mathcal{A}(\Omega) + \pi\,\mathcal{L}(\partial\Omega)^2 &= L_B^2\,\mathcal{A}(\Omega) + L_\Omega^2\,\mathcal{A}(B) \ &\leq 2L_\Omega L_B\,\mathcal{A}(\Omega,B) \ &= 2\pi\,\mathcal{L}(\partial\Omega)^2. \end{aligned}$$

35. Better Inequality for Polygons.



Figure: K Circumscribes Unit Circle with Same Normals as P.

Let P be a convex polygon with n sides and let \mathbf{n}_i denote its outward unit normal vectors. Let K be the polygon that circumscribes the unit circle having the same normal directions as P.



Figure: Area of Sector from \mathbf{n}_i to \mathbf{n}_{i+1} .

The total area of K is larger than circle so $\mathcal{A}(K) > \pi$.

Let *OQRS* denote the sector of *K* from the \mathbf{n}_i side to the n_{i+1} side. Let $\alpha_i = \angle(\mathbf{n}_i, \mathbf{n}_{i+1})$. It is also the exterior angle from *QR* to *RS* and twice the angle $\alpha_i/2 = \angle(QOR)$. Thus the area of the triangle

$$\mathcal{A}(\textit{QOR}) = rac{1}{2}bh = rac{1}{2} an\left(rac{lpha_i}{2}
ight)\cdot 1$$

which equals $\mathcal{A}(SOR)$ by reflection across line OR.

Hence, adding over all n sectors, the area of K is

$$\mathcal{A}(\mathcal{K}) = \sum_{i=1}^{n} \tan\left(\frac{\alpha_i}{2}\right).$$
 (14)

37. Compute the Mixed Area of P and K.



Figure: Enlarge all support numbers for Polygon P by r. Get $P \boxplus rK$.

 $\mathcal{A}(P \boxplus rK) = \mathcal{A}(P) + r \mathcal{L}(\partial P) + r^2 \mathcal{A}(K) \text{ so that}$ $2 \mathcal{A}(P, K) = \mathcal{L}(\partial P). \tag{15}$

Theorem (Lhuilier's Inequality.)

Let P be a compact convex polygon with n sides, whose exterior angles are α_i for i = 1, ..., n. Then

$$\mathcal{L}(\partial P)^2 \ge 4 \mathcal{A}(P) \sum_{i=1}^n \tan\left(\frac{\alpha_i}{2}\right).$$

Equality holds if P circumscribes a circle.

Proof. Let K the polygon circumscribing the unit circle with the same normal directions as P. By (15) and Brunn's Inequality (11),

$$\mathcal{L}(\partial P)^2 = 4 \mathcal{A}(P, K)^2 \ge 4 \mathcal{A}(P) \mathcal{A}(K) = 4 \mathcal{A}(P) \sum_{i=1}^n \tan\left(\frac{\alpha_i}{2}\right) \quad (16)$$

as desired. Equality implies equality in Brunn's Inequality. Then, up to translation, P is homothetic to K, so P circumscribes a circle.

Theorem (Jensen's Inequality.)

Let ϕ : $(a, b) \rightarrow \mathbb{R}$ be a strictly convex function and $\xi_i \in (a, b)$ be points and $\mu_i > 0$ be weights for i = 1, ..., n. Then

$$\phi\left(\frac{\sum_{i=1}^{n}\mu_{i}\xi_{i}}{\sum_{i=1}^{n}\mu_{i}}\right) \leq \frac{\sum_{i=1}^{n}\mu_{i}\phi(\xi_{i})}{\sum_{i=1}^{n}\mu_{i}}.$$
(17)

Equality holds if and only if $\xi_1 = \cdots = \xi_n$.

Proof. Consider the points $(\xi_i, \phi(\xi_i))$ with weights μ_i . The center of mass of these points $(\bar{\xi}, \bar{\eta}) = \left(\frac{\sum_{i=1}^n \mu_i \xi_i}{\sum_{i=1}^n \mu_i}, \frac{\sum_{i=1}^n \mu_i \phi(\xi_i)}{\sum_{i=1}^n \mu_i}\right)$

lies in their convex hull, thus is above the curve: $\phi(\bar{\xi}) \leq \bar{\eta}$. Let λ be linear function supporting ϕ so that $\psi = \phi - \lambda$ has a strict minimum at $\bar{\xi}$. Then equality holding in (17) implies equality in $\psi(\bar{\xi}) \sum \mu_i \leq \sum \mu_i \psi(\xi_i)$ which implies $\xi_i = \bar{\xi}$ for all i.



Figure: Center of Gravity of Points on a Convex Curve is Above the Curve.

40. Smallest Circumscribing Polygon with *n* Sides is Regular.

Theorem (The *n*-gon of least area containing a circle is regular.)

Let K be a compact polygon containing the unit circle with n or fewer sides. Then

 $\mathcal{A}(K) \geq n \tan\left(\frac{\pi}{n}\right)$.

Equality holds if and only if K is a regular n-gon.

Proof. Area is reduced by pulling in sides, so we may assume K circumscribes the circle. By (14), such an *n*-gon K has

$$\mathcal{A}(K) = \sum_{i=1}^{n} \operatorname{tan}\left(\frac{\alpha_i}{2}\right),$$

where $0 < \alpha_i < \pi$ are the exterior angles whose total is $\sum \alpha_i = 2\pi$. The function $\phi(\xi) = \tan(\xi/2)$ is convex on $(0, \pi)$. Using Jensen's inequality with weights $\mu_i = 1/n$,

$$\frac{1}{n}\sum_{i=1}^{n}\tan\left(\frac{\alpha_{i}}{2}\right)\geq\tan\left(\frac{1}{2n}\sum_{i=1}^{n}\alpha_{i}\right)=\tan\left(\frac{\pi}{n}\right).$$

Equality holds if and only if $\alpha_i = \frac{2\pi}{n}$ for all *i*.

Theorem (Isoperimetric Inequality for *n*-gons.)

Let P be a compact polygon with n or fewer sides. Then

$$\mathcal{L}(\partial P)^2 \ge 4n \tan\left(\frac{\pi}{n}\right) \mathcal{A}(P).$$

Equality holds if and only if P is a regular n-gon.

Proof. We have observed that we can replace P by its convex hull \hat{P} , which is a polygon with n or fewer sides. Applying (16)

$$\mathcal{L}(\partial P)^2 \geq \mathcal{L}(\partial \hat{P})^2 \geq 4 \,\mathcal{A}(\hat{P}, \mathcal{K})^2 \geq 4 \,\mathcal{A}(\hat{P}) \,\mathcal{A}(\mathcal{K}) \geq 4 \,\mathcal{A}(P) \,\mathcal{A}(\mathcal{K}),$$

where K is the circumscribing polygon with the same normals as \hat{P} . The Minimum Area theorem for circumscribing K finishes the inequality. Equality implies $P = \hat{P}$ and P is regular. 42. How Much Greater is the Polygon Isoperimetric Constant?

The Polygonal Isoperimetric Constant $I_n = 4n \tan\left(\frac{\pi}{n}\right)$ The ratio to the usual Isoperimetric Constant $\frac{I_n}{4\pi}$

п	Polygon Isop. Const. \mathcal{I}_n	Ratio of Isop. Consts. $\frac{\mathcal{I}_n}{4\pi}$
3	20.78460969099	1.65398668628
4	16.0000000000	1.27323954474
5	14.53085056017	1.15632834699
6	13.85640646061	1.10265779085
8	13.25483399599	1.05478617516
12	12.86156123674	1.02349052335
16	12.73039151234	1.01305236834
20	12.67075522600	1.00830666346
100	12.57050641898	1.00032911687
∞	12.56637061436	1.0000000000

Table: Polygonal Isoperimetric Constants

43. Stable Versions of Geometric Inequalities.

The Isoperimetric Inequality and the Brunn-Minkowski Inequality say that some geometric quantity has a lower bound for all domains and if equality holds then the domain is special. *E.g.*, the Isoperimetric Ratio

$$\mathcal{I}(\Omega):=rac{\mathcal{L}(\partial\Omega)^2}{4\pi\,\mathcal{A}(\Omega)}-1\geq 0$$

for all $\Omega \in \mathcal{K}^2$ and "=" holds if and only if Ω is a circle.

A modern trend is to study a Stability Inequality version: If the quantity is close to its lower bound then the domain is close to special. *E.g.*, we'll show for the Isoperimetric Ratio

$$\mathcal{I}(\Omega) \ge 6\pi \, \mathcal{D}_2^2(\Omega, D_z) \tag{18}$$

where D_z is the circular Steiner disk of Ω and \mathcal{D}_2 is a standardized \mathcal{L}^2 measure of the distance. In particular, if \mathcal{I} vanishes, Ω equals its Steiner Disk, and thus is circular. The stability inequality implies the equality case: the special nature of Ω when equality holds.

The Steiner Point $z = (a_1, b_1)$ is the support-weighted average normal direction

$$a_1 = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \cos \theta \, d\theta$$
$$b_1 = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \sin \theta \, d\theta$$

The Steiner Disk D_z is centered at z with the same average width as Ω , which by Cauchy's Formula, $\mathcal{L}(\partial D_z) = \mathcal{L}(\partial \Omega)$. Expanding in Fourier Series

$$h_{\Omega}(heta) \sim \sum_{k=0}^{\infty} a_k \cos k heta + b_k \sin k heta$$

then the support function of D_z is

$$h_D(\theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta.$$



Figure: Steiner Point z and Steiner Disk D_z of Ω .

It is a fact that for $\Omega \in \mathcal{K}^2$ then its Steiner Point $z \in \Omega$.

Theorem (Wirtinger's inequality)

Let $f(\theta)$ be a piecewise $C^1(\mathbb{R})$ function with period 2π (for all θ , $f(\theta + 2\pi) = f(\theta)$). Suppose $\int_0^{2\pi} f(\theta) d\theta = 0$. Then

$$\int_0^{2\pi} f(\theta)^2 \, d\theta \le \int_0^{2\pi} \left(f'(\theta) \right)^2 \, d\theta. \tag{19}$$

Equality holds iff for some constants a_1 , b_1 ,

 $f(\theta) = a_1 \cos \theta + b_1 \sin \theta.$

Suppose in addition $\int_0^{2\pi} f(\theta) \cos \theta \, d\theta = \int_0^{2\pi} f(\theta) \sin \theta \, d\theta = 0$. Then

$$\int_0^{2\pi} f(\theta)^2 \, d\theta \leq \frac{1}{4} \int_0^{2\pi} \left(f'(\theta) \right)^2 \, d\theta. \tag{20}$$

Equality holds iff for some constants a_2 , b_2 ,

$$f(\theta) = a_2 \cos 2\theta + b_2 \sin 2\theta.$$

46. Proof of Wirtinger's Inequality.

Under the second hypotheses

$$f(\theta) = \sum_{k=2}^{\infty} \left\{ a_k \cos k\theta + b_k \sin k\theta \right\}.$$

Since the sines and cosines are complete, the Parseval equation holds

$$\int_0^{2\pi} f^2 = \pi \sum_{k=2}^{\infty} \left(a_k^2 + b_k^2 \right).$$
(21)

The Fourier Series for the derivative is given by

$$f'(\theta) \sim \sum_{k=2}^{\infty} \{-ka_k \sin k\theta + kb_k \cos k\theta\}$$

Since f' is square integrable, Bessel's inequality gives

$$\pi \sum_{k=1}^{\infty} k^2 \left(a_k^2 + b_k^2 \right) \le \int_0^{2\pi} (f')^2.$$
 (22)

Wirtinger's inequality is deduced form (21) and (22) since

$$\frac{1}{4}\int_0^{2\pi} (f')^2 - \int_0^{2\pi} f^2 \ge \pi \sum_{k=2}^{\infty} \left(\frac{k^2}{4} - 1\right) \left(a_k^2 + b_k^2\right) \ge 0.$$

Equality implies that for $k \ge 3$, $\left(\frac{1}{4}k^2 - 1\right)\left(a_k^2 + b_k^2\right) = 0$ so $a_k = b_k = 0$, thus f takes the form $f(\theta) = a_2 \cos 2\theta + b_2 \sin 2\theta$.

47. The \mathcal{L}^2 Measure of Closeness of two Convex Bodies.

Let $A, B \in \mathcal{K}^2$ and h_A and h_B their support functions. We define

$$\delta_2(A,B) = \left[\int_0^{2\pi} \left(h_A(\theta) - h_B(\theta)\right)^2 d\theta\right]^{\frac{1}{2}} = \|h_A - h_B\|_{\mathcal{L}^2}.$$

 δ_2 is a metric on \mathcal{K}^2 . Its completeness follows from

Theorem (Blaschke Selection Theorem)

Let $\{A_n\} \subset \mathcal{K}^2$ be a sequence of bodies and $B \in \mathcal{K}^2$ such that $\delta_2(A_n, B)$ is bounded. Then there is a body $A_{\infty} \in \mathcal{K}^2$ and a subsequence $\{n_j\}_{j \in \mathbb{N}}$ with $n_j \to \infty$ as $j \to \infty$ and

$$\delta_2\left({{\it A}_{{\it n}_j}},{{\it A}_{\infty }}
ight)
ightarrow 0 \qquad {\it as } j
ightarrow \infty.$$

Theorem (Brunn's Stability Inequality)

Let $A, B \in \mathcal{K}^2$ be arbitrary convex bodies in the plane. Then the mixed area satisfies

$$\mathcal{A}(A,B)^2 - \mathcal{A}(A) \mathcal{A}(B) \ge \frac{3}{2} L_B^2 \mathcal{A}(A) \mathcal{D}_2^2(A,B)$$
(23)

where D_2 is the standardized \mathcal{L}^2 distance, (25).

Proof. We prove the case that A and B are C^1 convex bodies with support functions h_A and h_B . Translate each set so that the origin is the Steiner Point

$$z(A)=z(B)=0.$$

The averages of h_A and h_B are proportional to boundary lengths L_A , L_B by Cauchy's Formula

$$\frac{1}{2\pi} \int_0^{2\pi} h_A \, d\theta = \frac{L_A}{2\pi}, \qquad \qquad \frac{1}{2\pi} \int_0^{2\pi} h_B \, d\theta = \frac{L_B}{2\pi}.$$

Let

$$f(\theta) = L_B h_A(\theta) - L_A h_B(\theta).$$

Because Steiner Points are zero,

$$\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,\cos\theta\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,\sin\theta\,d\theta=0.$$

50. Proof of Brunn's Stability Inequality..

Applying Wirtinger's Inequality (20) to f,

$$4L_B^2 \int h_A^2 - 8L_A L_B \int h_A h_B + 4L_A^2 \int h_B^2 =$$

$$4 \int (L_B h_A - L_A h_b)^2 = 4 \int f^2 \leq \int \dot{f}^2 = \int (L_B \dot{h}_A - L_A \dot{h}_b)^2$$

$$= L_B^2 \int \dot{h}_A^2 - 2L_A L_B \int \dot{h}_A \dot{h}_B + L_A^2 \int \dot{h}_B^2$$

Thus

$$2L_{A}L_{B}\left(\int h_{A}h_{B} - \dot{h}_{A}\dot{h}_{B}\right) - L_{B}^{2}\left(\int h_{A}^{2} - \dot{h}_{A}^{2}\right) - L_{A}^{2}\left(\int h_{B}^{2} - \dot{h}_{B}^{2}\right)$$
$$\geq 3\int \left(L_{B}^{2}h_{A}^{2} - 2L_{A}L_{B}h_{A}h_{B} + L_{A}^{2}h_{B}^{2}\right) = 3\int \left(L_{B}h_{A} - L_{A}h_{B}\right)^{2}$$

Hence

$$2L_{A}L_{B}\mathcal{A}(A,B) - L_{B}^{2}\mathcal{A}(A) - L_{A}^{2}\mathcal{A}(B) \geq \frac{3}{2}L_{A}^{2}L_{B}^{2}\delta_{2}^{2}\left(\frac{1}{L_{A}}A,\frac{1}{L_{B}}B\right).$$
 (24)

51. Proof of Brunn's Stability Inequality...

We give an asymmetrical argument, slightly different than before. From (24),

$$L_{A}^{2}\left(\mathcal{A}(A,B)^{2}-\mathcal{A}(A)\mathcal{A}(B)\right)$$

= $\mathcal{A}(A)\left(2L_{A}L_{B}\mathcal{A}(A,B)-L_{B}^{2}\mathcal{A}(A)-L_{A}^{2}\mathcal{A}(B)\right)$
+ $\left(L_{A}\mathcal{A}(A,B)-L_{B}\mathcal{A}(A)\right)^{2}$
 $\geq \frac{3}{2}L_{A}^{2}L_{B}^{2}\mathcal{A}(A)\delta_{2}^{2}\left(\frac{1}{L_{A}}A,\frac{1}{L_{B}}B\right)$

so inequality (23) follows.

We call the dimensionless Standardized Distance which compares bodies up to homothety

$$\mathcal{D}_2(A,B) = \delta_2\left(\frac{1}{L_A}A, \frac{1}{L_B}B\right).$$
(25)

Using the isoperimetric ratio, we see that the problem of what is the greatest area for fixed length or what is the least length for fixed area are equivalent.

Theorem (Stability Isoperimetric Inequality.)

Let C be a simple closed curve in the plane whose length is L and that encloses the region Ω of area A. Then the following inequalities hold for the isoperimetric ratio and the isoperimetric difference

$$\mathcal{I}(\Omega) = \frac{L^2}{4\pi A} - 1 \ge 6\pi \mathcal{D}_2^2\left(\widehat{\Omega}, \widehat{D_z}\right), \qquad (26)$$

$$\Delta(\Omega) = L^2 - 4\pi A \ge 6\pi \,\delta_2^2\left(\widehat{\Omega}, \widehat{D_z}\right),\tag{27}$$

where δ_2 is the \mathcal{L}^2 distance, \mathcal{D} is the standardized distance (25), $\widehat{\Omega}$ is the convex hull and $\widehat{D_z}$ is the Steiner Disk of the convex hull.

Proof. We have already observed that the isoperimetric quantities are decreased under convex hull $\mathcal{I}(\Omega) \geq \mathcal{I}(\widehat{\Omega})$ and $\Delta(\Omega) \geq \Delta(\widehat{\Omega})$. For (26), we put $A = \widehat{\Omega}$ and let B be the unit disk in (23)

$$\mathcal{I}(A) = \frac{L_A^2 - 4\pi \mathcal{A}(A)}{4\pi \mathcal{A}(A)} = \frac{4\left(\mathcal{A}(A, B)^2 - \mathcal{A}(A) \mathcal{A}(B)\right)}{4\pi \mathcal{A}(A)}$$
$$\geq \frac{6\mathcal{A}(A)L_B^2 \delta_2^2 \left(\frac{1}{L_A} A, \frac{1}{L_B} B\right)}{4\pi \mathcal{A}(A)}$$
$$= 6\pi \mathcal{D}_2^2 (A, B).$$

Since they are standardized to unit length in \mathcal{D}_2 , scaling the ball doesn't matter: $\mathcal{D}_2(A, B) = \mathcal{D}_2(A, \widehat{D_z})$.

For (27), we put
$$B = \widehat{\Omega}$$
 instead and let A be the unit disk in (23)

$$\Delta(B) = L_B^2 - 4\pi \mathcal{A}(B) = 4 \left(\mathcal{A}(A, B)^2 - \mathcal{A}(A) \mathcal{A}(B) \right)$$

$$\geq 6 \mathcal{A}(A) L_B^2 \, \delta_2^2 \left(\frac{1}{L_A} A, \frac{1}{L_B} B \right)$$

$$= 6\pi \, \delta_2^2 \left(\widehat{D_z}, \widehat{\Omega} \right)$$

since the circumference of the Steiner Disk is L_B so $\widehat{D_z} = \frac{L_B}{L_A}A$.

Alternatively, we can conclude the Isoperimetric Inequality directly from Wirtinger's Inequality.

Recall Green's theorem. If p and q are differentiable functions on the plane and Γ is a piecwise C^1 curve bounding the region Ω then

$$\oint_{\Gamma} p \, dx + q \, dy = \iint_{\Omega} (q_x - p_y) \, dx \, dy.$$

If we take q = x and p = 0 then Green's theorem says

$$\oint_{\Gamma} x \, dy = \mathcal{A}(\Omega). \tag{28}$$

The same formula can be used to make sense of area even for curves that are merely rectifiable, namely, those whose length is the limit of lengths of approximating polygonal curves. 56. Hurwitz's proof of the Isoperimetric Inequality.



Figure: Adolf Hurwitz 1859–1919.

Proof. We suppose that the boundary curve has length L is parameterized by arclength, thus given by two piecewise C^1 and L periodic functions x(s), y(s) that satisfy

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1.$$

We convert to 2π periodic functions

$$f(heta) = x\left(rac{L heta}{2\pi}
ight), \qquad g(heta) = y\left(rac{L heta}{2\pi}
ight)$$

so that writing "'" $= d/d\theta$ gives

$$(f')^2 + (g')^2 = \frac{L^2}{4\pi^2}.$$
 (29)

We now simply to estimate the area integral (28). Using $\int g' d\theta = 0$, Wirtinger's Inequality and (29),

$$2A = 2\int_0^{2\pi} fg' d\theta = 2\int_0^{2\pi} (f - \bar{f})g' d\theta$$
$$= \int_0^{2\pi} (f - \bar{f})^2 + (g')^2 - (f - \bar{f} - g')^2 d\theta$$
$$\leq \int_0^{2\pi} (f')^2 + (g')^2 d\theta = \int_0^{2\pi} \frac{L^2}{4\pi^2} d\theta = \frac{L^2}{2\pi},$$

which is the isoperimetric inequality.

58. Hurwitz's proof of the Isoperimetric Inequality...

Equality forces equality in Wirtinger's Inequality so

$$f(\theta) = \bar{f} + a\cos\theta + b\sin\theta$$

for some constants a, b. Equality also forces the dropped term to vanish

$$\int_0^{2\pi} (f-\bar{f}-g')^2 \, d\theta = 0$$

so that

$$g'=f-\overline{f}.$$

Hence

$$g(\theta) = \bar{g} + a\sin\theta - b\cos\theta.$$

Hence (29) implies

$$a^2+b^2=\frac{L^2}{4\pi^2}$$

so (x(s), y(s)) is a circle of radius $L/2\pi$.

59. Minkowski's Proof of Brunn's Inequality.

Alternatively, here is Minkowsi's proof of (10) using induction. The inequality is proved for finite unions of rectangles first and then a limiting process gives the general statement. Suppose that $A = \bigcup_{i=1}^{n} R_i$ and $B = \bigcup_{i=1}^{m} S_i$ where R_i and S_i are pairwise disjoint open rectangles, that is $R_i \cap R_i = \emptyset$ and $S_i \cap S_i = \emptyset$ if $i \neq j$. The proof is based on induction on $\ell = m + n$. For $\ell = 2$ there are two rectangles as in (1). The area $\mathcal{A}((a,b)\times(c,d)\boxplus(e,f)\times(g,h))=\mathcal{A}((a+e,b+f)\times(c+g,d+h))$ = (b - a + f - e)(d - c + h - g)= (b-a)(d-c) + (f-e)(h-g) + (b-a)(h-g) + (f-e)(d-c) $\geq (b-a)(d-c) + (f-e)(h-g) + 2\sqrt{(b-a)(h-g)(f-e)(d-c)}$ $=\left(\sqrt{(b-a)(d-c)}+\sqrt{(f-e)(h-g)}
ight)^2$ $=\left(\sqrt{\mathcal{A}((a,b) imes(c,d))}+\sqrt{\mathcal{A}((e,f) imes(g,h))}
ight)^2,$

where we have used the Arithmetic-Geometric Mean Inequality

$$\frac{1}{2}(|X|+|Y|)-\sqrt{|X||Y|}=\frac{1}{2}\left(\sqrt{|X|}-\sqrt{|Y|}\right)^2\geq 0.$$

Now assume the induction hypothesis: suppose that (10) holds for $A = \bigcup_{i=1}^{n} R_i$ and $B = \bigcup_{i=1}^{m} S_i$ with $m + n \le \ell - 1$. For A and B so that $m + n = \ell$, we may arrange that n > 2. Then some vertical or horizontal plane, say $x = x_1$, can be placed between two rectangles. Let $R'_i = R_i \cap \{(x, y) : x < x_1\}$ and $R''_i = R_i \cap \{(x, y) : x > x_1\}$ and put $A' = \bigcup_i R'_i$ and $A'' = \bigcup_i R''_i$. By choice of the plane, the number of nonempty rectangles in #A' < n and #A'' < n, but both A' and A'' are nonempty. Select a second plane $x = x_2$ and set $S'_i = S_i \cap \{(x, y) : x < x_2\}$ and $S''_i = S_i \cap \{(x, y) : x > x_2\}$ and put $B' = \bigcup_i S'_i$ and $B'' = \bigcup_i S''_i$. Note that $\#B' \leq m$ and $\#B'' \leq m$. x_2 can be chosen so that the area fraction is preserved

$$heta = rac{\mathcal{A}(\mathcal{A}')}{\mathcal{A}(\mathcal{A}') + \mathcal{A}(\mathcal{A}'')} = rac{\mathcal{A}(\mathcal{B}')}{\mathcal{A}(\mathcal{B}') + \mathcal{A}(\mathcal{B}'')}.$$

By definiton of Minkowski sum, $A \boxplus B \supset A' \boxplus B' \cup A'' \boxplus B''$. Furthermore, observe that $A' \boxplus B'$ is to the left and $A'' \boxplus B''$ is to the right of the plane $x = x_1 + x_2$, so they are disjoint sets. Now we may use the additivity of area and the induction hypothesis on $A' \boxplus B'$ and $A'' \boxplus B''$.

$$\begin{split} \mathcal{A}(A \boxplus B) &\geq \mathcal{A}(A' \boxplus B') + \mathcal{A}(A'' \boxplus B'') \\ &\geq \left(\sqrt{\mathcal{A}(A')} + \sqrt{\mathcal{A}(B')}\right)^2 + \left(\sqrt{\mathcal{A}(A'')} + \sqrt{\mathcal{A}(B'')}\right)^2 \\ &= \theta \left(\sqrt{\mathcal{A}(A)} + \sqrt{\mathcal{A}(B)}\right)^2 + (1 - \theta) \left(\sqrt{\mathcal{A}(A)} + \sqrt{\mathcal{A}(B)}\right)^2 \\ &= \left(\sqrt{\mathcal{A}(A)} + \sqrt{\mathcal{A}(B)}\right)^2. \end{split}$$

Thus the induction step is complete.

Finally every compact region can be realized as the intersection of a decreasing sequence of open sets $A_n \supset A_{n+1}$ so that $A = \bigcap_n A_n$. A_n can be taken as the interiors of a union of finitely many closed squares. For each $\varepsilon = 2^{-n} > 0$ consider the closed squares in the grid of side ε which meet the set. Then the interior of the union of these squares is A_n . Removing the edges of the squares along gridlines A'_n results in a set with the same area. The result follows since Lebesgue measure of the limit is limit of the Lebesgue measure for decreasing sequences. Since the Minkowski sum of a decreasing set of opens is itself a decreasing set of opens, it follows that

$$\begin{split} \sqrt{\mathcal{A}(A \boxplus B)} &= \lim_{n \to \infty} \sqrt{\mathcal{A}(A_n \boxplus B_n)} \geq \lim_{n \to \infty} \sqrt{\mathcal{A}(A'_n \boxplus B'_n)} \\ &\geq \lim_{n \to \infty} \left(\sqrt{\mathcal{A}(A'_n)} + \sqrt{\mathcal{A}(B'_n)} \right) = \sqrt{\mathcal{A}(A)} + \sqrt{\mathcal{A}(B)} \end{split}$$

and we are done. \Box

Thanks!